Advanced Accelerator physics - Part 5

The betatron and synchrotron motion:

Particle motion in circular accelerator or in a linac can be decomposed into transverse and longitudinal motion around a reference orbit. The transverse motion is also called betatron motion, and the longitudinal motion is called synchrotron motion, i.e. the motion relative to a synchronous particle.

Transverse betatron motion

The Hamiltonian for charged particles in the presence of Electromagnetic fields is given by

\[ H = \vec{P} \cdot \vec{v} - L = e\left[m^2c^2 + (\vec{P} - e\vec{A})^2\right]^{1/2} + e\Phi, \]

\[ \dot{x} = \frac{\partial H}{\partial P_x}, \quad \dot{P}_x = -\frac{\partial H}{\partial x}, \quad \text{etc}, \]

(1)

where \((x, P_x, \ldots)\) are pairs of conjugate coordinates, and \((\Phi, A)\) are the scalar and vector potentials of the electromagnetic field. Expanding the particle coordinates around a reference orbit in Frenet-Serret coordinate system, we find

\[ \dot{s}(s) = \frac{d\vec{r}(s)}{ds}, \quad \dot{x}(s) = -\rho(s)\frac{d\dot{s}(s)}{ds}, \quad \dot{z}(s) = \dot{x}(s) \times \dot{s}(s), \]

\[ \dot{x}'(s) = \frac{1}{\rho(s)}\dot{s}(s) + \tau(s)\dot{z}(s), \quad \dot{y}'(s) = -\tau(s)\dot{x}(s), \]

\[ \vec{r}(s) = \vec{r}_0(s) + x\dot{x}(s) + z\dot{z}(s). \]

(2)

The coordinate transformation to the curvilinear coordinate is carried out via canonical transformation by using \((x, s, z)\) coordinates. The resulting Hamiltonian and Hamilton’s equations of motion with time \(t\) as independent variable become

\[ H = e\Phi + c \left\{ m^2c^2 - \frac{(p_s - eA_s)^2}{(1 + x/o)^2} + (p_x - eA_x)^2 + (p_z - eA_z)^2 \right\}^{1/2} \]

\[ \dot{s} = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial s}; \quad \dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{p}_x = -\frac{\partial H}{\partial x}; \quad \dot{z} = \frac{\partial H}{\partial p_z}, \quad \dot{p}_z = -\frac{\partial H}{\partial z}. \]

(3)

Using the path length \(s\) of the reference orbit for the independent variable, the Hamiltonian with \((x, p_x; z, p_z; t, -H)\) as conjugate phase space coordinates becomes

\[ \dot{H} = - \left(1 + \frac{x}{\rho}\right) \left[ \frac{(H - e\Phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2 \right]^{1/2} - eA_s \]

\[ \dot{t}' = \frac{\partial H}{\partial H} \quad H' = -\frac{\partial \dot{H}}{\partial t}; \quad x' = -\frac{\partial \dot{H}}{\partial p_x}, \quad p'_x = \frac{\partial \dot{H}}{\partial x}; \quad z' = -\frac{\partial \dot{H}}{\partial p_z}, \quad p'_z = \frac{\partial \dot{H}}{\partial z}. \]

(4)
The energy and momentum of a particle are given by $E = H - e\Phi$ and $p = (E^2/c^2 - m^2c^2)^{\frac{1}{2}}$. Since the total momentum is much larger than that of the transverse motion around the closed orbit, we can approximate the Hamiltonian as

$$\tilde{H} \approx -p \left(1 + \frac{x}{\rho}\right) + \frac{1 + x/\rho}{2p} \left[(p_x - eA_x)^2 + (p_z - eA_z)^2\right] - eA_s$$  

(5)

The scale factors for the Frenet-Serret coordinate system are $h_x = 1$, $h_s = 1 + x/\rho$, $h_z = 1$. The differential path length is $d\ell^2 = h_x^2dx^2 + h_s^2ds^2 + h_z^2dz^2$, and the differential operators are

$$\nabla \Phi = \frac{\partial \Phi}{\partial x} \hat{x} + \frac{1}{h_s} \frac{\partial \Phi}{\partial s} \hat{s} + \frac{\partial \Phi}{\partial z} \hat{z},$$

$$\nabla \cdot \vec{A} = \frac{1}{h_s} \left[\frac{\partial (h_s A_1)}{\partial x} + \frac{\partial A_2}{\partial s} + \frac{\partial (h_s A_3)}{\partial z}\right],$$

$$\nabla \times \vec{A} = \frac{1}{h_s} \left[\frac{\partial A_3}{\partial s} - \frac{\partial (h_s A_2)}{\partial z}\right] \hat{x} + \left[\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right] \hat{s} + \frac{1}{h_s} \left[\frac{\partial (h_s A_2)}{\partial x} - \frac{\partial A_1}{\partial s}\right] \hat{z},$$

$$\nabla^2 \Phi = \frac{1}{h_s} \frac{\partial}{\partial x} h_s \frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial s} h_s \frac{\partial \Phi}{\partial s} + \frac{\partial}{\partial z} h_s \frac{\partial \Phi}{\partial z}.$$  

(6)

Using Hamilton’s equation:

$$x' = \frac{\partial \tilde{H}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{H}}{\partial x}, \quad z' = \frac{\partial \tilde{H}}{\partial p_z}, \quad p_z' = -\frac{\partial \tilde{H}}{\partial z}.$$  

(7)

We find

$$x'' = \frac{p + x}{\rho^2} = \pm \frac{B_z}{B_\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2, \quad z'' = \mp \frac{B_x}{B_\rho} \frac{p_0}{p} \left(1 + \frac{x}{\rho}\right)^2.$$  

(8)

Here, the top/bottom signs correspond to positive and negative charge respectively. With dipole and quadrupole field:

$$B_z = \mp B_0 + \frac{\partial B_z}{\partial x} x = \mp B_0 + B_z x, \quad B_x = \frac{\partial B_z}{\partial x} z = B_1 z,$$  

(9)

We obtain Hill’s equation of motion:

$$x'' + K_x(s)x = 0, \quad z'' + K_z(s)z = 0,$$

$$K_x = \frac{1}{\rho^2} \mp K_1(s), \quad K_z = \pm K_1(s), \quad K_1(s) = B_1(s)/B_\rho.$$  

(10)
**Floquet transformation**

Horizontal and vertical Hill’s equation can be derived from a pseudo-Hamiltonian:

\[ H = \frac{1}{2} y'^2 + \frac{1}{2} K(s) y^2, \]

Here \((y, y')\) represent phase space coordinates in either horizontal or vertical direction. We can carry out Floquet transformation to the Hamiltonian:

\[
y' = -\frac{y}{\beta} (\tan \psi - \frac{\beta'}{2}), \quad \quad \quad \quad F_1(y, \psi) = \int_0^y y'dy = -\frac{y^2}{2\beta} (\tan \psi - \frac{\beta'}{2})
\]

\[ J = -\frac{\partial F_1}{\partial \psi} = \frac{y^2}{2\beta} \sec^2 \psi = \frac{1}{2\beta} [y^2 + (\beta y' + \alpha y)^2]. \]

(11)

Here \((J, \psi)\) are action-angle variables, \((\alpha \text{ and } \beta)\) are Courant-Snyder parameters. The new Hamiltonian is

\[ \tilde{H} = H - \frac{\partial F_1}{\partial s} = \frac{J}{\beta}. \]

(12)

Hamilton’s equations of motion in the action angle become

\[ \psi' = \partial \tilde{H} / \partial J = 1 / \beta(s), \quad \text{and} \quad \frac{dJ}{ds} = -\frac{\partial \tilde{H}}{\partial \psi} = 0. \]

The first equation recovers the phase advance vs the betatron amplitude function, and the second equation shows that the action is invariant. This is the Courant-Snyder invariant. The phase space coordinates are related to the action-angle variables with

\[ y = \sqrt{2\beta J} \cos \psi, \quad y' = -\sqrt{\frac{2J}{\beta}} [\sin \psi + \alpha \cos \psi], \]

(13)

We transform the position coordinate \(s\) to the orbital angle \(\theta\) so that the time is uniformly measured every turn by the generating function:

\[ F_2(\psi, \bar{J}) = \left( \psi - \int_0^s \frac{d\psi}{\beta} + \nu \theta \right) \bar{J} \quad \implies \quad \bar{\psi} = \psi - \int_0^s \frac{ds}{\beta} + \nu \theta, \quad \bar{J} = J, \]

(14)

The new Hamiltonian is \( \tilde{H} = \nu J \). We note that the linear betatron motion is simple harmonic motion. The Hamiltonian for 2D betatron motion becomes

\[ \tilde{H} = \nu_x J_x + \nu_z J_z. \]
Nonlinear beam dynamics

In accelerator, there is an intrinsic nonlinearity in the coordinate transformation into Frenet-Serret coordinate system. This intrinsic nonlinearity is most important for small storage rings with small bending radius $\rho$ [see e.g. H. Xu, et al., PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS 17, 070101 (2014)].

Since the focusing strength depends on particle momentum, the betatron tunes depend on particle momentum, chromatic effect. The chromaticity increases with machine size and betatron tunes. Correction of chromaticity requires sextupoles located in dispersive section. Dynamic aperture becomes an important issue in the accelerator design. This effect is particularly important for high luminosity colliders and the high brightness electron storage rings. The Hamiltonian for particle motion with sextupoles becomes

$$H = \frac{1}{2} \left[ x'^2 + K_x x'^2 + z'^2 + K_z z'^2 \right] + V_3(x, z, s),$$
$$V_3(x, z, s) = \frac{1}{6} K_2(s)(x^3 - 3xz^2), \quad K_2(s) = \frac{B_2(s)}{B\rho}.$$  (1)

The nonlinearity can cause betatron tunes to depend on betatron amplitude, and dynamic aperture reduction due to nonlinear resonances. The dynamics aperture for synchrotron radiation light source storage rings is an extremely challenging problem in accelerator physics. It is a current top difficult problem to be solved.

We will deal with a simpler problem that the nonlinearity can be dealt with perturbation, where the nonlinear detuning and dynamic aperture can be carried out by canonical perturbation and resonance driving terms calculations. We carry out Floquet transformation and obtain

$$V_3 = -\frac{\sqrt{2}}{4} J_x^{1/2} J_z \beta_x^{1/2} \beta_z K_2(s) \left[ 2 \cos \Phi_x + \cos(\Phi_x + 2 \Phi_z) + \cos(\Phi_x - 2 \Phi_z) \right]$$
$$+ \frac{\sqrt{2}}{12} J_x^{3/2} \beta_x^{3/2} \beta_z K_2(s) \left[ 3 \cos \Phi_x + 3 \cos \Phi_x \right],$$
$$\Phi_x = \dot{\phi}_x + \chi_x(s) - \nu_x \theta, \quad \chi_x = \int_0^s \frac{ds}{\beta_x}, \quad \Phi_z = \phi_z + \chi_z(s) - \nu_z \theta, \quad \chi_z = \int_0^s \frac{ds}{\beta_z}.$$  (2)

