Lecture 2: Modeling Accelerators – Calculation of lattice functions and parameters

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Outline

• Closed orbit
• Transfer matrix, tunes, Optics functions
• Chromatic effects: dispersion function, chromaticity
• Radiation integrals and lattice parameters
• Characterization of nonlinear dynamics
• Dynamic aperture and momentum aperture from tracking
Calculation of closed orbit

• Closed orbit is the fixed point of the one-turn map \((M)\)

\[ M(X_c) = X_c \]

Here the map is evaluated with tracking.

If the one-turn map is linear, i.e.

\[ X_1 = M(X_0) = R(0)X_0 + \Delta \]

where \(R(0)\) is the transfer matrix on the reference orbit and \(\Delta\) is the accumulated coordinate displacement after one turn, then the closed orbit is

\[ X_c = (I - R(0))^{-1}\Delta \]

For the general nonlinear case, \(X_c\) can be found numerically with iteration.

Closed orbit differs from the design orbit when the guiding magnetic fields deviate from the ideal values (field errors or misalignment) or there is an energy error.
• Closed orbit with fixed momentum deviation $\delta$ (useful for computing optics functions).

Starting with $X_0 = (0,0,0,0,0,\delta)^T$, at iteration $n$

$$Y_n = M(X_n)$$

Also calculate $R_4(X_n)$, the $4 \times 4$ transverse transfer matrix about $X_n$. Then the next iteration moves to

$$X_{n+1}^{(4)} = X_n^{(4)} + (I - R_4(X_n))^{-1}(Y_n - X_n)^{(4)}$$

where superscript "$(4)$" indicates vector with the first 4 coordinates, the longitudinal coordinates are fixed at $(z = 0, \delta)$.

• Closed orbit with fixed frequency – situation in real machine.
  – Closed orbit in full 6D phase space.
  – The above iterative procedure is valid, using $6 \times 6$ matrix and full vector.
  – RF cavity, radiation damping can be turned on.
Transfer matrix

- Each element may generate its own transfer matrix internally. The transfer matrix for a section or a full ring is the product of the transfer matrices of the elements.

\[
R_{0\rightarrow N} = R_{0\rightarrow 1}R_{1\rightarrow 2} \ldots R_{N-1\rightarrow N}
\]

- The transfer matrix for an element, a section or a full ring can also be obtained from tracking with numerical differential.

To find the transfer matrix around orbit \(X_0 = (x_0, p_{x0}, y_0, p_{y0}, z_0, \delta_0)^T\) (e.g., the reference orbit \(X_0 = (0,0,0,0,0,0)^T\)), tracking particles

\[
X_1(-\Delta) = M(X_0 - \Delta), \quad \text{Here } M \text{ represents the map from location 0 to 1,}
\]

\[
X_1(+\Delta) = M(X_0 + \Delta), \quad \text{which is evaluated with tracking.}
\]

\[
\Delta_1 = (\varepsilon, 0, 0, 0, 0, 0)^T, \text{ with small quantity } \varepsilon \sim 10^{-6} \sim 10^{-8}.
\]

Then the column corresponding to coordinate 1 (\(x\)) is

\[
R_{1:6,1} = \frac{X_1(+\Delta) - X_1(-\Delta)}{2\varepsilon}
\]

Similarly for the other columns.
TRANSPORT map

• TRANSPORT map is an extension of transfer matrix to second order (Taylor map).

\[ Y_\alpha = R_{\alpha\beta} X_\beta + T_{\alpha\beta\gamma} X_\beta X_\gamma, \quad \alpha, \beta, \gamma = 1 \ldots 6, \]

Summation of identical indices is assumed. Note by convention (making \( T \) symmetric), \( T_{\alpha\beta\gamma} = T_{\alpha\gamma\beta} \).

• For an element with a constant Hamiltonian (no \( s \)-dependence), the second order TRANSPORT map \( T \) can be derived from second and third order polynomials in the Hamiltonian.

• TRANSPORT map is generally not symplectic. But it can be useful for one-pass system.

