Truncated Power Series Expansion
Motivation

- It is useful to find nonlinear map to very high orders. The map can be used to extract beam dynamics quantities.

\[
\begin{pmatrix}
W \\
\begin{pmatrix}
x \\
p_x \\
y \\
p_y \\
ct \\
\delta
\end{pmatrix}_i
\end{pmatrix} \rightarrow \begin{pmatrix}
V \\
\begin{pmatrix}
x \\
p_x \\
y \\
p_y \\
ct \\
\delta
\end{pmatrix}_f
\end{pmatrix}
\]

Complicate nonlinear lattice
Very time-consuming to track
A faster compromise?

• We want to establish a formula (maybe complicate) to describe such complex, time-consuming.

• Usually no exact analytical solution.

\[
\begin{pmatrix}
  x \\
  p_x \\
  y \\
  p_y \\
  ct \\
  \delta
\end{pmatrix}_f = M \begin{pmatrix}
  x \\
  p_x \\
  y \\
  p_y \\
  ct \\
  \delta
\end{pmatrix}_i + error
\]
TPSA

• Truncated power series algorithm is suitable for such goal.
• It is a mathematical treatment, no physics inside.
• Express the map as polynomials:

\[ W = \sum_{j}^{\Omega} M_j V^j + O(V^{\Omega+1}) \]

• Convergence? Which order to stop?
Determine the coefficient

• Coefficient can be determined by the derivative with respect to the elements of initial vector.
• For N-dimensional problem, the coefficient for the ith element of order j gives
  \[ M_{i,j} = j_1 + j_2 + \cdots + j_N \sim \frac{\partial^j W_i}{\partial V_1^{j_1} \partial V_2^{j_2} \cdots \partial V_N^{j_N}} \]
• How to find them effectively?
• The answer is TPSA, introduced by M. Berz in 1989
1-D case

• Lets assume we need to construct 1D Taylor map:

\[ y(x) = y(x_0) + (x - x_0)y'(x_0) + \cdots + \frac{1}{\Omega!}(x - x_0)^\Omega y^{(\Omega)}(x_0) \]

• To get the value of all the derivatives at \( x_0 \), the simple and usual way is:

\[ y'(x_0) = \frac{y(x_0 + \epsilon) - y(x_0)}{\epsilon} \]
\[ y''(x_0) = \frac{y(x_0 + \epsilon) + y(x_0 - \epsilon) - 2y(x_0)}{\epsilon^2} \]

• Does not guarantee accuracy.
TPSA method

• Let’s start with an example

\[ y(x) = \frac{1}{x^2 + 1} \quad y'(x) = -\frac{2x}{(x^2 + 1)^2} \]

• We know that \( y(1) = 0.5, y'(1) = -0.5 \)

• With the following rule, we plug in \( x = (1, 1) \)

\[
\begin{align*}
(a, b) + (c, d) &= (a + c, b + d) \\
(a, b) \cdot (c, d) &= (ac, ad + bc) \\
(a, b)^{-1} &= (1/a, -b/a^2)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{(1, 1)^2 + 1} &= \frac{1}{(1, 2) + 1} \\
&= \frac{1}{(2, 2)} \\
&= \left(\frac{1}{2}, \frac{-1}{2}\right)
\end{align*}
\]
Why TPSA works?

• TPSA may give the value of the derivatives without calculate the expression.

• In the 1-D, first derivative example, we substitute \( x \) with \((a,1)\). Since \( (x,x') |_{x=a}=(a,1) \)

• The rule also comes naturally,

\[
\begin{align*}
  y(x) &= f(x) + g(x), \quad y'(x) = f'(x) + g'(x) \\
  y(x) &= f(x)g(x), \quad y'(x) = f(x)g'(x) + g(x)f'(x) \\
  y(x) &= 1/f(x), \quad y' = -f'(x)/f^2(x)
\end{align*}
\]
1D, higher orders

• Substitute $x$ with $(a, 1, 0, 0, \ldots, 0)$, a $\Omega+1$ dimension, we will get $(y(a), y'(a), \ldots, y^{(\Omega)}(a))$

• Multiplication rule:

$$(c_0, c_1, \ldots, c_n) = (a_0, a_1, \ldots, a_n) \times (b_0, b_1, \ldots, b_n)$$

$$= (a_0 b_0, a_1 b_0 + b_1 a_0, a_2 b_0 + 2a_1 b_1 + a_0 b_2, \ldots)$$

$$c_m = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} a_k b_{m-k}$$

Binomial Coefficient!
Homework

• Assuming $n > 0$, find the rules for

$$ (a, a_1)^n \quad (a, a_1)^{-n} $$

• For 2$^{nd}$ order TPSA, find the rule for

$$ (a, a_1, a_2)^{-1} $$
Special functions

• We have demonstrated the TPSA for arithmetic operations. How about special functions such as \( \sin(x) \), \( \exp(x) \), \( \log(x) \), etc.

• Answer: use Taylor expansion:

\[
f(x) = f(x_0) + (x - x_0)f'(x_0) + \cdots + \frac{1}{\Omega!}(x - x_0)^\Omega f^{(\Omega)}(x_0)
\]

• Note that

\[
[(x_0, x_1, x_2, \ldots) - x_0]^m = (0, x_1, x_2, \ldots)^m
\]

\[
= (0, 0, \ldots, X, X)
\]

With \( m \) leading zeros
Special function, cont’d

• Therefore, for \(X = (x_0, x_1, x_2, \ldots, x_n)\), the arbitrary function \(f(x)\):

\[
f(x) = f((x_0, 0, \ldots)) + \sum_{n=1}^{\Omega} \frac{(0, x_1, x_2, \ldots)^n f^{(n)}((x_0, 0, \ldots))}{n!}
\]

• Let’s do exercise for
  
  – \(\text{Exp}(X), \text{Log}(X)\) and \(\sin(X)\)
  
  – And special cases, \(\exp(x=(x_0, 1, 0, \ldots))\) and \(\sin(x=(x_0, 1, 0, \ldots))\)
Extension to multi-dimension

• Now consider the independent variables has more than one dimension, $y = y(x_1, x_2, x_3, \ldots, x_n)$
• One order has multiple terms
• How many terms for N dimension, M orders?
• Two variable example $(x_1, x_2)$:

$$(a_0, a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, a_{0,2}, a_{3,0}, a_{2,1}, a_{1,2}, a_{0,3}, \ldots)$$
Application in accelerator

• Basic application
  – Express 6D coordinate as its initial value.
  – 6 Dimension with some preset orders
  – Linear order gives the 6-by-6 matrix
  – Derive linear/nonlinear properties, optics, nonlinear tune dependence, nonlinear driving terms, etc.

• Advanced application
  – Include magnet parameter in to account, perfect for optimizing the machine.
TPSA Library

• The initial library written in Fortran is widely used.
• A C++ version by L.Y. Yang is now used in PTC.
• A good library should
  – Accept arbitrary order and dimension
  – Optimize the memory usage.
  – Optimize the calculation
  – speed.
Reference

• Alex Chao, Lecture Notes on Special Topics in Accelerator Physics.