Since the perturbation is a periodic function of the circumference, we can expand the perturbation into Fourier harmonics:

$$H = \nu_x J_x + \nu_z J_z + \sum_\ell \left\{ G_{3,0,\ell} J_x^{3/2} \cos(3\phi_x - \ell\theta + \xi_{3,0,\ell}) + G_{1,2,\ell} J_x^{1/2} \cos(\phi_x + 2\phi_z - \ell\theta + \xi_{1,2\ell}) + G_{1,-2,\ell} J_x^{1/2} \cos(\phi_x - 2\phi_z - \ell\theta + \xi_{1,-2\ell}) + G_{1,0,3,0,\ell} J_x^{3/2} \cos(\phi_x - \ell\theta + \eta_{3,0,3,0,\ell}) + g_{1,0,1,2,\ell} J_x^{1/2} J_z \cos(\phi_x - \ell\theta + \eta_{3,0,1,2,\ell}) \right\}.$$  (3)
The resonance strengths are

\[
G_{3,0,\ell} e^{j\xi_{3,0,\ell}} = \frac{\sqrt{2}}{2\pi} \int \beta_x^{3/2} K_2(s) e^{j[3x_\ell(s) - (3\nu_\ell - \ell)\theta]} ds
\]

\[
G_{1,\pm 2,\ell} e^{j\xi_{1,\pm 2,\ell}} = \frac{\sqrt{2}}{8\pi} \int \beta_x^{1/2} \beta_z K_2(s) e^{j[\chi_\ell(s) \pm 2\chi_\ell(s) - (\nu_\ell \pm 2\nu_\ell - \ell)\theta]} ds,
\]

\[
g_{1,0,3,0,\ell} e^{jn_{1,0,3,0,\ell}} = \frac{\sqrt{2}}{8\pi} \int \beta_x^{3/2} K_2(s) e^{j[\chi_\ell(s) - (\nu_\ell - \ell)\theta]} ds
\]

\[
g_{1,0,1,2,\ell} e^{jn_{1,0,1,2,\ell}} = \frac{\sqrt{2}}{4\pi} \int \beta_x^{1/2} \beta_z K_2(s) e^{j[\chi_\ell(s) - (\nu_\ell - \ell)\theta]} ds.
\]

(4)

**The third order resonance at** \(3\nu_\ell = \ell\)

The Hamiltonian is important in accelerator physics. We first talk about the 3rd order Hamiltonian at \(3\nu_\ell = \ell\):

\[
H \approx \nu_\ell J_x + G_{3,0,\ell} J_x^{3/2} \cos (3\phi_x - \ell\theta + \xi).
\]

(5)

This Hamiltonian describes much important physics in slow-extraction, and beam manipulations. One can carry out Floquet transformation to the resonance rotating frame with \(\delta = \nu_\ell - \frac{\ell}{3}\)

\[
F_2 = (\phi_x - \frac{\ell}{3}\theta + \frac{\xi}{3})J_x, \quad \Rightarrow \quad \phi = \phi_x - \frac{\ell}{3}\theta + \frac{\xi}{3}, \quad J = J_x.
\]

(6)

\[
H = \delta J + G_{3,0,\ell} J_x^{3/2} \cos 3\phi,
\]

\[
\dot{\phi} \equiv \frac{d\phi}{d\theta} = \delta + \frac{3}{2} G_{3,0,\ell} J_x^{1/2} \cos 3\phi, \quad \dot{J} \equiv \frac{dJ}{d\theta} = 3G_{3,0,\ell} J_x^{3/2} \sin 3\phi.
\]

(7)

**Fixed points**

Fixed points of a dynamics system are associated phase space points with zero velocity fields, i.e. \(\dot{\phi} = 0\) and \(\dot{J} = 0\). The fixed points for the third order resonance with zero detuning parameter are

\[
J_x^{1/2}_{\text{UPF}} = \left| \frac{2\delta}{3G_{3,0,\ell}} \right| \quad \text{with} \quad \begin{cases}
\phi_{\text{FP}} = 0, \pm 2\pi/3 & \text{if } \delta/G_{3,0,\ell} < 0 \\
\phi_{\text{FP}} = \pm \pi/3, \pi & \text{if } \delta/G_{3,0,\ell} > 0
\end{cases}
\]

(8)

The action of unstable fixed points depends on the proximity parameter \(\delta\). These fixed points are drawn in the left plot below as a function of the proximity parameter \(\delta\). These UFPs are symmetric with respect to the proximity parameter \(\delta\), i.e. the beam encounter the resonance above or below the resonance. One can expand the equation of motion around the unstable fixed point \(K = J - J_{\text{UPF}}\) to find

\[
\dot{K} - 3\delta^2 K - 6\frac{\delta^2}{J_{\text{UPF}}} K^2 = 0
\]
Particle motion around the UFP is exponential or faster than exponential. If one defines the normalized conjugate coordinate as \( X = \sqrt{J/J_{\text{UFP}}} \cos \phi \), \( P = -\sqrt{J/J_{\text{UFP}}} \sin \phi \), the separatrix becomes 3 straight lines given by

\[
[2X - 1] \left[ P - \frac{1}{\sqrt{3}} (X + 1) \right] \left[ P + \frac{1}{\sqrt{3}} (X + 1) \right] = 0
\]

The fixed points are intersections of these separatrices, which cut through the phase space and divide the phase space into sections. Particles outside the stable area will move to infinity. As the tune passes through the resonance, the phase space ellipses, in the resonance rotating frame, rotate by 120° and particle motion changes direction shown in the right plots below. The betatron tune of particles on the separatrix sits exactly on resonance, and thus the stable fixed points are at infinity.

Nonlinear magnets give rise to betatron tune dependence on the amplitude of particle motion. In first order approximation, the betatron tunes can depend on the betatron amplitude. In perturbation theory, we have

\[
Q_x = \nu_x + \alpha_{xx} J_x + \alpha_{xz} J_z + \cdots, \quad Q_z = \nu_z + \alpha_{xz} J_x + \alpha_{zz} J_z + \cdots
\]  

(9)

Nonlinear detuning may arise from canonical perturbation to the sextupole field or octupoles. With nonlinear detuning, the Hamiltonian for the 3\text{rd} order resonance becomes

\[
H = \delta J + \frac{1}{2} \alpha J^2 + GJ^{3/2} \cos 3\phi.
\]

The fixed points of the Hamiltonian are

\[
\frac{\alpha J_{\text{UFP}}^{1/2}}{G} = \left\{ \begin{array}{ll}
-\frac{3}{4} - \frac{3}{4} \sqrt{1 - \frac{16\alpha \delta}{9G^2}} & \phi = \gamma, \pm 2\pi/3 \quad \dot{\gamma} < 0 \quad \text{(UFP)} \\
+\frac{3}{4} - \frac{3}{4} \sqrt{1 - \frac{16\alpha \delta}{9G^2}} & \phi = \gamma, \pm 3\pi/2 \quad \theta \leq \delta \leq \frac{9G^2}{16\alpha} \quad \text{(UFP)} \\
+\frac{3}{4} - \frac{3}{4} \sqrt{1 - \frac{16\alpha \delta}{9G^2}} & \phi = \gamma, \pm \pi/3 \quad \dot{\gamma} \leq \frac{9G^2}{16\alpha} \quad \text{(SFP)}
\end{array} \right.
\]

(10)

Here we use \( \alpha = \alpha_x \) and \( G = G_{3,0,1} \) to simplify our notation. Note that the appearance of stable fixed points. These fixed points are plotted in the left plot above. The fixed points are now bend over to one side because the betatron tune depends on the amplitude.
The plot below shows the evolution of phase space as the betatron tune moves through the resonance.

As the betatron tune moves across the resonance, bifurcation occurs at \( \frac{a \delta}{G^2} = \frac{9}{16} \). The bifurcation creates equal number of stable and unstable fixed points. Third order resonances are normally dangerous in the beam dynamics, but they are also used in slow-extraction.

The orientation of the phase space ellipse at the electrostatic septum (ES) depends on the locations of sextupoles in accelerator. The evolution of the beam and particles receiving kick from the ES depends on the values of the betatron amplitude function and the phase advance between the ES and the magnetic septum.
An example of the 3rd order resonance experiments of is given below:

Left: The measured Poincar’e map of the normalized phase-space coordinates \((x, p_x)\) of betatron motion near a third-order resonance \(3y = 11\) at the IUCF cooler ring. Particles outside the separatrix survive only about 100 turns. Tori for particles inside the separatrix are distorted by the third order resonance. The orientation of the Poincar’e map, determined by sextupoles, rotates at a rate of betatron phase advance along the ring. The right plot shows the Poincar’e map in action-angle variables \((J, \phi)\). The solid lines are Hamiltonian tori of Eq. (2.245) with \(\delta = -0.0060, |G_{3,0,11}| = 2.2 (\text{pm})^{-1/2}\), and \(\alpha_{xx} = 0\), which gives \(J_{\text{UFP}} = 3.3 \text{ pm}\). Equally good fix can be obtained by using the measured \(\alpha_{xx} \approx 650 (\text{pm})^{-1}\) to obtain \(J_{\text{UFP}} = 3.1 \text{ pm}\), and \(J_{\text{SFP}} \approx 27.6 (\text{pm})\).

Other 3rd order resonance

The Hamiltonian in Eq. (3) has many other 3rd order resonances. The sum resonance is located at \(\nu_x + 2\nu_z = \ell\). This is a dangerous resonance because the amplitude of horizontal and vertical motion can simultaneously increase without bound. In the design of accelerator, one tries to avoid this resonance at any cost.

The third term in Eq. (3) corresponds to the Walkinshaw resonance at \(\nu_x - 2\nu_z = \ell\). This resonance is very important to cyclotron design. For AVF or separate-sector cyclotrons, the isochronous condition requires \(\nu_x \sim \gamma\), the Lorentz energy factor, while \(\nu_z < 1\) for week focusing. There is a possibility for the Walkinshaw resonance to occur. The Hamiltonian near a Walkinshaw resonance is

\[
H = \nu_x J_x + \nu_z J_z + \frac{1}{2} \alpha_{xx} J_x^2 + \alpha_{xz} J_x J_z + \frac{1}{2} \alpha_{zz} J_z^2 + G_{1,-2,\epsilon} J_x^{1/2} J_z \cos(\phi_x - 2\phi_z - \ell \theta + \xi_{1,-2,\epsilon}) + \cdots
\]

(11)

The Hamiltonian is canonically transformed to the rotating frame using the generating function

\[
F_2(\phi_x, \phi_z, J_1, J_2) = (\phi_x - 2\phi_z - \ell \theta + \xi_{1,-2,\epsilon})J_1 + \phi_z J_2
\]

(12)

The coordinate transformation is

\[
\phi_1 = \phi_x - 2\phi_z - \ell \theta + \xi_{1,-2,\epsilon}, \quad J_x = J_1
\]

\[
\phi_2 = \phi_z, \quad J_z = -2J_1 + J_2.
\]

The new Hamiltonian becomes
\[ \tilde{H} = H_1(J_1, \phi_1, J_2) + H_2(J_2) \]

\[ H_1(J_1, \phi_1, J_2) = \delta J_1 + \frac{1}{2} \alpha_{11} J_1^2 + \alpha_{12} J_1 J_2 + G_{1,-2,\epsilon} J_1^{1/2} (J_2 - 2J_1) \cos(\phi_1) \]

\[ H_2(J_2) = \nu_z J_2 + \frac{1}{2} \alpha_{2z} J_2^2 \]

\[ \alpha_{11} = \alpha_{xx} - 4\alpha_{xz} + 4\alpha_{zz} \]

\[ \alpha_{12} = \alpha_{xz} - 2\alpha_{zz} \]

\[ \alpha_{22} = 4\alpha_{zz} \]

With proximity parameter \( \delta = \nu_x - 2\nu_z - \ell \). Since the Hamiltonian is independent of \( \phi_2 \), the action \( J_2 \) is invariant. Hamilton’s equation becomes

\[
\frac{dJ_1}{d\theta} = -\frac{\partial \tilde{H}}{\partial \phi_1} = G_{1,-2,\epsilon} J_1^{1/2} (J_2 - 2J_1) \sin(\phi_1),
\]

\[
\frac{d\phi_1}{d\theta} = +\frac{\partial \tilde{H}}{\partial J_1} = \delta + \alpha_{12} J_2 + \alpha_{11} J_1 + G_{1,-2,\epsilon} \frac{J_2 - 6J_1}{2J_1^{1/2}} \cos(\phi_1).\]


Particle dynamics obey Eq. (15) at constant \( J_2 \) and \( H_1 \), which are invariants if the betatron tunes are not changed. The fixed points of the Hamiltonian \( H_1 \) are obtained by equating Eq. (15) to zero. Two unstable fixed points (UFPs) are located at the intersections between the Courant-Snyder (CS) circle \((2J_1 = J_2)\) and the coupling arc. The plot below shows experimental measurement of phase space points and the separatrix (right plot) by Ellison et al.