• Combination of TRANSPORT maps \( (R = R^{0\rightarrow 2}, T = T^{0\rightarrow 2}) \).

\[ R_{\alpha\beta} = R^2_{\alpha\gamma} R^1_{\gamma\beta}, \]
\[ T_{\alpha\beta\gamma} = R^2_{\alpha\delta} T^1_{\delta\beta\gamma} + T^2_{\alpha\delta\zeta} R^1_{\delta\beta} R^1_{\zeta\gamma}. \]

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Calculating TRANSPORT map with numerical differential

For $\beta = \gamma$, tracking particles (from location 0 to 1)

\[ X_1(0) = M(X_0), \]
\[ X_1(-\Delta) = M(X_0 - \Delta), \]
\[ X_1(\Delta) = M(X_0 + \Delta), \]

Example: $\Delta_{\beta=1} = (\epsilon, 0, 0, 0, 0, 0)^T$, with small quantity $\epsilon \sim 10^{-3} \sim 10^{-4}$.

Then the elements corresponding to coordinate 1 ($x$), $T_{\alpha 11}$, ($\alpha = 1..6$) is

\[ T_{11} = \frac{X_1(+\Delta) + X_1(-\Delta) - 2X_1(0)}{2\epsilon^2} \]

For $\beta \neq \gamma$, tracking particles (assuming $\beta = 1, \gamma = 2$)

\[ X_1(-, -) = M(X_0 + \Delta_{-, -}), \quad \Delta_{-, -} = (-\epsilon, -\epsilon, 0, 0, 0, 0)^T \]
\[ X_1(-, +) = M(X_0 + \Delta_{-, +}), \quad \Delta_{-, +} = (-\epsilon, +\epsilon, 0, 0, 0, 0)^T \]
\[ X_1(+, -) = M(X_0 + \Delta_{+, -}), \quad \Delta_{+, -} = (+\epsilon, -\epsilon, 0, 0, 0, 0)^T \]
\[ X_1(+, +) = M(X_0 + \Delta_{+, +}), \quad \Delta_{+, +} = (+\epsilon, +\epsilon, 0, 0, 0, 0)^T \]

Then

\[ T_{\beta \gamma} = T_{\gamma \beta} = \frac{1}{8\epsilon^2} (X_1(+, +) + X_1(-, -) - X_1(+, -) - X_1(-, +)) \]
Linear optics for a storage ring (uncoupled)

- Assuming there is no $x - y$ coupling, the one-turn transfer matrix looks like

$$ R = \begin{pmatrix} M_x & 0 & E \\ 0 & M_y & 0 \\ F & 0 & M_z \end{pmatrix} $$

where $M_x$, $M_y$ and $M_z$ are $2 \times 2$ matrices. The horizontal and vertical optics functions can be determined from $M_x$ and $M_y$, respectively.

**Caution:** When using tracking to compute linear optics, elements with time dependence or elements that change momentum deviation (like RF cavity) should be turned off.

- Courant-Snyder parameterization of the one turn matrix

$$ M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \cos \Phi + \alpha \sin \Phi & \beta \sin \Phi \\ -\frac{1 + \alpha^2}{\beta} \sin \Phi & \cos \Phi - \alpha \sin \Phi \end{pmatrix} $$

With $\Phi = 2\pi \nu$, and $\nu$ the tune.
Tune and Twiss functions

- The tune can be determined from
  \[ \cos \Phi = \frac{M_{11} + M_{22}}{2}, \quad \sin \Phi = \text{sign}(M_{12}) \sqrt{M_{12}M_{21} - \frac{1}{4}(M_{11} - M_{22})^2} \]
  \[ \nu = \frac{1}{2\pi} \tan^{-1} \frac{\sin \Phi}{\cos \Phi}. \]

- Twiss functions at the location of the one-turn matrix
  \[ \beta_0 = \frac{M_{12}}{\sin 2\pi \nu}, \quad \alpha_0 = \frac{(M_{11} - M_{22})}{2 \sin 2\pi \nu} \]

