Lecture 3: Modeling Accelerators – Fringe fields and Insertion devices

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Outline

• Fringe field effects
  – Dipole
  – Quadrupole

• Modeling of insertion devices

• Radiation damping and quantum excitation
  – Damping, excitation, Ohmi envelope

• Longitudinal tracking with acceleration
Magnetic field profile and the hard-edge model

- Most accelerator modeling codes use the hard-edge model for magnet – constant Hamiltonian.
- Real magnets always have a smooth transition at the edges – fringe fields.

Hard edge model
- Field or gradient is a constant equal to the average value inside the magnet body.
- The effective length is the integrated strength divided by the field or gradient.

The hard-edge model is non-maxwellian!
Vector potential with fringe field

- With cylindrical symmetry, the vector potential of a normal multipole \( n = 1 \) for dipole, etc.) with fringe field (with Coulomb gauge \( \nabla \cdot A = 0 \)) on a straight geometry is

\[
A_r = \frac{\cos n\theta}{2n!} \sum_{p=0}^{\infty} \frac{1}{n+p+1} G_{n,2p+1}(s) r^{2p+n+1},
\]

\[
A_\theta = \frac{\sin n\theta}{2n!} \sum_{p=0}^{\infty} \frac{1}{n+p+1} G_{n,2p+1}(s) r^{2p+n+1},
\]

\[
A_s = -\frac{\cos n\theta}{n!} \sum_{p=0}^{\infty} G_{n,2p}(s) r^{2p+n}.
\]

With \( x = r \cos \theta \) and \( y = r \sin \theta \) and

\[
G_{n,2p}(s) = (-1)^p \frac{n!}{4^p (n+p)! p!} \frac{d^{2p} G_{n,0}(s)}{ds^{2p}}, \quad \text{and} \quad G_{n,2p+1}(s) = \frac{dG_{n,2p}(s)}{ds}.
\]

Example: for dipole \( n = 1 \), assuming \( G_{1,0} = B_0 \Theta(z) \), then

\[
A_x = \frac{x^2-y^2}{4} B_0 \Theta'(z) + O(4), \quad A_y = \frac{xy}{2} B_0 \Theta'(z) + O(4)
\]

\[
A_z = -B_0 \Theta(z) x + \frac{1}{8} B_0 \Theta''(z) x (x^2 + y^2) + O(5).
\]

Correspondingly, the magnetic field is

\[
B_x = -B_0 \Theta''(z) \frac{xy}{4}, \quad B_y = B_0 \Theta(z) - \frac{1}{8} B_0 \Theta''(z) (x^2 + 3y^2)
\]

\[
B_z = B_0 \Theta'(z) y.
\]
Approximate field profile with Enge function

- The longitudinal profile function can be approximated well with the Enge function

\[ \theta(z) = \frac{1}{1 + e^{\sum_{i=0}^{n} a_i \left(\frac{s-s_0}{G}\right)^i}} \]

Where \( s_0 \) is at the effective edge, \( G \) is the full gap.

Effective edge at \( s_0 = 0.7473286 \) m
Gap \( G = 0.05 \) m.
Coefficients:
\( a = [0.0053, -0.0856, 0.4409, -0.9460, 2.0744, 0.2416] \)
At the entrance edge:

\( y \) and \( Y \) axises pointing out of page.

Suppose points 1, 2 are boundaries of the fringe field, point 0 is the entrance face of the bend magnet. For the hard-edge model, the drift ends at the \((x, y)\) plane at 0, followed by a sector dipole whose entrance face coincides with the plane.

The difference between reality and the hard-edge model is captured by the following transformations:

1. At 0, in drift space, propagate particles to the \((X, Y)\) plane.
2. In drift space, back propagate to \((X, Y)\) plane at 1.
3. In actual fields, propagate to point 2.
4. In sector dipole, back propagate to point 0.
5. At 0, in sector dipole, back propagate particles to the original \((x, y)\) plane.

Coordinate transformations between \((x, y, s)\) system and \((X, Y, Z)\) system are performed at steps (1) and (5).