![Plot](image_url)

The separatrix is the Hamiltonian torus that pass through the UFP, i.e.
\[
\frac{1}{2} (J_2 - 2J_1) \left\{-\delta - \frac{1}{2} \alpha_1 \left( J_1 + \frac{J_2}{2} \right) - \alpha_{12} J_2 + 2G_{1,-2} \epsilon_1^{1/2} \cos(\phi_1) \right\} = 0
\] (16)

Separatrices at other various conditions have been obtained by K. Symon in the AIP conference proceedings. Experimental measurement of Hamiltonian flow in the resonance rotating frame is given below [see Ellison et al PRE 50, 4051 (1994) for experimental conditions].

When the betatron tunes are ramped, \(J_2\) remains an invariant. In reality, particle motion is under the influence of many other resonances; \(J_2\) is quasi-invariant. Since the rate of resonance crossing is normally small, \(H_1\) changes slowly. Particle motion also follows a quasiconstant \(H_1\) contour.

We study the effects of the Walkinshaw resonance on a beam of particles. When the betatron tunes ramp through a Walkinshaw resonance, all fixed points move across the beam, and the beam distribution will evolve as well. For simplicity, we consider a beam with bi-Gaussian distribution

\[
\rho_2(J_x, J_z) = \frac{1}{\epsilon_x \epsilon_z} \exp\left\{-\frac{J_x}{\epsilon_x} - \frac{J_z}{\epsilon_z} \right\}
\] (17)

where \(\epsilon_x\) and \(\epsilon_z\) are, respectively, the horizontal and vertical rms emittances of the beam. We transform \(J_x\) and \(J_z\) are transformed to \(J_1\) and \(J_2\). The invariant distribution function in \(J_2\) can be obtained by integrating over \(J_1\):

\[
\rho_1(J_2) = \frac{1}{2\epsilon_x - \epsilon_z} \left[ \exp\left(-\frac{J_2}{2\epsilon_x}\right) - \exp\left(-\frac{J_2}{\epsilon_z}\right) \right].
\] (18)

As the betatron tunes ramp through the resonance, the action \(J_2\) is invariant, and the distribution function \(\rho(J_2)\) is invariant. The first moment \(\langle J_2 \rangle = 2\epsilon_x + \epsilon_z\) is also invariant, i.e. \(2\Delta\epsilon_x = -\Delta\epsilon_z\). Since \(J_2\) varies from particle to particle, it is advantageous to study the beam distribution in the variable \(u = J_1/J_2\). The transformed beam distribution is
\[ \rho_{2a}(u, J_2) = \frac{J_2}{\epsilon_x \epsilon_z} \exp \left\{ - \left( \frac{u + \frac{1 - 2u}{\epsilon_x}}{\epsilon_z} \right) J_2 \right\} \]  

(19)

where the variables \( u \) in \([0, \frac{1}{2}]\) and \( J_2 \) in \([0, \infty)\). In this representation, all particles in the beam have the same CS circle at \( u = J_1/J_2 \). We also note that, when \( 2\epsilon_x = \epsilon_z \), \( \rho(u, J_2) \) is independent of \( u \) for all \( J_2 \). Integrating over \( J_2 \), we find the 1D distribution as

\[ \rho_{1a}(u) = \frac{\epsilon_x/\epsilon_z}{[\epsilon_x/\epsilon_z + (1 - 2\epsilon_x/\epsilon_z)u]^2} \Theta \left( \frac{1}{2} - u \right) \]  

(20)

where \( \Theta \) is the Heaviside step function. The plot below shows \( \rho_{1a}(u) \) for the bi-Gaussian distribution with various ratios of the initial horizontal to vertical emittances. When \( \epsilon_{xi} > \frac{1}{2} \epsilon_{zi} \), there are more particles at higher \( J_1 \) actions, and we expect that the horizontal emittance will decrease in crossing a Walkinshaw resonance. Conversely, when \( \epsilon_{xi} < \frac{1}{2} \epsilon_{zi} \), the horizontal emittance will increase in crossing the Walkinshaw resonance. At the condition \( \epsilon_{xi} = \frac{1}{2} \epsilon_{zi} \), the distribution function is uniform, and we expect no emittance exchange during the crossing of a Walkinshaw resonance.

![Graph showing \( \rho_{1a}(u) \) distribution for different \( \epsilon_{xi}/\epsilon_{zi} \) ratios.](image)

When the betatron tunes ramp through a \( \nu_x - 2\nu_z = \ell \) resonance with \( \epsilon_{xi} > \frac{1}{2} \epsilon_{zi} \), there are more particles with higher horizontal actions. They are drawn along the coupling arc toward the center of the CS circle with their horizontal actions reduced and vertical actions increased. Emittance exchange obey the property \( 2\Delta \epsilon_x = -\Delta \epsilon_z \). We define the fractional emittance growth (FEG)

\[ \text{FEG} = \left| \frac{\Delta \epsilon_x}{\epsilon_{xi}} \right| + \left| \frac{\Delta \epsilon_z}{\epsilon_{zi}} \right| = \left| \frac{\Delta \epsilon_x}{\epsilon_{xi}} \right| \left( \frac{2\epsilon_{zi}}{\epsilon_{zi} + 1} \right) = \left| \frac{\Delta \epsilon_z}{\epsilon_{zi}} \right| \left( \frac{\epsilon_{zi}}{2\epsilon_{xi} + 1} + 1 \right) \]  

(21)

The maximum emittance change is \( \Delta \epsilon_x = \epsilon_{xi} \), or \( \Delta \epsilon_z = \epsilon_{zi} \). Thus we expect a maximum FEG for asymptotic value. The maximum FEG is also independent of the detuning parameters. The left plot below shows the emittance evolution for multiparticle simulations with various detuning parameters listed on the graph.
The FEG depends on the resonance strength, initial emittance and the resonance crossing rate. It appears that FEG obeys a scaling law, depending only on $G \sqrt{\frac{\epsilon_{xl}}{d(\nu_x-2\nu_z)/dn}}$. Note that the FEG reaches an asymptotic value depending only on the ratio of the initial beam emittances, and independent of the accelerator parameters. The scaling law of the asymptotic FEG vs the emittance ratio is plotted below.

The asymptotic FEG scaling law obeys Eq. (21) and the FEG is zero at $\epsilon_{z1} = \frac{1}{2} \epsilon_{z1}$ as expected. The scaling law works also for other beam distributions as demonstrated in the Monte-Carlo simulations.
For a beam with equal initial horizontal and vertical emittance $\varepsilon_0$ passing through a strong Walkinshaw resonance, the final emittances will be $\varepsilon_x = \frac{1}{2} \varepsilon_0$ and $\varepsilon_y = 2 \varepsilon_0$. If the vertical aperture is not an issue, the smaller horizontal emittance can pass through a smaller magnetic or electric septum gap. If this resulting beam is made to pass through the same resonance again at a similarly strong strength, the horizontal emittance and vertical emittance will be exchanged again and restored to their original values; i.e., the final beam emittances are $\varepsilon_x = \varepsilon_y = \varepsilon_0$. All these predictions can be tested experimentally in cyclotrons or circular accelerators.

The concept of nonlinear coupling can be applied to diffraction limited storage rings. In synchrotron radiation light sources, the vertical emittance is small. Nonlinear coupling resonances may be used to reduce the horizontal emittance and produce round beams. Such concept has yet to be fully explored.
III. Nonlinear space charge beam dynamics

For fast rampsing synchrotrons, the longitudinal space charge effect is normally small due to substantial rf voltage in the acceleration process. The bunch length change is small. The effect of transverse space charge effect can be approximated by a two-dimensional potential, characterized by the beam sizes $\sigma_x$ and $\sigma_z$ which depend on $s$. The space charge potential in the Frenet-Serret coordinate system becomes

$$V_{sc}(x, z) = \frac{K_{sc}}{2} \int_0^\infty \frac{\exp\left[-\frac{x^2}{2\sigma_x^2 + t} - \frac{z^2}{2\sigma_z^2 + t}\right] - 1}{\sqrt{(2\sigma_x^2 + t)(2\sigma_z^2 + t)}} \, dt$$

(1)

The space charge force is proportional to a dimensionless quantity called space charge permeance $K_{sc} = 2N\epsilon_0/\beta^2 \gamma^5$, where $N = N_B e/(\pi)\sigma_s$ is the charge density per unit length; $N_B e$ is the bunch charge, $\beta$ and $\gamma$ are the relativistic Lorentz factors; and $\gamma^5 = e^2/4\pi\epsilon_0 m c^2$ is the particle classical radius, $\epsilon_0$ is the permittivity of free space; $\sigma_x$, $\sigma_z$, and $\sigma_s$ are the rms bunch lengths. The classical radii are $2.82 \times 10^{-15}$ m for electrons and $1.535 \times 10^{-18}$ m for protons. The space charge force is especially severe in low energy proton or ion beams because their $\beta$ and $\gamma$ are small.

The space-charge force on each particle is obtained by Hamilton’s equation. Thus each beam particle passing through a length $\Delta s$ experiences a space-charge kick

$$\frac{\Delta x'}{\Delta s} = -\frac{\partial V_{sc}}{\partial x}, \quad \frac{\Delta z'}{\Delta s} = -\frac{\partial V_{sc}}{\partial z}$$

We expand the space-charge potential in Taylor series in order to study the systematic space charge resonances:

$$V_{sc}(x, z) = -\frac{K_{sc}}{2} \left\{ \frac{x^2}{\sigma_x^2 + \sigma_z^2} + \frac{z^2}{\sigma_x^2 + \sigma_z^2} - \frac{1}{4\sigma_x^2 \sigma_z^2} \left[ \frac{2 + r}{3} \right] \right. \left. \frac{x^2}{x^2 + \sigma_x^2} \frac{z^2}{z^2 + \sigma_z^2} + \frac{2}{3} \sigma_x^2 \sigma_z^2 \right\} + \frac{1}{72\sigma_x^2 \sigma_z^2} \left[ \frac{8 + 9r + 3r^2}{3} \right] \left[ \frac{3(3 + r)}{r^3} \right] + \frac{1}{5r^4} + \frac{1}{6} + \ldots$$

(2)

with $r = \sigma_z/\sigma_x$. The first term inside the curly brackets represents the linear force, which gives rise to linear space charge (Laslett) tune shift. The second and the third terms drive the 4th and 6th order resonances. The linear space charge tune shift parameters become

$$\xi_{sc,x/z} \equiv |\Delta \nu_{sc,x/z}| = \begin{cases} \frac{K_{sc}}{4\pi} \int \frac{1}{\sigma_x(\sigma_x + \sigma_z)} \beta_x ds \quad \text{roundbeam} & \frac{2\pi R K_{sc}}{8\pi \epsilon_{rms}} \\ \frac{K_{sc}}{4\pi} \int \frac{1}{\sigma_z(\sigma_x + \sigma_z)} \beta_z ds & \end{cases}$$

(3)

Particles at the center of the beam has a betatron tune shift $-\xi_{sc,x/z}$, and large betatron amplitude particles have small betatron tune shift. Since particles at different betatron amplitudes have different betatron tune shift, the space charge force produces an incoherent tune spread $\xi_{sc}$. The defocusing space charge force causes each particle to have its own tunes depending on its position in the beam. The betatron tune spread causes by the space charge is called the incoherent space charge tune shift. This tune shift can cause particle to encounter nonlinear resonances.