- Propagation of Twiss functions to another location (knowing transfer matrix \( M_{0 \to 1} \))
  \[ M = M_{0 \to 1} \]
  \[ \beta_1 = \frac{1}{\beta_0} \left( (M_{11} \beta_0 - M_{12} \alpha_0)^2 + M_{12}^2 \right), \]
  \[ \alpha = -\frac{1}{\beta_0} \left( (M_{11} \beta_0 - M_{12} \alpha_0)(M_{21} \beta_0 - M_{22} \alpha_0) + M_{12} M_{21} \right). \]

- Phase advance
  \[ \Delta \Phi_{01} = \Phi_1 - \Phi_0 = \tan^{-1} \frac{M_{12}}{M_{11} \beta_0 - M_{12} \alpha_0} \]

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**Linear optics (coupled)**

- With $x - y$ coupling, the $4 \times 4$ matrix of the transverse phase space is

$$T_{4\times4} = \begin{pmatrix} M & m \\ n & N \end{pmatrix}$$

where $M, N, m, n$ are all $2 \times 2$ matrices. The $x, y$ motion are coupled when $m, n \neq 0_2$. In this case, the tunes derived from $M$ and $N$ are inaccurate.

- Decoupling: with a coordinate transformation $Y = V^{-1}X$, such that the transfer matrix for $Y$ is block diagonal (decoupled), i.e., for $Y_1 = UY_0$, with

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \text{and} \quad T_{4\times4} = VUV^{-1}$$

The transformation matrix can be computed.

$$V = \begin{pmatrix} \gamma I & C \\ -C^+ & \gamma I \end{pmatrix}$$

where $C^+$ is the symplectic conjugate of $C$ (required such that $V$ is symplectic)

$$C^+ = \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix}$$
Remarkably, elements of the $V$ matrix can be directly computed from the original $T_{4 \times 4}$ matrix.

The reverse matrix is

$$V^{-1} = \begin{pmatrix} \gamma I & -C \\ C^+ & \gamma I \end{pmatrix}$$

Symplecticity of $V$ also leads to

$$\gamma^2 + |C| = 1$$

The reverse matrix is

$$V^{-1} = \begin{pmatrix} \gamma I & -C \\ C^+ & \gamma I \end{pmatrix}$$

- Tunes and Twiss functions can be computed from decoupled transfer matrices $A$ and $B$ (blocks in the $U$ matrix) in the same manner as the uncoupled case.

$$\gamma = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{(\text{Tr}[M-N])^2}{(\text{Tr}[M-N]) + 2||H||}}}, \quad \text{and} \quad C = \frac{-H \text{sgn}(\text{Tr}[M-N])}{\gamma \sqrt{(\text{Tr}[M-N])^2 + 4||H||}},$$

where $H = m + n^+$.
Dispersion function and chromaticity

- By definition dispersion is the differential closed-orbit w.r.t. momentum deviation.

\[ D = \frac{\partial X_c^{(4)}}{\partial \delta} \]

The first order and second order dispersion functions are defined through

\[ X_c(\delta) = X_c(0) + D_1 \delta + \frac{1}{2} D_2^2 \delta^2 + O(\delta^3) \]

where \( D_1 = (D_{1x}, D'_{1x}, D_{1y}, D'_{1y})^T \) and similarly for \( D_2 \).

- Numerical calculation by off-momentum closed orbit (fixed momentum).
- Chromaticities are calculated with numerical differential of the tunes w.r.t. momentum deviation.
Dispersion function from transfer matrix

- The one-turn transfer matrix is

\[
R = \begin{pmatrix}
M_{4 \times 4} & E_{4 \times 2} \\
F_{2 \times 4} & M_z
\end{pmatrix}
\]

With

\[
E_{4 \times 2} = \begin{pmatrix}
0 & R_{16} \\
0 & R_{26} \\
0 & R_{36} \\
0 & R_{46}
\end{pmatrix}
\]

The closed orbit with an infinitesimal momentum deviation \( \delta \) would be \( X_c (\delta) = (D, 0,1) \delta \). From \( RX_c = X_c \)