The differences in phase space coordinates before and after the above procedure is the fringe field effects.
Edge focusing

At the entrance edge:
\[ M_x = \begin{pmatrix} 1 & 0 \\ h \tan \beta_1 & 1 \end{pmatrix}, \]
\[ M_y = \begin{pmatrix} 1 & 0 \\ -h \tan(\beta_1 - \psi_1) & 1 \end{pmatrix}, \]

At the exit edge
\[ M_x = \begin{pmatrix} 1 & 0 \\ h \tan \beta_2 & 1 \end{pmatrix}, \]
\[ M_y = \begin{pmatrix} 1 & 0 \\ -h \tan(\beta_2 - \psi_2) & 1 \end{pmatrix}, \]

Definition of entrance and exit angles.
(K. Brown, SLAC-75)

Correction for the vertical plane due to finite extent of fringe field:
\[ \psi_{1,2} = K h G (1 + \sin^2 \beta_{1,2}) / \cos \beta_{1,2} \]

With \( G \) the full gap and
\[ K = \int \frac{B_y(z)(B_0 - B_y(z))}{g B_0^2} \, dz \]
Fringe field effect of quadrupoles

- The Hamiltonian for a quadrupole with fringe field

From \( H \approx -\left(1 + \delta\right) - \frac{1}{2(1+\delta)}(p_x^2 + p_y^2) + \frac{1}{1+\delta}(p_x a_x + p_y a_y)\) - \(a_s\), ignore \(\delta\), apply vector potential for quadrupole, get

\[
H = \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}K(s)(x^2 - y^2) - \frac{1}{4}K'(s)(x^2 - y^2)(xP_x + yP_y) - \frac{1}{12}K''(s)(x^4 - y^4) + O(X^6),
\]

- Linear effect (for the entrance edge)

\[
H = H_0 + \tilde{H}(s),
\]

With the hard-edge model

\[
H_0 = \begin{cases} 
\frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}K_0(x^2 - y^2), & s \geq s_0 \\
\frac{1}{2}(P_x^2 + P_y^2), & s < s_0
\end{cases}
\]

and the perturbation term

\[
\tilde{H}(s) = \frac{1}{2} \tilde{K}(s)(x^2 - y^2)
\]

with

\[
\tilde{K}(s) = \begin{cases} 
K(s) - K_0, & s \geq s_0 \\
K(s), & s < s_0
\end{cases}
\]
• **Linear effect as a thin element** \( F_Q \)

Transfer matrix

\[
M(1 \rightarrow 2) = M_D(1 \rightarrow 0)F_Q M_Q(0 \rightarrow 2)
\]

The generating function for map* \( F_Q = e^{f_2} \) is calculated to be

\[
f_2 = \frac{I_1}{2}(xp_x - yp_y)
\]

With integral \( I_1 \) defined as

\[
I_1 = \int_{-D}^{+D} \tilde{K}(s)(s - s_0) \, ds
\]

The transfer matrix is \( \text{diag}(e^{I_1}, e^{-I_1}, e^{-I_1}, e^{I_1}) \).

At the exit edge, integral \( I_1 \) has opposite sign. So the two edges tend to cancel. The cancellation is not complete for quadrupoles with finite length. The net effect includes a tune shift (always negative)

\[
\Delta \nu_x = -\frac{K_0^2 L \beta_x}{2\pi} \left| I_1 \right|, \quad \Delta \nu_y = -\frac{K_0^2 L \beta_y}{2\pi} \left| I_1 \right|
\]

*The Lie map \( e^{:f_2:} = 1 + :f_2: + \frac{1}{2!} :f_2:^2 + \cdots \) is an operator, where \( :f_2: \cdot g = [f_2, g] \), Poisson bracket. The Lie map for a constant Hamiltonian is \( e^{-:LH:} \).**
Nonlinear effect of quadrupole fringe field

- Apply the same approach to find the generating function for nonlinear effects (the map is $e^{if_4}$). For the entrance edge (change sign for the exit edge).

$$f_4 = -\frac{1}{12}K_0(x^3p_x + 3xy^2p_x - y^3p_y - 3x^2yp_y)$$

This cannot be symplectically integrated. However, the generating function for a skew quadrupole is integrable.

$$f_{4,\text{skew}} = \frac{a_2}{6}(x^3p_y + y^3p_x)$$

$\exp(\alpha y^3p_x) x = x - \alpha y^3,$

$\exp(\alpha y^3p_x) p_y = p_y - 3\alpha y^3p_x,$

Similarly for the $x^3p_y$ term. These are two kicks!