In 2005, X. Huang carried out a systematic measurement of space charge effect at the Fermilab Booster, which is rapid-cycling synchrotron at 15 Hz for proton acceleration from 400 MeV to 8 GeV kinetic energy. Fermilab Booster had a turn-by-turn ionization-profile monitor since 1994 [W. S. Graves, Ph.D. thesis, University of Wisconsin, Madison, 1994]. The data of beam emittance in the first 4000 revolution is shown in the left plot below for the normalized vertical rms emittance from 70 revolution to 4000 revolution for all data sets with 2-turn injection to 18-turn injection. Note the red curve is for 12-turn injection which marks the border of two kinds of emittance growth behavior.

![Graph showing emittance growth](image)

Similarly, the right plot above shows the horizontal $\sigma^2_x$ for 4-turn and 12-turn injection. The beam widths below the transition energy show adiabatic damping. The beam widths above the transition-energy show quadrupole mode oscillation. The extraction of the horizontal emittance is complicated by dispersion function and beam momentum spread. After carefully a analysis, one finds that the horizontal emittance remains nearly constant vs the increase in intensity.

Modeling of the Fermilab emittance measurement had been carried out systematically [Huang et al., Phys. Rev. ST Accel. Beams 9, 014202 (2006)] and found that the emittance growth was due to the combined effect of Montague resonance, and random skew-quadrupole induced different and sum resonances.

Recently, H. Huang et al. carried out emittance measurements at the AGS [Proceedings of IBIC2013, Oxford, UK, 492] showing that emittance growth for various beam charge. There is no firm confirmation of these results, particularly, the final emittance at top energy is independent of the beam intensity from $0.5 \times 10^{11}$ to $2 \times 10^{11}$ particles per bunch. In fact, the Laslett parameter is less than 0.02. It is difficult to understand to these experimental results.

A mechanism of emittance growth is the systematic nonlinear space charge resonance. This resonance has been observed in the KEP-PS. The KEK PS has 4 superperiods with circumference 340 m and the operational betatron tunes $v_x = 7.15$ and $v_z = 5.25$. The fourth-order systematic resonance will occur at all integers. Measurement of the beam profile at high intensity is shown in the left plot below at the betatron tune of $v_x = 7.05$ at 500 MeV [S. Igarashi et al.,

It is important to note that the resonance at $\nu_x=7.0$ is the systematic $4^{\text{th}}$ order space charge resonance shown in Eq. (2). The systematic resonance is such that $4 \nu_x$ is divisible by the superperiod of the accelerator. The space charge potential of Eqs. (1) and (2) is a periodic function with superperiod $P$. Its Fourier harmonic is zero except those harmonics of integer multiple of $P$. The KEK PS experiment is a case of systematic $4^{\text{th}}$ order resonance.

The systematic space charge resonance is particularly important for accelerators with many superperiods, e.g. the non-scaling FFAG accelerators [S.Y. Lee, Phys. Rev. Lett. 97, 104801 (2006)]. The reason is that the space charge kick of each superperiod adds up coherently at the systematic resonance. The numerical simulation shows that phase space evolution as the betatron tunes ramp through the $4^{\text{th}}$ and the $6^{\text{th}}$ order resonances.
The KEK experiment is a stationary beam without ramping. Filamentation of particles encountering resonance causes emittance growth. These particles have small actions. The distribution will become that shown at the left plot of Fig. 1.

On the other hand, the non-scaling FFAG accelerator needs to pass through systematic space charge resonances. Emittance growth for such accelerators may be inevitable. The 4th order resonance during the acceleration has also been observed in LINAC [L. Groening, et al., PRL 102, 234801 (2009)] shown in the Figs. 3 and 4 below, where $\sigma_0$ stands for phase advance per cell for a beam without space charge detuning. Earlier works would have attribute the emittance growth to the envelope instability.

Although the space charge nonlinear resonance may not induce beam loss, the emittance growth will produce beam halo and cause beam loss in later stage of the accelerator.
Space-Charge Dominated Beams in Synchrotrons

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The equations of motion for space-charge dominated beams in synchrotron are derived. We find that the space-charge force generates an identical defocusing function to the betatron motion and dispersion function. The self-consistent envelope equation obeys the Kapchinskij–Vladimirskij-type equation similar to that of the linear transport system. We employ these results to analyze the stability of the crystalline beams, and discuss the implication on the high intensity proton driver for the neutron spallation sources. [S0031-9007(98)06214-0]

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Intense charge particle beams have many applications. Some of these applications are proton drivers for the neutron spallation sources, the energy amplifier, the secondary beam sources such as muons, pions, and kaons, and heavy ion beam drivers for fusion energy \([1,2]\). Thus the stability of the space-charge dominated beam is an important topic in beam physics. There are two methods commonly used in the study of the stability of space-charge dominated beams. The linearized Vlasov equation method studies the threshold of an equilibrium distribution perturbatively \([3]\), while the particle-core model studies the core stability using the envelope equation, and the particle stability using Hill’s equation \([4]\). The linearized Vlasov equation approach has been shown to provide accurate description of the threshold behavior of collective modes. On the other hand, the particle-core model has been successfully used to describe the halo formation using parametric nonlinear resonances. Numerical simulations have been found to agree well with both the linearized Vlasov equation theory and the particle-core model in linear transport systems.

In the past, space-charge dominated beams were studied mainly in linacs, where the beam energy is low, and thus the space-charge force is important \([5,6]\). Since the synchrotron can accumulate linac beams and attain a much higher line density, the space-charge force can be as important. So far, the stability of space-charge dominated beams has not been fully explored in synchrotrons due to the complication of the dispersion functions, where we rely solely on multiparticle simulations. Deriving an envelope equation for a space-charge dominated beam in synchrotron is therefore an important timely task because the synchrotron is considered as the prevailing scheme for the bunch compression in the neutron spallation sources. The task is to find a self-consistent distribution and the envelope equation for the space-charge dominated beams in synchrotrons. This paper attempts to solve the task and discuss the effect of the strong space-charge force on the lattice functions.

In the curvilinear Frenet-Serret coordinate system with \((x, s, z)\) as unit basis vectors, the particle motion can be described by the phase space coordinate \((x, p_x, z, p_z, t, -E)\) [7], where the synchrotron phase space coordinates \((t, E)\) are the time and the energy of the particle, \(x, z\) are betatron coordinates, and \(p_x, p_z\) are the corresponding conjugate coordinates. We consider only transverse force, where we choose \(A_x = A_z = 0\) for the vector potential. The Hamiltonian up to the second order in \(p_x, p_z\) is

\[
\hat{H} = -\left(1 + \frac{x}{\rho}\right)p_x + \left(1 + \frac{z}{\rho}\right)\left(\frac{p_x^2 + p_z^2}{2\rho}\right) - eA_s,
\]

(1)

where \(e\) and \(p\) are the charge and the momentum of the particle, \(\rho\) is the radius of curvature, and \(A_s = (1 + x/\rho)|A|\cdot \mathbf{s}\) is a component of the vector potential with \(B_z = [1/(1 + x/\rho)](\partial A_s/\partial x), B_x = [-1/(1 + x/\rho)](\partial A_s/\partial z)\). We expand the momentum about the reference value \(p_0\) and obtain

\[
\Delta p = p - p_0 = \frac{\Delta E}{\beta_0 c} - \frac{1}{2p_0\beta_0c\gamma_0}\left(\Delta E\right)^2 - \frac{eV_{sc}}{\beta_0 c},
\]

(2)

where \(\Delta E = E - E_0\) is the energy deviation, and \(\beta_0\) and \(\gamma_0\) are the Lorentz factors of the particle. Here \(V_{sc}\) is the self-consistent space-charge scalar potential that satisfies the Poisson equation

\[
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)V_{sc} = -\frac{e}{\epsilon_0} \iint dp_x dp_z \nabla \mathbf{s} d(-\Delta E) f(x, z, p_x, p_z, -\Delta E; \mathbf{s}),
\]

(3)

where the distribution function \(f\) obeys the Vlasov equation \(df/ds = 0\). The equilibrium distribution function in a synchrotron must also be a periodic function of \(s\), and is generally a function of the effective Hamiltonian that includes the space-charge mean field potential. For a few special distributions, self-consistent space-charge potential can be expressed in analytic form, e.g., the Kapchinskij-Vladimirskij (KV) distribution \([8]\) in a linear transport system.
Now we consider the case of a coasting beam without synchrotron motion so that the longitudinal electric field is zero. The vector potential is given by

\[
A_x = B_0 x + \frac{B_0}{2 \rho} x^2 + \frac{1}{2} B_1 (x^2 - z^2) + A_{sc}, \tag{4}
\]

where \(B_1 = \partial B_z / \partial x\) is the focusing function evaluated at the reference orbit, and \(A_{sc}\) is the vector potential due to the space-charge force. Since the magnetic force is equal to \(-\beta_0^2\) times the electric force, we obtain \(A_{sc} = \beta_0^2 V_{sc}/\beta_0 c\). Substituting the vector potential into the Hamiltonian, one obtains

\[
F_2(x, p_x, z, p_z, t, -\Delta E) = \left( x - D_x \frac{\Delta E}{\beta_0^2 E_0} \right) p_x + \left( z - D_z \frac{\Delta E}{\beta_0^2 E_0} \right) p_z - (E_0 + \Delta E) t + x \frac{D'_x}{\beta_0^2 c} \Delta E - \frac{1}{2} D_x D'_x p_x + z \frac{D'_z}{\beta_0^2 c} \Delta E - \frac{1}{2} D_z D'_z p_z + \frac{e}{\beta_0 c \gamma_0} V_{sc},
\]

and making a scale change to coordinates \(\tilde{p}_x = p_x/p_0, \tilde{p}_z = p_z/p_0, \) and \(W = \Delta E/p_0, \) the new Hamiltonian becomes

\[
\tilde{H}_1 = -\frac{W}{\beta_0 c} + \frac{\tilde{p}_x^2}{2} + \frac{1}{2} K_x \tilde{x}^2 + \frac{\tilde{p}_z^2}{2} + \frac{1}{2} K_z \tilde{z}^2 + \frac{W}{\beta_0 c} \tilde{x} \left( D''_x + K_x D_x - \frac{1}{\rho} \right) + \frac{W}{\beta_0 c} \tilde{z} \left( D''_z + K_z D_z \right) + \frac{1}{2} \left( \frac{W}{\beta_0 c} \right)^2 \left[ \frac{1}{\gamma_0} + D_x \left( D''_x + K_x D_x - \frac{2}{\rho} \right) + D_z \left( D''_z + K_z D_z \right) \right] + \frac{e}{\beta_0 c \rho \gamma_0} V_{sc}.
\]