One gets

\[
M_{4 \times 4} D + R:6 = D
\]

where \( R:6 = (R_{16}, R_{26}, R_{36}, R_{46})^T \). Therefore

\[
D = (I - M_{4 \times 4})^{-1} R:6
\]

- Propagation of dispersion

\[
D(s) = M_{4 \times 4} (s_0 \rightarrow s) D(s_0) + R:6 (s_0 \rightarrow s)
\]
Momentum compaction factor

- Momentum compaction factor (MCF) is defined
  \[ \alpha_c = \frac{1}{C} \frac{d\Delta C(\delta)}{d\delta} = \frac{1}{C} \int \frac{D(s)}{\rho} ds \]

  Higher order momentum compaction factors
  \[ \Delta C(\delta) = \alpha_1 \delta + \frac{1}{2} \alpha_2 \delta^2 + \ldots \]

- MCF is related to one-turn transfer matrix element \( R_{56} \)
  \[ -\alpha_c C = R_{56} - \mathcal{H} \sin 2\pi \nu_x \]

  where \( \mathcal{H} = (D_x^2 + (\alpha D_x + \beta_x D_x')^2)/\beta_x \) is the local dispersion invariant.
  Usually \( \alpha_c = -R_{56}/C \) is a good approximation.

- Calculating MCF with tracking.
  1. Calculate the closed orbit for \( \delta = -\epsilon \) and \( \delta = \epsilon \) (using \( \epsilon \sim 10^{-8} - 10^{-6} \)) with fixed momentum deviation, to get \( X_c(-\epsilon) \) and \( X_c(+\epsilon) \).
  2. Set the \( z \) coordinate in \( X_c(\pm \epsilon) \) to 0. Then track the closed orbits for one turn \( X_1(-\epsilon) = M(X_c(-\epsilon)), X_1(+\epsilon) = M(X_c(+\epsilon)) \).
  3. The MCF is
  \[ \alpha_c = \frac{(X_1(+\epsilon) - X_1(-\epsilon))_5}{2\epsilon} \]
  2nd order MCF can be computed similarly.
Radiation integrals.

- Radiation integrals are related to important lattice parameters for electron storage rings.
- Definition

\[
I_1 = \oint \frac{D}{\rho} \, ds, \quad I_2 = \oint \frac{1}{\rho^2} \, ds, \quad I_3 = \oint \frac{1}{|\rho|^3} \, ds,
\]

\[
I_4 = \oint \frac{D}{\rho} \left( \frac{1}{\rho^2} + 2K \right) \, ds, \quad \text{with} \quad K = b_2 = \frac{1}{B \rho} \frac{\partial B_y}{\partial x}
\]

\[
I_5 = \oint \frac{H}{|\rho|^3} \, ds.
\]

- For \( I_1, I_4 \) and \( I_5 \), integration needs to consider variation of dispersion and Twiss functions (for \( I_5 \)) inside the magnets. This can be done analytically, with

\[
D(s) = D_0 \cos \sqrt{K_x} s + \frac{D_0'}{\sqrt{K_x}} \sin \sqrt{K_x} s + \frac{1 - \cos \sqrt{K_x} s}{\rho_K x}, \quad \text{for} \quad K_x = K + \frac{1}{\rho^2} > 0
\]

\[
D(s) = D_0 \cosh \sqrt{-K_x} s + \frac{D_0'}{\sqrt{-K_x}} \sinh \sqrt{-K_x} s + \frac{1 - \cos \sqrt{-K_x} s}{-\rho_K x}, \quad \text{for} \quad K_x < 0,
\]

and formulas for \( \beta_x(s) \) and \( \alpha_x(s) \).
Lattice parameters

Note: $\alpha_x$ and $D'_x$ change at the dipole edges due to edge focusing!

$$D'_0 = D'_{0-} + \frac{D_0}{\rho} \tan \psi_1, \alpha_{x0} = \alpha_{x0-} - \frac{\beta_{x0}}{\rho} \tan \psi_1$$

where $\psi_1$ is the entrance angle and subscript ‘−’ indicate values before the edge ($D_0$ and $\beta_x$ are continuous at the edge).