Therefore, we can model the nonlinear effects of quadrupole fringe field with a symplectic map by rotating $\frac{\pi}{4}$, applying the skew quad fringe kick, and rotating back.
Example

Magnetic field

\[ B_x = B_1 [y \Theta'(z) - \frac{1}{12} \Theta'''(z)(3x^2 y + y^3)] \]
\[ B_y = B_1 [x \Theta(z) - \frac{1}{12} \Theta'''(z)(x^3 + 3xy^2)] \]
\[ B_z = \text{sgn}(z) B_1 \Theta'(z) xy \]

SPEAR3 quadrupole fringe field

\[ I_{la} = \frac{I_1}{k_0} = 0.61 \times 10^{-3} \text{ m}^2, \quad \Delta = \sqrt{\frac{6I_1}{k_0}} = 0.060 \text{ m} \]

<table>
<thead>
<tr>
<th>Case</th>
<th>Element</th>
</tr>
</thead>
<tbody>
<tr>
<td>QuadLinearPass</td>
<td>Quad</td>
</tr>
<tr>
<td>QuadLinearFPass</td>
<td>Quad</td>
</tr>
<tr>
<td>Field Pass + Quad</td>
<td>[Drift QuadField Drift]</td>
</tr>
</tbody>
</table>
On the mid-plane, the magnetic field

\[ B_y = B_0 \cos kz \]

Trajectory

\[ x(z) = x_0 - A \cos kz \]

with amplitude

\[ A = \frac{1}{k^2 \rho} = \frac{K \lambda_u}{\gamma 2\pi}. \]

Wiggler parameter

\[ K = \frac{eB_0}{kmc} = 0.934B_0[T]\lambda_u[cm]. \]

with \( k = \frac{2\pi}{\lambda_u}. \)
The field is periodic in the longitudinal direction.

Suppose the vertical field is

\[ B_y = B(x, y) \cos kz \]

with symmetry \( B(x, y) = B(-x, y) \) and \( B(x, y) = B(x, -y) \).

It can be shown that to satisfy \( \nabla \cdot B = 0 \) and \( \nabla \times B = 0 \), we need

\[ \frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} - k^2 B = 0 \]

The other field components

\[ B_x = \frac{\partial}{\partial x} \int B(x, y) dy \cos kz, \]
\[ B_z = (-k) \int B(x, y) dy \cos kz. \]

Symmetry leads to \( B_x(x = 0) = 0, B_x(y = 0) = 0 \) and \( B_z(y = 0) = 0 \).

The Halbach wiggler field model

\[ B_y = B_0 \cosh k_x x \cosh k_y y \cos kz, \]
\[ B_x = \frac{k_x}{k_y} B_0 \sinh k_x x \sinh k_y y \cos kz, \]
\[ B_z = -\frac{k}{k_y} B_0 \cosh k_x x \sinh k_y y \cos kz. \]

with \( k_x^2 + k_y^2 = k^2 \).
Effects of ideal ID on the beam

- The Hamiltonian for the Halbach model (to 4th order)

\[
H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{4k^2\rho^2} (k_x^2 x^2 + k_y^2 y^2) + \frac{1}{12k^2\rho^2} (k_x^4 x^4 + k_y^4 y^4 + 3k_x^2 k_y^2 x^2 y^2) - \frac{\sin ks}{2k\rho} (p_x (k_x^2 x^2 + k_y^2 y^2) - 2k_x p_y x y).
\]

Assume \( k_x = 0 \) (ideal planar wiggler with wide poles)

\[
H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{y^2}{4\rho^2} - \frac{k \sin ks}{2\rho} p_x y^2 + \frac{k^2}{12\rho^2} y^4 + O(X^5)
\]

For the ideal planar wiggler, the effect on beam is on the vertical plane.