Note that \(\tilde{x}\) is the betatron coordinate around the off-momentum closed orbit. Since the equilibrium distribution is a function of \(x\) and \(z\), the mean field Coulomb potential is given by

\[
V_{sc} = V_{sc,0} + \frac{1}{2} V_{sc,xx} (\tilde{x} + D_x \frac{W}{\beta_0 c})^2 + V_{sc,xz} (\tilde{x} + D_x \frac{W}{\beta_0 c}) (\tilde{z} + D_z \frac{W}{\beta_0 c}) + \frac{1}{2} V_{sc,zz} (\tilde{z} + D_z \frac{W}{\beta_0 c})^2 + \ldots,
\]

where \(V_{sc,0}\) is a constant term, and

\[
V_{sc,xx} = \frac{\partial^2 V_{sc}}{\partial \tilde{x}^2}, \quad V_{sc,xz} = \frac{\partial^2 V_{sc}}{\partial \tilde{x} \partial \tilde{z}}, \quad V_{sc,zz} = \frac{\partial^2 V_{sc}}{\partial \tilde{z}^2},
\]

are partial derivatives of the space-charge potential evaluated at the reference orbit. Eliminating the cross terms in the Hamiltonian, the equations for the dispersion functions are given by

\[
D''_x + \left( K_x + \frac{e V_{sc,xx}}{\beta_0 c \rho \gamma_0^2} \right) D_x + \frac{e V_{sc,xz}}{\beta_0 c \rho \gamma_0^2} D_z = \frac{1}{\rho}, \tag{7}
\]

\[
D''_z + \left( K_z + \frac{e V_{sc,zz}}{\beta_0 c \rho \gamma_0^2} \right) D_z + \frac{e V_{sc,xz}}{\beta_0 c \rho \gamma_0^2} D_x = 0. \tag{8}
\]

Note that the space-charge mean field reduces the focusing strength and may introduce a linear coupling to the equations of motion. The new Hamiltonian becomes

\[
\tilde{H}_2 = \frac{1}{2} \left[ \tilde{p}_x^2 + \left( K_x + \frac{e V_{sc,xx}}{\beta_0 c \rho \gamma_0^2} \right) \tilde{x}^2 + \tilde{p}_z^2 + \left( K_z + \frac{e V_{sc,zz}}{\beta_0 c \rho \gamma_0^2} \right) \tilde{z}^2 + 2 \frac{e V_{sc,xz}}{\beta_0 c \rho \gamma_0^2} \tilde{x} \tilde{z} \right] - \frac{W}{\beta_0 c} + \frac{1}{2} \left( \frac{W}{\beta_0 c} \right)^2 \left[ \frac{1}{\gamma_0^2} - \frac{D_x}{\rho} \right] \quad \text{[9]}
\]

The synchrotron equation of motion is given by

\[
\frac{d(\Delta \tilde{t})}{ds} = \frac{1}{\beta_0 c} \left( \frac{D_x}{\rho} - \frac{1}{\gamma_0^2} \right) \frac{W}{\beta_0 c}, \quad \frac{d(-W)}{ds} = 0, \quad \text{[10]}
\]

where \(\Delta \tilde{t} = \tilde{t} - s/\beta_0 c\) is the relative time. For a coasting beam, \(W\) is constant. The corresponding momentum compaction factor for the space-charge dominated beam

\[
\alpha_{c,sc} = \frac{1}{C} \oint \frac{D_x}{\rho} \, ds, \quad \text{[11]}
\]

where \(C\) is the circumference of the synchrotron, has an identical form as that of the emittance dominated beams except that the dispersion function is modified by the space-charge mean field. The equations of betatron motion
are given by
\[ 
\ddot{x} + \left( K_x + \frac{e V_{\text{sc,xx}}}{\beta_0 e p_0 y_0} \right) \dot{x} + \frac{e V_{\text{sc,xx}}}{\beta_0 e p_0 y_0} x = 0, \quad (12) 
\]
\[ 
\ddot{z} + \left( K_z + \frac{e V_{\text{sc,xz}}}{\beta_0 e p_0 y_0} \right) \dot{z} + \frac{e V_{\text{sc,xz}}}{\beta_0 e p_0 y_0} z = 0. \quad (13) 
\]
We observe that Hill’s equations for betatron motion have an identical focusing function as that of the dispersion functions. Furthermore, the space-charge force may introduce linear coupling to betatron motion and gives rise to the vertical dispersion.

We now apply our formalism to analyze the crystalline beams in a storage ring as they are the ultimate form of space-charge dominated beams [10]. By using a properly tailored “tapered” cooling force, an ordered state of the crystal beam can be obtained by the molecular dynamics numerical simulations [11]. When the crystalline state is formed, the normalized temperatures, defined in Ref. [11], will be less than 10\(^{-4}\) in all degrees of freedom. Figure 1 shows the dispersion function \(x_{\text{co}}/\delta\), and the vertical closed orbit \(z_{\text{co}}/\langle z_{\text{co}} \rangle\) in one period of the lattice. The existence of an unique dispersion function shown in the upper curve of Fig. 1 indicates that the horizontal closed orbit of each particle is related to the dispersion function. The fact that \(z_{\text{co}}/\langle z_{\text{co}} \rangle = 1\) for all particles in the crystalline beam indicates that (1) the space-charge force has almost fully compensated the quadrupole focusing force, and (2) there is no vertical dispersion function and no inhomogeneous term in Eq. (13). Thus the linear coupling due to the space-charge force is small, i.e., \(V_{\text{sc,xz}} = 0\).

Neglecting the linear coupling and nonlinear contribution from the space-charge potential, the betatron Hamiltonian becomes
\[ 
\hat{H}_3 = \frac{1}{2} \left[ \ddot{p}_x^2 + \ddot{p}_z^2 + \left( K_x + \frac{e V_{\text{sc,xx}}}{\beta_0 e p_0 y_0} \right) \dot{x}^2 \right] + \left( K_z + \frac{e V_{\text{sc,xz}}}{\beta_0 e p_0 y_0} \right) \dot{z}^2. \quad (14) 
\]
and the KV distribution is a self-consistent solution [8]. In the KV model, we obtain
\[ 
V_{\text{sc,xx}} = -\frac{eN}{\pi \varepsilon_0} \frac{1}{a(a + b)}, 
\]
\[ 
V_{\text{sc,xz}} = -\frac{eN}{\pi \varepsilon_0} \frac{1}{b(a + b)}, \quad (15) 
\]
where \(N\) is the number of particles per unit length, and \(a\) and \(b\) are the horizontal and the vertical betatron beam envelopes. For a KV beam, the envelope equations become
\[ 
a'' + K_x a - \frac{e_x^2}{a^3} = \frac{2K_{\text{sc}}}{a + b}, \quad (16) 
\]
\[ 
b'' + K_z b - \frac{e_z^2}{b^3} = \frac{2K_{\text{sc}}}{a + b}, \quad (17) 
\]
where \(e_x\) and \(e_z\) are the beam emittances, \(K_{\text{sc}} = 2N\rho_0/\beta_0^3\gamma_0\) is the space-charge perveance, and \(\rho_0\) is the classical radius.

When the horizontal and vertical betatron tunes for the crystalline beam lattice are equal, the envelope equation for \(b = a\), in the smooth approximation, can be written as [12]
\[ 
a'' + \left( \frac{2\pi\nu}{C} \right)^2 a - \frac{e_x^2}{a^3} = \frac{K_{\text{sc}}}{a}, \quad (18) 
\]
where \(\nu\) is the betatron tune, and \(\varepsilon\) is the average emittance of the crystalline beam. The envelope tune is given by
\[ 
v_{\text{env}} = 2\nu \left[ 1 - \frac{\kappa}{\sqrt{1 + \kappa^2} + \kappa} \right]^{1/2}, \quad (19) 
\]
where \(\kappa\) is the normalized space-charge parameter given by \(\kappa = K_{\text{sc}}C/(4\pi\nu\varepsilon)\). Here we note that the envelope tune is \(2\nu\) for an emittance dominated (hot) beam and \(\sqrt{2}\nu\) for a space charge dominated (cold) beam. If the envelope tune encounters a systematic half-integer stop band (Mathieu instability), the envelope (or betatron amplitude function) will be unstable.

For a synchrotron made of \(P\) superperiods, systematic half-integer stop bands occur at \(P/2, P, 3P/2, \ldots\). Therefore, in order to maintain a crystalline beam, the lattice must satisfy \(\sqrt{2}\nu \leq P/2\) [11].

Figure 2 shows the beam temperature, obtained from the molecular dynamics calculations, vs \(\sqrt{2}\nu/P\). When the betatron tune of a cold beam reaches the envelope stop band at \(\sqrt{2}\nu/P = 1/2\), the betatron envelope for all off-momentum particles becomes unstable, and the temperature of the beam increases suddenly. When the betatron
The beam temperature obtained from MD simulations is lower in Tokyo. The arrow attached to the star symbol shows that of the TARN-11 storage ring at Institute of Nuclear Studies and the stars are obtained from the lattice with 6 superperiods function of the betatron tune. The circles and squares are obtained from MD simulations with several different lattices are plotted as a normalized unit [11] obtained from the molecular dynamics.

The envelope stop band of a cold beam and dashed lines at that of linear transport channels. Our theory is found to be consistent with the numerical results obtained from a molecular dynamics simulation for the crystalline beam. Our formalism can be extended readily to solve the space-charge dominated bunched beam problems.

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For high density beams in synchrotrons, the space-charge parameter is designed to satisfy the condition $\nu K \ll 0.4$, which is small in comparison with that of the linac beams or the crystalline beams. However, the beam particles stay in the synchrotron for a long time, and the accumulated effect can be as important. Here the systematic and random half-integer stop band for the envelope equation may play an essential role in the stability of the space-charge dominated beams. Comparison of numerical simulations with the theory presented in this Letter would be valuable.

In conclusion, we have derived the equations of motion for space-charge dominated beams in synchrotrons. We find that the space-charge defocusing field on the betatron coordinate $\tilde{x}$ and dispersion function $D_x$ are identical. The momentum compaction factor of a synchrotron with space-charge dominated beams can be calculated by Eq. (11) with the modified dispersion function. We find that the KV beam is also a self-consistent distribution for the space-charge dominated beams in synchrotrons. For a KV beam, the envelope equations of motion are identical to that of linear transport channels. Our theory is found to be consistent with the numerical results obtained from a molecular dynamics simulation for the crystalline beam. Our formalism can be extended readily to solve the space-charge dominated bunched beam problems.

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[12] As the density of a space-charge dominated beam becomes higher, the intrabeam scattering becomes more important, and the heating rate increases. However, when a crystalline structure is formed, the random scattering vanishes. The crystalline state corresponds to a state with vanishing small transverse and longitudinal emittances.
Synchro-Betatron Hamiltonian

The Hamiltonian in the Frenet-Serret coordinate system is

$$H_0 = -(1 + \frac{x}{\rho}) \left[ \left( \frac{E - e\Phi}{e} \right)^2 - m^2 c^2 - p_x^2 - p_z^2 \right]^{1/2} - eA_s \approx -(1 + \frac{x}{\rho}) \left( p - \frac{e\Phi}{\beta c} \right) + (1 + \frac{x}{\rho}) \left( \frac{p_x^2 + p_z^2}{2\rho} \right) - eA_s$$

(1)

where the orbital length $s$ is used as an independent variable, $p = \sqrt{(\frac{E}{c})^2 - (mc)^2}$ is the momentum of a particle, $\rho$ is the bending radius of the Frenet-Serret coordinate system, $\Phi$ is the scalar potential, $A_s$ is the longitudinal vector potential, and $(x, p_x, z, p_z, t, -E)$ are canonical phase-space coordinates. The static transverse magnetic field and the longitudinal varying electric field can be expressed as

$$B_z = \frac{1}{1 + (x/\rho)} \frac{\partial A_s}{\partial x}, \quad B_x = -\frac{1}{1 + (x/\rho)} \frac{\partial A_s}{\partial z},$$

$$E_s = -\frac{\partial A_s}{\partial t} = \sum_k V_k \delta_p(s - s_k) \sin(\omega_{rf} t + \phi_{0k}), \quad \omega_{rf}(s - s_k) = \sum_n \delta(s - s_k - 2\pi n R)$$