• Lattice parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy loss per turn</td>
<td>$U_0[\text{keV}] = 14.085E^4[\text{GeV}]I_2[m^{-1}]$</td>
</tr>
<tr>
<td>Momentum spread</td>
<td>$\sigma^2 = Cq\gamma^2 \frac{I_3}{2I_2 + I_4}$</td>
</tr>
<tr>
<td>Horizontal emittance</td>
<td>$\epsilon_x = Cq\gamma^2 \frac{I_5}{I_2 - I_4}$</td>
</tr>
<tr>
<td>Damping partition</td>
<td>$J_x = 1 - \frac{I_4}{I_2}, J_y = 1, J_z = 2 + \frac{I_4}{I_2}$</td>
</tr>
<tr>
<td>Damping time</td>
<td>$\tau_{x,y,z} = \frac{2E}{J_{x,y,z}U_0}T_0$</td>
</tr>
</tbody>
</table>
Precise tune determination from tracking data

- To characterize nonlinear dynamics of a storage ring, it is often useful to determine tunes from tracking or measured turn-by-turn data for beam with various energy error or oscillation amplitude.
- Simple FFT is not accurate (precision $\sim 1/N$)
- NAFF (numerical analysis of fundamental frequency) is a method to accurately determine the tune from turn-by-turn data.

Assume a discrete sampling of a quasi-periodic signal $s(t) = \cos(2\pi ft + \phi_0)$ ($f$ may have small secular drifting) is

$$s_n = s(nh) = \cos(2\pi vn + \phi_0), \quad n = 1, 2, ..., N$$

with sampling frequency $f_0 = 1/h$ and $\nu = f / f_0$. How to determine $f$ from the data?

NAFF: find the frequency $\bar{f} = \bar{\nu} f_0$ that maximizes the overlap between the power contents of $s_n$ and $\exp(-i2\pi \bar{f} t)$, i.e.

$$F(\bar{\nu}) = \left| \sum_{n=1}^{N} s_n e^{-i2\pi \bar{\nu} n} W(n) \right|$$

Then $\bar{f}$ is a precise approximation to $f$. 
The function $W(n)$ (window function) is a data filter, added to improve accuracy. A good choice is the $W(n) = \frac{\sin \pi n}{(N-1)^4}$.

- **Numerical experiment**

  Data $x(n) = \cos 2\pi vn + u_n$, $n = 1 \ldots N$, $u_n$ random drawn Gaussian distribution with $(\mu = 0, \sigma)$

<table>
<thead>
<tr>
<th>Error</th>
<th>No noise</th>
<th>$\sigma = 0.01$</th>
<th>$\sigma = 0.05$</th>
<th>$\sigma = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N=32</td>
<td>-5.7e-06</td>
<td>-1.3e-04</td>
<td>-0.0012</td>
<td>-0.0020</td>
</tr>
<tr>
<td>N=64</td>
<td>-5.9e-08</td>
<td>7.3e-05</td>
<td>-1.7e-04</td>
<td>-5.2e-04</td>
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<tr>
<td>N=128</td>
<td>2.5e-09</td>
<td>-1.3e-07</td>
<td>-1.2e-05</td>
<td>-2.8e-04</td>
</tr>
<tr>
<td>N=256</td>
<td>1.1e-10</td>
<td>-4.9e-06</td>
<td>-1.9e-05</td>
<td>-6.3e-05</td>
</tr>
</tbody>
</table>
Characterization of nonlinear dynamics

- The motion of particles with large oscillation amplitude (nonlinear dynamics) is critical to storage ring performance.
- Common indicators of beam nonlinear dynamics
  - Nonlinear chromaticity (tune shifts with momentum deviation)
  - Tune shifts with amplitude
  - Tune diffusion
  - Frequency map
  - Resonance driving terms
- Direct measures of nonlinear dynamics performance
  - Dynamic aperture
  - Momentum aperture
Nonlinear chromaticity