- Linear effect

Tune shift:

\[
\Delta \nu_y = \frac{\beta^* L}{8\pi \rho^2} + \frac{L^3}{96\pi \rho^2 \beta^*}
\]

where \( L \) is wiggler length, \( \beta^* \) is beta function at ID center (minimum)

- Nonlinear effects

- Octupole-like effect causing amplitude-dependent tune shift, \( \propto \frac{B_0^2 L}{\lambda_u^2} \)
Effects of an imperfect wiggler

Kick to the beam (by field integral on trajectory)

\[
\Delta x' = \frac{1}{B \rho} \int B_y(x(z), y, z)dz
\]

\[
= \frac{1}{B \rho} \int B_y(x, y, z)dz + \frac{1}{B \rho} \int \frac{\partial B_y}{\partial x} \Delta x(z)dz
\]

Static \hspace{1cm} \text{dynamic}

Static field integrals (on straight path):
First integral \( I_{1y}(x) = \int B_y(x, 0, z)dz, \quad \Delta x' = I_{1y}/B \rho \)
Second integral \( I_{2y}(x) = \int dz \int_{-\infty}^{z} B_y(x, 0, \tilde{z})d\tilde{z}, \quad \Delta x = I_{2y}/B \rho \)
(similarly for \( I_{1x} \) and \( I_{2x} \))

Quadrupole int. \( I_q = \int \frac{\partial B_y}{\partial x} dz, \quad \Delta v_x = \frac{\beta x I_q}{4\pi}, \quad \Delta v_y = -\frac{\beta y I_q}{4\pi} \)

Skew quad int. \( I_{sq} = \int \frac{\partial B_x}{\partial x} dz, \quad \text{coupling} \)

Sextupole int. \( I_{sex} = \int \frac{\partial^2 B_y}{\partial x^2} dz, \quad \text{nonlinear dynamics} \)

…

The field integrals can be obtained from measurements. These static effects can be modeled as multipole kicks.

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Dynamic effects from field roll-off

Kick from the dynamic effect
\[ \Delta x' = \frac{1}{B \rho} \int \frac{\partial B_y}{\partial x} \Delta x(z) dz, \]
with
\[ \Delta x(z) = -\frac{1}{k^2 \rho} \cos kz, \quad \frac{\partial B_y}{\partial x} = \frac{\partial B(x,0)}{\partial x} \cos kz, \quad \text{(on the mid-plane)} \]
So the kick is
\[ \Delta x' = -\frac{1}{2} \frac{L}{(B \rho)^2 k^2} \frac{B_0 \partial B(x,0)}{\partial x} \]

The dynamic kick effect is particularly severe for elliptically-polarized undulator (EPU)

Field roll-off of the planar phase for an EPU

Dynamic integral from the field roll-off
Modeling of effects of IDs on the beam

There are methods of symplectic integration for \( s \)-dependent Hamiltonian when analytic forms of vector potential is available (not discussed here).

A commonly used method for ID modeling is the kick map method:

\[
(x, y) \rightarrow (\Delta x', \Delta y')
\]

The kicks can be derived from the potential (P. Elleaume)

\[
\Psi(x, y) = \int dz \left( \left( \int_{-\infty}^{z} B_x(x, y, \tilde{z}) d\tilde{z} \right)^2 + \left( \int_{-\infty}^{z} B_y(x, y, \tilde{z}) d\tilde{z} \right)^2 \right)
\]

with the kicks given by

\[
\Delta x' = -\frac{1}{2(B\rho)^2} \frac{\partial \Psi(x,y)}{\partial x},
\]

\[
\Delta y' = -\frac{1}{2(B\rho)^2} \frac{\partial \Psi(x,y)}{\partial y},
\]

Comparison of the Elleaume kick map with numerical integration (Runge-Kutta)
Example of ID field (EPU for MAX-II)

Orbit for 1.5 GeV electrons

I. Blomqvist
References

4. E. Forest and Milutinovic, NIMA 269, 474-482 (1988)
6. X. Huang, et al, Lattice modeling for SPEAR3, IPAC 2010
8. P. Elleaume, A new approach to the electron beam dynamics in undulators and wigglers, EPAC 92, p. 661 (1992)