(2)

where $\delta_p(s - s_k) = \sum_n \delta(s - s_k - 2\pi n R)$ is a periodic delta function with period $2\pi R$, $V_k$ is the rf voltage, $\omega_{rf}$ is the angular frequency of the rf field, and $\phi_{0k}$ is the initial phase of the $k$th cavity. Thus the rf accelerating field can be represented by

$$A_{s,rf} = \frac{1}{\omega_{rf}} \sum_k V_k \delta_p(s - s_k) \cos(\omega_{rf} t + \phi_{0k}).$$

(3)

The Hamiltonian is an implicit function of energy $E$. Let $E = E_0 + \Delta E$ and $p = p_0 + \Delta p$, where $E_0$ and $p_0$ are the energy and momentum of the reference particle. We obtain

$$\frac{\Delta p}{p_0} \approx \frac{\Delta E}{\beta^2 E_0} - \frac{1}{2\gamma^2} \left( \frac{\Delta E}{\beta^2 E_0} \right)^2, \quad \frac{\Delta E}{\beta^2 E_0} \approx \frac{\Delta p}{p_0} + \frac{1}{2\gamma^2} \left( \frac{\Delta p}{p_0} \right)^2.$$

(4)

Expanding the dipole field $B_z$ in power series with $B_z = B_0 + B_1 x + \cdots$, where $B_1 = \partial B_z / \partial x$, we obtain

$$A_s = B_0 x + \frac{B_1}{2\rho} x^2 + \frac{1}{2} B_1 (x^2 - z^2) + \cdots + A_{s,rf} + A_{s,sc},$$

(5)

where $A_{s,rf}$ stands for the vector potential of rf cavities. The mean field space charge force of the beam particles can be represented by a scalar and vector potentials $\Phi = V_{sc}$ and $A_{s,sc} = \beta^2 V_{sc}/\beta c$. Substituting the scalar and vector potentials into the Hamiltonian, we find

$$H_0 = -p_0 - p_0 \frac{\Delta E}{\beta^2 E_0} + p_0 \frac{1}{2\gamma^2} \left( \frac{\Delta E}{\beta^2 E_0} \right)^2 - p_0 \frac{\Delta E}{\beta^2 E_0} \frac{x}{\rho} + \frac{p_x^2 + p_z^2}{2p_0} + \frac{p_0}{2} (K_x x^2 + K_z z^2) - eA_{s,rf}$$

(6)

up to second order in phase-space coordinates, where $K_x = 1/\rho^2 - B_1/B\rho$ is the focusing function for the horizontal plane, and we used the identity condition $B_0 = -p_0/e\rho$, which signifies the expansion of $x$ around the closed orbit at the reference energy.
The next step is to transform the coordinate system onto the closed orbit for a particle with off-energy $\Delta E$. This procedure cancels the cross-term proportional to $(\Delta E/\beta^2E_0) \cdot x$ in the Hamiltonian. Using the generating function

$$F_2(x, \bar{p}_x, t, -\Delta \xi) = (x - D \frac{\Delta E}{\beta^2E_0})\bar{p}_x - (E + \Delta E)t + x \frac{D'}{\beta c} \Delta E - \frac{1}{2} D D' p_0 \left(\frac{\Delta E}{\beta^2 E_0}\right)^2$$

where the new phase-space coordinates are

$$\bar{p}_x = p_x - D' \frac{\Delta E}{\beta c}, \quad \bar{x} = x - D \frac{\Delta E}{\beta^2E_0}, \quad \Delta E = E - E_0, \quad \bar{t} = t + \left(\frac{D}{\beta^2 E_0} \bar{p}_x - \frac{D'}{\beta c} \bar{x}\right),$$

with dispersion function satisfying the equation: $D'' + K_x D = 1/\rho$. The resulting Hamiltonian is

$$H_1 = -p_0 \frac{\Delta E}{\beta^2 E_0} - \frac{1}{2} p_0 \left(\frac{D}{\rho} - \frac{1}{\gamma^2}\right) \left(\frac{\Delta E}{\beta^2 E_0}\right)^2 + \frac{p_x^2}{2 p_0} + \frac{p_0}{2} K_x x^2 - \frac{dE_0}{ds} \left(\frac{t}{\beta^2 E_0} \bar{p}_x + \frac{D'}{\beta c} \bar{x}\right) - eA_{s, rf}$$

Here, the components for vertical betatron motion in the Hamiltonian are not explicitly shown for simplicity. Note that $\bar{x}$ is the betatron phase-space coordinate around the off-momentum closed orbit, and the rf vector potential is

$$eA_{s, rf} = \frac{1}{\omega_{rf}} \sum_k eV_k \delta_p (s - s_k) \cos \left[\omega_{rf} \left(\bar{t} - \frac{D}{\beta^2 E_0} \bar{p}_x + \frac{D'}{\beta c} \bar{x}\right) + \phi_{0k}\right]$$

We expand the standing wave of the rf field into a traveling wave, i.e.

$$\delta_p (s - s_k) \cos (\omega_{rf} \bar{t} + \phi_{0k}) = \frac{1}{4\pi R} \sum_{n=-\infty}^{\infty} \left[c^n e^{i(n\theta - \omega_{rf} \bar{t} + \phi_{0k} - n\phi_k)} + c^n e^{i(n\theta + \omega_{rf} \bar{t} + \phi_{0k} - n\phi_k)}\right]$$

Keeping only terms that synchronize the beam arrival time with $n = \pm h$, we obtain

$$\delta_p (s - s_k) \cos (\omega_{rf} \bar{t} + \phi_{0k}) = \frac{1}{2\pi R} \cos (\omega_{rf} \bar{t} - \frac{h s}{R} + \phi_{0k} + h\phi_k)$$

where $\phi_{0k} + h\phi_k$ should be an integer multiple of $2\pi$. Using the generating function and coordinate transformation:

$$F_2 = x \bar{p}_x + (\omega_{rf} \bar{t} - \frac{h s}{R}) W,$$

$$p_x = \bar{p}_x, \quad x = \bar{x}, \quad W = -\frac{\Delta E}{\omega_{rf}}, \quad \phi = \omega_{rf} \bar{t} - \frac{h s}{R}$$

we obtain the Hamiltonian

$$H_2 = -\frac{\omega_{rf}^2}{2\beta^2 c E_0} \left(\frac{D}{\rho} - \frac{1}{\gamma^2}\right) W^2 + \frac{p_x^2}{2 p_0} + \frac{p_0}{2} K_x x^2 - \frac{1}{2\pi R \omega_{rf}} \sum_k eV_k \cos (\phi - \frac{\omega_{rf} p_x}{\beta c p_0} + \frac{D'}{\beta c} \bar{x})$$

$$+ \frac{1}{2\pi R \omega_{rf}} \sum_k eV_k (\phi - \frac{\omega_{rf} p_x}{\beta c p_0} + \frac{D'}{\beta c} \bar{x}).$$

Making a scale change to canonical phase-space coordinates with
we obtain the Hamiltonian

\[ H_3 = \frac{H_2}{p_0} = \frac{1}{2} (x'^2 + K_x x'^2) - \frac{1}{2} \left( \frac{E}{p} - \frac{1}{\gamma^2} \right) \left( \frac{\omega_r W}{\beta^2 E_0} \right)^2 \]

\[ - \frac{1}{2\pi \hbar \beta^2 E_1} \sum_k eV_k \cos \left( \phi - \frac{\hbar}{R} D x' + \frac{\hbar}{R} D' x \right) \]

\[ - \frac{\sin \phi_x}{2\pi \beta^2 E_0} \sum_k eV_k \left( \phi - \frac{\hbar}{R} D x' + \frac{\hbar}{R} D' x \right). \]

Since \( \varphi \) and \( (x, x') \) are coupled through dispersion function \( D, D' \) in rf cavities, synchrotron and betatron motions are coupled. This is called synchro-betatron coupling (SBC). If a resonance condition is encountered, it is called synchro-betatron resonance (SBR). The synchrotron phase-space coordinates are chosen naturally to be \((R\varphi/\hbar, -\Delta p/p_0)\), and the Hamiltonian in 6D phase-space coordinates becomes

\[ H_1 = \frac{1}{2} (x'^2 + K_x x'^2) + \frac{1}{2} (z'^2 + K_z z'^2) - \frac{1}{2} \left( \frac{D}{\rho} - \frac{1}{\gamma^2} \right) \left( \frac{\Delta p}{p_0} \right)^2 \]

\[ - \frac{1}{2\pi \hbar \beta^2 E_1} \sum_k eV_k \cos (\phi - \frac{\hbar}{R} D x' + \frac{\hbar}{R} D' x) \]

\[ - \frac{\sin \phi_x}{2\pi \beta^2 E_0} \sum_k eV_k (\phi - \frac{\hbar}{R} D x' + \frac{\hbar}{R} D' x). \]

(11)

In a straight section without dipole, the phase advance of dispersion function is the same as that of the betatron motion. Using the normalized coordinates for the betatron and dispersion coordinates:

\[ \frac{1}{\sqrt{\beta_x}} x = \sqrt{2J_1} \cos \Phi, \quad \frac{1}{\sqrt{\beta_x}} D = \sqrt{2J_1} \cos \Phi_d, \]

\[ \sqrt{\beta_x} x' + \frac{\alpha_x}{\sqrt{\beta_x}} x = -\sqrt{2J_1} \sin \Phi, \quad \sqrt{\beta_x} D' + \frac{\alpha_x}{\sqrt{\beta_x}} D = -\sqrt{2J_1} \sin \Phi_d. \]

(12)

We find \( xD' - x'D = 2\sqrt{J_1} \sin(\Phi - \Phi_d) \). The phase of arriving particle depends on the betatron motion, and this cause synchro-betatron motion. LEP has observed synchro-betatron resonance in particle beam experiments [see e.g. Phys. Rev. E49, 5706 (1994)]. Experimental measurements of beam size as a function of the horizontal betatron tune across the \( v_x - 2Q_x = \ell \) resonance were performed in LEP [S. Myers, in Measurements of Synchro-Betatron Resonances at LEP, Proceedings of the Sixth ICFA Beam Dynamics Workshop on Synchro-Betatron Coupling Resonances, Madeira, Portugal, 1993 (CERN, Geneva)]. The left two panels of the plot below show the rms beam sizes vs the horizontal betatron tune, which was scanned from 0.1 to 0.3 while the vertical tune was kept at 0.20 and the synchrotron tune was 0.076. The rms beam sizes obtained from downward tune scan were shown upside down to exhibit possible asymmetry between the upward and the downward tune scans. The increase in beam size at \( v_x \sim 0.152 \) might result from the SBR at \( v_x - 2Q_x = \ell \). The difference between the upward and the downward scans may result from the hysteresis phenomena in resonances.
When an electron beam encounters a dipole-like resonance, the beam will oscillate around a time-dependent closed orbit. The measured beam size is the average of beam profile around the reference orbit, i.e. \( \sigma_{\text{measured}}^2 = \beta_x I_{x,\text{stf}} + \beta_x \epsilon_x \), where \( \sqrt{\beta_x \epsilon_x} \) is the off-resonance beam size, and \( I_{x,\text{stf}} \) is the rms action of the SFP of all particles. The right panel shows \( \sqrt{\beta_x I_{x,\text{stf}}} \) deduced from the data. The data can be fitted by a theoretical model and provide an explanation to the hysteresis effect. The SBR Hamiltonian for \( \nu_x - m_s Q_x = \ell \) resonance with effective resonance strength \( \tilde{g} \) can be used to fit the data shown as solid line above:

\[
H = \nu_x I_x + \frac{1}{2} \alpha_{zz} I_z^2 - \frac{\eta}{|\eta|} (Q_s I_s + \frac{1}{2} \alpha_{ss} I_s^2) + \tilde{g} \frac{I_z^{1/2}}{2} \cos(\phi_x + m_s \psi_s - \ell \theta + \gamma)
\]

Accelerators for EDM experiments

Recently, there is a push for electric dipole moment (EDM) experiments. These accelerators employ all electric field to avoid beam polarization. At a magic energy, the electric field does not produce spin precession in the storage ring. There were design of all electric field rings; conceptual design proposals. Some of these works are listed as follows:


Beam dynamics of all electric ring has recently arouse interests in accelerator physics community. A relevant problem is the synchro-betatron coupling for particles with non-zero betatron motion. Although there are analysis on possible all electric ring has recently been published [see e.g. S.R. Mane, NIMA 758, 77 (2014)]. The question on the existence of the dispersion function is questionable. Transverse betatron motion can change the energy of the particle due to transverse electric field. Will the dispersion function to be defined in the same way? The betatron motion is coupled to energy in a very different fashion in this all electric ring due to the radial electric field. This is a question yet to be solved.