- First order chromaticities are usually corrected to a small positive number (with sextupoles). The higher order chromaticities may dominate the $\nu \sim \delta$ behavior.
- Tracking setup: correct linear chromaticity to zero, turn off cavity (fixed momentum deviation), small initial $x$, $y$ offsets, track 256 to 1024 turns.
- Example: SPEAR3
Amplitude-dependent tune shifts

• Nonlinear elements cause dependence of betatron tunes over oscillation amplitude.

\[

\nu_x(J_x, J_y) = \nu_{x0} + \left[ \frac{\partial \nu_x}{\partial (2J_x)} \right]_0 2J_x + \left[ \frac{\partial \nu_x}{\partial (2J_y)} \right]_0 2J_y + O(J^2)
\]

\[

\nu_y(J_x, J_y) = \nu_{y0} + \left[ \frac{\partial \nu_y}{\partial (2J_x)} \right]_0 2J_x + \left[ \frac{\partial \nu_y}{\partial (2J_y)} \right]_0 2J_y + O(J^2)
\]

where \( J_x = (x^2 + (ax + \beta x')^2)/\beta \), and likewise for \( J_y \).

In general, \( \frac{\partial \nu_x}{\partial (2J_y)} = \frac{\partial \nu_y}{\partial (2J_x)} \).

The coefficients can be computed with formulas.

• Numerical calculation of amplitude-dependent tune shifts.

Track particles with small initial \( y \) offset and a series of \( x \) offsets, obtain tunes for each initial \( x \) offset, fit \( \nu_x \) and \( \nu_y \) vs. \( J_x \) with linear model. Repeat for \( J_y \) dependence.
An example: SPEAR3
The coefficients are
\[
\begin{pmatrix}
\frac{\partial v_x}{\partial (2J_x)} & \frac{\partial v_x}{\partial (2J_y)} \\
\frac{\partial v_y}{\partial (2J_x)} & \frac{\partial v_y}{\partial (2J_y)}
\end{pmatrix}
= \begin{pmatrix}
1900 & 2130 \\
2110 & 1760
\end{pmatrix}
\]
A more complete revelation nonlinear behavior is the frequency map analysis (FMA).

With initial transverse position coordinates $x$ and $y$ distributed on a grid (all other coordinates set to zero), find the corresponding betatron tunes and tune diffusion.

$$(x, y) \to (\nu_x, \nu_y)$$
Evaluation of dynamic aperture

- Dynamic aperture (DA) is the (single particle dynamics) stability region around the reference orbit. Large DA is critical for storage ring performance (injection efficiency and lifetime).

- DA is evaluated in simulation with tracking.
  - Long term tracking is needed (one damping time or more for electron storage rings).
  - 6D phase space tracking with radiation damping.
  - To be realistic, model may include insertion device effects, systematic errors and random errors of magnets.
  - Multiple random error seeds are used.
  - Physical apertures may also be included in the model.
  - Tracking may start at the injection point (septum magnet).
Initial particle coordinates
Initial particle distribution

\[ p_x = p_y = 0, \delta = 0 \] for on-momentum case.

\( x, y \) distributed on grid or rays

Search for the first lost particle along each ray from the inner side outward.
Evaluation of momentum aperture

- Momentum aperture (MA): the largest initial momentum deviation error for a particle that will survive in the ring.
- MA is the most important factor that determines the Touschek lifetime of a beam.
  - Touschek lifetime, $\tau_T \sim \sigma_x \sigma_y \sigma_z \delta^3 / D(\xi)$
- MA may vary with location.
  - RF bucket height is an overall limit for the ring. Transverse motion and aperture may place a tighter limit.
  - Estimate with a linear model (with local equivalent aperture $A_x$)
    
    $$-A_x < (D_x \pm \sqrt{\beta_x H}) \delta < A_x$$
  - Higher order dispersion, coupling and nonlinear dynamics complicate the calculation.
- Particle tracking is the usual way of determining MA in simulation.
• Tracking for momentum aperture
  – Lattice model settings are the same as DA tracking
  – MA is evaluated at locations all around the ring, or locations of a period.
  – Particles are launched with a range of energy errors.
References