Synchrotron Hamiltonian

Now, we review the basic synchrotron Hamiltonian. For synchronization, the required rf frequency must be integer times the revolution frequency, i.e. \( \omega_{rf} = h \omega_0 \), where \( h \) is called the harmonic number, and the angular revolution frequency obeys \( \omega_0 R_0 = \beta_0 c \) = particle speed of the synchronous particle with \( R_0 \) being the average radius of the accelerator. Since the rf field is
changing with time, the energy gain for the reference particle per passage is the maximum rf voltage multiplied by a transit time factor $T$:

$$
\Delta E = e\mathcal{E}_0 \beta c \int_{-\mathcal{E}_0/2\beta c}^{\mathcal{E}_0/2\beta c} \sin(h\omega_0 t + \phi_s) dt = e\mathcal{E}_0 g T \sin \phi_s, \quad T = \frac{\sin(hg/2R_0)}{(hg/2R_0)}
$$

(13)

The effective voltage is $V_{rf} = \mathcal{E}_0 g T$. The energy gain rate is $\dot{E} = f_0 e V_{rf} \sin \phi_s$, where $f_0$ is the revolution frequency of the synchronous particle. Now, we consider a non-synchronous particle with

$$
\omega = \omega_0 + \Delta \omega, \quad \phi = \phi_s + \Delta \phi, \quad \theta = \theta_s + \Delta \theta, \quad p = p_0 + \Delta p, \quad E = E_0 + \Delta E
$$

The rf phase of the particle is related to its orbiting angle. The difference in angular frequency become

$$
\Delta \phi = \phi - \phi_s = -h \Delta \theta; \quad \Delta \omega = \frac{d}{dt} \Delta \theta = -\frac{1}{h} \frac{d}{dt} \Delta \phi = -\frac{1}{h} \frac{d \phi}{dt}
$$

(14)

The energy gain rate of the non-synchronous particle is $\dot{E} = f e V_{rf} \sin \phi$, where $f$ is the revolution frequency. The equation of the energy difference becomes

$$
\frac{d}{dt} \frac{(\Delta E)}{\omega_0} = \frac{1}{2\pi} eV (\sin \phi - \sin \phi_s); \quad \text{or} \quad \delta = \frac{\omega_0}{2\pi \beta^2 E} eV (\sin \phi - \sin \phi_s)
$$

(15)

Here the $\delta = \Delta p/p_0$ is the fractional off momentum parameter of the particle. From Eq. (14), we find that the rate rf phase depends on different of revolution frequencies, which in turn depends on the speed of the particle and the path-length of the particle in the accelerator. We find

$$
\dot{\phi} = \hbar \omega_0 \eta \delta = \frac{\hbar \omega_0^3 \eta}{\beta^2 E} \left( \frac{\Delta E}{\omega_0} \right)
$$

$$
\eta(\delta) = \eta_0 + \eta_1 \delta + \eta_2 \delta^2 + \cdots
$$

$$
\eta_0 = (\alpha_0 - \frac{1}{\gamma_0}),
$$

$$
\eta_1 = \frac{3\alpha_0^2}{2\gamma_0^2} + \alpha_1 - \alpha_0 \eta_0,
$$

$$
\eta_2 = -\frac{3\alpha_0^2 (5\beta_0^2 - 1)}{2\gamma_0^2} + \alpha_2 - 2\alpha_0 \alpha_1 + \frac{\alpha_1}{\gamma_0} + \alpha_0^2 \eta_0 - \frac{3\beta_0^2 \alpha_0}{2\gamma_0^2}
$$

(16)

Here $\alpha_0 + \alpha_1 \delta + \alpha_2 \delta^2 + \cdots$ is the momentum compaction factor of the accelerator, $\eta$ is the phase slip factor. Unless $\eta_0$ is very small, e.g. high energy accelerators, synchrotron radiation light sources, or specially designed accelerators, one can use only up to the $0^\text{th}$ order with $\eta = \eta_0$. Equations (15) and (16) form the basic synchrotron equation of motion. These coupled equations can be derived from the Hamiltonian:

$$
\frac{d\phi}{dt} = \frac{\hbar \omega_0^2 \eta}{\beta^2 E} \left( \frac{\Delta E}{\omega_0} \right), \quad \frac{d(\Delta E/\omega_0)}{dt} = \frac{1}{2\pi} eV (\sin \phi - \sin \phi_s),
$$

$$
H = \frac{1}{2} \frac{\hbar \omega_0^2}{\beta^2 E} \left( \frac{\Delta E}{\omega_0} \right)^2 + \frac{eV}{2\pi} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s]
$$

(17a)

or
\[
\frac{d\phi}{dt} = \hbar \omega_0 \eta \delta, \quad \frac{d\delta}{dt} = \frac{\omega_0 e V}{2\pi \beta^2 E} (\sin \phi - \sin \phi_s),
\]
\[
H = \frac{1}{2} \hbar \omega_0 \eta \delta^2 + \frac{\omega_0 e V}{2\pi \beta^2 E} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s].
\] (17b)

The phase space area of a Hamiltonian flow is invariant. In reality, the energy gain of particle in an accelerator is located at the rf location, while the rf phase slip is changed mainly in the rest of the accelerator. The synchrotron motion is more appropriately expressed in terms of the mapping equations:

\[
\Delta E_{n+1} = \Delta E_n + eV (\sin \phi_n - \sin \phi_s),
\]
\[
\phi_{n+1} = \phi_n + \frac{2\pi \hbar \eta}{\beta^2 E} \Delta E_{n+1}.
\] (18)

The synchrotron motion calculation can more easily be carried out in mapping equations. The Hamiltonian of Eq. (17) can provide useful analysis of properties of synchrotron motion. The rf potential form a potential well for stable particle motion. The rf bucket properties are:

| Bucket Area | \( \left( \frac{\Delta \phi}{\omega_0} \right) \) | \( \left( \frac{\Delta \delta}{\alpha_b(\phi_s)} \right) \) | \( \left( \frac{\beta^2 E V}{2\pi \omega_0^2 \hbar |\eta|} \right)^{1/2} \) |
|--------------|---------------------------------|---------------------------------|---------------------------------|
| 16 \( \left( \frac{\beta^2 E V}{2\pi \omega_0^2 \hbar |\eta|} \right)^{1/2} \) | \( \alpha_b(\phi_s) \) | \( \alpha_b(\phi_s) \) | \( \alpha_b(\phi_s) \) |
| Bucket Height | 2 \( \left( \frac{\beta^2 E V}{2\pi \omega_0^2 \hbar |\eta|} \right)^{1/2} \) | \( Y(\phi_s) \) | \( Y(\phi_s) \) |
| \( \sin \phi_s \) | \( \phi_n \) | \( \pi - \phi_s \) | \( Y(\phi_s) \) | \( \alpha_b(\phi_s) \) |
| 0.00 | -180.30 | 180.00 | 1.0000 | 1.0000 |
| 0.10 | -118.30 | 174.26 | 0.9208 | 0.8041 |
| 0.20 | -93.71 | 168.46 | 0.8402 | 0.6611 |
| 0.30 | -73.59 | 162.54 | 0.7577 | 0.5388 |
| 0.40 | -55.36 | 156.42 | 0.6729 | 0.4305 |
| 0.50 | -38.39 | 150.00 | 0.5852 | 0.3333 |
| 0.60 | -21.88 | 143.13 | 0.4936 | 0.2460 |
| 0.70 | -4.48 | 135.57 | 0.3967 | 0.1679 |
| 0.80 | 14.59 | 126.87 | 0.2919 | 0.0991 |
| 0.90 | 37.77 | 115.84 | 0.1731 | 0.0408 |
| 1.00 | 90.30 | 90.00 | 0.0000 | 0.0000 |

Stable particle motion in the bucket carries out synchrotron motion. The synchrotron tune for small amplitude is given by

\[
\omega_s = \omega_0 \sqrt{\frac{\hbar e V |\eta_0| \cos \phi_s}{2\pi \beta^2 E}} = \frac{e}{R} \sqrt{\frac{\hbar e V |\eta| \cos \phi_s}{2\pi E}}, \quad Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{\hbar e V |\eta_0| \cos \phi_s}{2\pi \beta^2 E}}.
\] (19)

A beam is composed of particles with different synchrotron amplitudes. The distribution function depends on the initial condition and accelerator properties. For a beam with given phase space area, its bunch height and bunch length are related by
\begin{align*}
\hat{A} &= \pi \delta \hat{\phi} = \hbar A \left( \frac{\omega_0}{\beta^2 E} \right), \\
\hat{\delta} &= A^{1/2} \left( \frac{\omega_0}{\pi \beta^2 E} \right)^{1/2} \left( \frac{\hbar e V \cos \phi_s}{2 \pi \beta^2 E |\eta|} \right)^{1/4} \\
\hat{\delta} \frac{\hat{\theta}}{\hat{\delta}} = \left( \frac{\hbar e V \cos \phi_s}{2 \pi \beta^2 E |\eta|} \right)^{1/2} = Q_s, \\
\hat{\theta} &= \frac{1}{\hbar} \hat{\phi} = A^{1/2} \left( \frac{\omega_0}{\pi \beta^2 E} \right)^{1/2} \left( \frac{2 \pi \beta^2 E |\eta|}{\hbar e V \cos \phi_s} \right)^{1/4}
\end{align*}

Here $A$ is the phase space area in eV-sec, and $\hat{A}$ has no dimension as defined in Eq. (20). From Eq. (20), the normalized phase coordinates for the synchrotron Hamiltonian is $(\phi_s, -(\hbar |\eta|/Q_s) \delta)$. Using the properties of accelerator, we also find the scaling properties of the bunch parameters:

\begin{align*}
\hat{\delta} &\sim A^{1/2} V^{1/4} \hbar^{1/4} |\eta|^{-1/4} \gamma^{-3/4}, \\
\hat{\theta} &\sim A^{1/2} V^{-1/4} \hbar^{-1/4} |\eta|^{1/4} \gamma^{-1/4}
\end{align*}

The synchrotron tune of a particle depends on its amplitude as shown in the measurement below at the IUCF Cooler. The inlet shows the phase ellipse of a particle and its tune. The synchrotron tune is shown as solid line to compare with experimental measured synchrotron tune.

\begin{equation}
\tilde{G}_s(J) = \frac{\partial H_0}{\partial J} = \frac{\pi \nu_s}{2K(k)} = \nu_s \left( 1 - \frac{J}{8} - \frac{3J^2}{256} - \cdots \right)
\end{equation}

Here the elliptical integral and the action are related by

\begin{align*}
K(k) &= \frac{\pi}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \cdots \right] \\
J &= 2k^2 \left( 1 + \frac{1}{8} k^2 + \frac{3}{64} k^4 + \cdots \right) \\
k &= \sin(\hat{\phi}/2)
\end{align*}

In the following, I would like to describe an experiment that used the properties of synchrotron motion to uncover the horizontal betatron emittance [see X. Huang, S.Y. Lee, K.Y. Ng, and Y. Su, Emittance measurement and modeling for the Fermilab Booster, the Physical Review Special Topics in Accelerators and Beams 9, 014202 (2006)].
can also be used to fit the data well. This model describes emittance growth due to an initial nonequilibrium distribution of the beam. Since both $\tilde{b}_1$ and $\tilde{b}_2$ parameters are small, the resulting fits are equivalent to the model of Eq. (6).

**B. Horizontal emittance**

The horizontal beam width reflects both the horizontal emittance and the longitudinal off-momentum distribution as seen in Eq. (3). Figure 13 shows $\sigma_x^2$ for 4-turn and 12-turn injection for the entire ramping cycle. The off-momentum width becomes more important around transition where the bunch length is shortened. The beam width starts oscillating after transition because of the longitudinal phase-space mismatch.

Since the horizontal beam width is a quadrature of contributions from the betatron motion and the rms off-momentum oscillation, we cannot directly calculate the normalized horizontal emittance unless we can isolate and remove the contributions of the off-momentum contribution. In the following, we will discuss a method by taking the advantage of different scaling property as a function of the beam momentum. We will show that the momentum scaling property can effectively separate the...
transverse horizontal betatron emittance and the off-momentum width.

1. Below transition energy

One way to obtain the transverse betatron component in \( \sigma_x^2 \) is to subtract the “known” off-momentum width component, i.e., \( \sigma_{x,b}^2 = \sigma_x^2 - D^2 \sigma_b^2 \). We can do so by assuming the beam fills up the rf bucket during the adiabatic capture at injection. The bunch shape then follows the evolution of rf bucket as a matched beam, which can be determined by the known rf voltage \( V \) and rf synchronous phase \( \phi_s \). Figure 14 shows the rf voltage (RFSUM), recorded with console program during the experiment.

Another way is to make use of the difference of the scaling rules of the transverse and longitudinal components [2]. Both terms of Eq. (3) change in the cycle as the momentum is ramped up. However, they change in different scaling rules:

\[
\sigma_x^2 = \beta_x \epsilon_{\text{rms}} + D^2 \sigma_b^2 = a A(t) + b B(t)
\]

with

\[
a = \epsilon_{\text{rms}}^2 \frac{\beta_x}{\beta_0 \gamma_0}, \quad A(t) = \frac{\beta_x}{\beta_{x,0}} \frac{\beta_0 \gamma_0}{\beta \gamma}, \quad b = D^2 \sigma_{b,0}^2, \quad B(t) = \frac{\gamma_0 \sqrt{\gamma_0 |\eta_0| V_0 |\cos \phi_{s,0}|}}{\gamma \sqrt{|\eta| V |\cos \phi_s|}},
\]

where we have used the scaling rule of \( \sigma_b \sim V^{1/4} |\eta|^{-1/4} \gamma^{-3/4} \) [2]. The scaling functions \( A(t) \) and \( B(t) \) are shown in Fig. 16.

The normalized horizontal emittance can be considered as growing linearly if it has the same behavior as the vertical emittance, so we further assume \( a = a_0 + a_1 t \). The invariant longitudinal phase-space area is preserved during acceleration, i.e., \( b = b_0 \). This assumption is reflected in the fact that the bucket area is nearly constant during the first 8000 revolutions in the booster acceleration cycle. If there is any longitudinal phase-space dilution, these particles will be squeezed out of the bucket. We thus try to fit the horizontal width with

\[
\delta_{\text{rms}}^2(t) = \frac{\gamma_0 \sqrt{\gamma_0 |\eta_0| V_0 |\cos \phi_{s,0}|}}{\gamma \sqrt{|\eta| V |\cos \phi_s|}}.
\]

FIG. 14. (Color) The rf voltage \( V \) (RFSUM) and the synchronous phase \( \phi_s \) recorded with console program during this experiment.

FIG. 15. Top: The bucket area throughout the booster cycle. The bucket area is about 0.08 eV-s at the beginning of the ramping cycle. Bottom: The rms momentum width \( \sigma_b \) (solid) and rms bunch length \( \sigma_b \) (dashed), assuming phase-space area \( A = 0.08 \text{ eV-s} \). The rms value is obtained by dividing the maximum \( \delta \) and \( \phi \) by \( \sqrt{6} \).

FIG. 16. Scaling function \( A(t) \) and \( B(t) \), obtained with realistic booster lattice model and rf parameters.
To avoid the nonlinear emittance blowup in the first several milliseconds of the cycle and the nonadiabatic region near transition, we fit $\sigma_x^2$ from 3001 revolution to 9200 revolution to the model of Eq. (11) to obtain constant parameters $a_0$, $a_1$, and $b_0$ for each data set. We then convert these parameters to horizontal emittance or rms momentum width according to Eqs. (9) and (10). An example of fitting curves is shown in Fig. 17.

This model does not fit the data of high-intensity data very well because the calibrated result of the linear parametrization scheme may deviate from the actual beam size for high-intensity beams (see Fig. 2 in Ref. [10]). The horizontal beam size $\sigma_x$ (see Fig. 13) drops from 3.5 mm (at 3001 revolution) to 2.5 mm at around 7000 revolution. For a beam with $\sigma_x = 3$ mm at a total charge of $4 \times 10^{12}$ particles (10-turn injection), the calibration error is as large as 5%. As the beam size shrinks due to adiabatic damping, the calibration error of IPM data gets even worse. The scaling property is particularly sensitive to errors in the IPM data calibration. Thus we fit data sets with 10-turn injection or less. The resulting normalized emittance, its growth rate and the rms momentum width are shown in Fig. 18. The standard deviations of the noises in $\sigma_x^2$, along with the covariance matrix of the fitting, are used to estimate the error bars of these parameters.

The results give reasonable values of emittances and rms momentum widths at 3001 revolution. The horizontal normalized emittances are found to be about 2 $\pi$ mm mrad. It seems that the explosive emittance blowup observed in the vertical plane (see Sec. II A), due to the space-charge effects, is absent in the horizontal data. This horizontal emittance at 3001-revolution number is about the same as that of the initial vertical emittance (see the top-left plot of Fig. 12). The emittance growth rate, defined as

$$\alpha_x = \frac{\Delta \epsilon_x}{\Delta N},$$

is about 0.8 $\pi$ mm mrad per $10^4$ revolutions. The horizontal growth rate is about the same as the vertical growth rate shown in Fig. 9. The rms momentum width is about 1.0 $\times$ $10^{-3}$, which is smaller than the value 1.4 $\times$ $10^{-3}$ as predicted in Fig. 15. The momentum width for these data sets
are nearly equal, indicating the rf buckets are nearly filled with particles at injection.

We have measured the momentum spread using the resistive wall monitor with a high resolution scope for 4-turn and 11-turn injection. The recorded peaks on the beam current signal are fitted to elliptic model [13] to derive the bunch lengths. The bunch lengths become the momentum spread according to the phase-space ellipses. The results are shown in Fig. 19. The rms momentum spread at 3001 revolution, deduced from the wall monitor data, is $1.25 \times 10^{-3}$.

From the above analysis, we find that the scaling law can successfully be used to separate the horizontal betatron and off-momentum components in the horizontal beam width. The essential error arises from the fact the IPM is located at a small $\beta_x$ location, and thus it is intrinsically more prone to errors. The resulting horizontal emittance and rms off-momentum width deduced from the IPM data agree well with other independent measurements.

2. Across the transition energy

The rms momentum width starts to grow rapidly as the beam gets near transition, where the phase slip factor $\eta$ becomes small. The bunch shape cannot follow the rf bucket when it is very close to transition and the longitudinal motion is nonadiabatic. The adiabatic time and non-

\[ \tau_{\text{ad}} = \left( \frac{\pi \beta^2 m c^2 \gamma f}{3 \sqrt{2} \omega \gamma \sigma_{\phi} \sigma_{\delta}} \right)^{1/3} \approx 0.187 \text{ ms}, \]

\[ \tau_{\text{nl}} = \frac{\gamma T}{2} \frac{\eta_1 \delta}{2} = \frac{\gamma T}{2} \frac{\beta_1^2 + \gamma^2 \alpha_1}{2} \delta = 0.100 \text{ ms}, \]

where we use $\gamma = 437 \text{ s}^{-1}$, $\gamma^2 \alpha_1 = 1.0$, and $\delta = \sqrt{6} \sigma_\delta$. The maximum rms momentum width at transition is obtained by using the nonadiabatic formula

\[ \sigma_{\delta,|\gamma = \gamma_T|} = \frac{\gamma T}{3 \sqrt{6} \beta_t \tau_{\text{ad}}^{1/2}} \left( \frac{2 A}{3 m c^2 \gamma} \right)^{1/2} \approx 0.502 \frac{\gamma T}{\beta_t \tau_{\text{ad}}} \left( \frac{A}{m c^2 \gamma} \right)^{1/2}. \tag{13} \]

Here $A$ is the rms phase-space area of the beam in eV-s. We obtain $\tau_{\text{ad}} = 0.187 \text{ ms}$, and $\tau_{\text{nl}} = 0.100 \text{ ms}$, and the total growth due to nonlinear longitudinal motion is $G = \exp \left( \frac{\tau_{\text{nl}}}{\tau_{\text{ad}}^{3/2}} \right) \approx 1.30 [2]$. Thus the phase-space growth due to the nonlinear motion is small. Figure 19 shows clearly that the measured rms bunch length and off-momentum spread agree very well with those derived with a constant phase-space area for the 4-turn injection case.

3. Microwave instability

The beam near transition energy can also suffer microwave instability. The emittance growth factor can be estimated as $G = \exp(S)$, where \[ \frac{S}{\pi} = \frac{N R}{n} \frac{1}{2} \left( \frac{\sigma_{\phi}^2}{\sigma_{\phi}} \right) \left( \frac{\sigma_{\delta}^2}{\sigma_{\delta}} \right) \] where $A = \pi \sigma_{E} \sigma_{r}$ is the rms phase-space area in (eV-s), $F = 0.207$ is the form factor, $N_B$ is the number of particles in a bunch, and $n = R/b = 1500$ is the mode number. Assuming a broadband impedance of $|Z_b|/n = 20 \Omega$, $6 A = 0.08 \text{ eV-s}$, $\sigma_{\phi} = 0.00232$ is the rms momentum spread at the longitudinal center of the bunch, and $N_B = 6 \times 10^{10}$, we find $G = 2.2$, which is small because the microwave instability growth is the growth factor on the very small Schottky noise.

4. Bunch mismatch oscillations in the synchrotron phase space

The post-transition beam width oscillations may arise from the mismatch across transition due to longitudinal space-charge potential as pointed out by Sorensen [16]. In a linearized approximation, the longitudinal Hamiltonian around the transition-energy region is

\[ H(\phi, \delta) = \frac{\hbar}{2} \frac{\eta_1 \delta}{4 \pi \beta^2 E} \left( V \cos \phi_s + \frac{h c g_0 Z_b N_B}{2 \gamma R \sigma_{\phi}^3} \right) \]

\[ \times (\phi - \phi_s)^2, \tag{15} \]

where $h$ is the harmonic number, $g_0 = 1 + 2 \ln^2 \frac{b}{a}$ is the geometric factor, $Z_b$ is the impedance of vacuum, $N_B$ is
the number of particles per bunch, and \(R\) is the mean radius of the synchrotron. Note that the space-charge force has a defocusing effect below transition energy (\(\cos \phi_s \geq 0\)) and a focusing effect above the transition energy (\(\cos \phi_s \leq 0\)). It causes sudden change of the shape of matched ellipse and thus the mismatch between the beam bunch and the ellipse. Figure 20 shows the rf voltage (RFSUM) and the voltage induced by the space-charge impedance for 12-turn injection. The voltage mismatch due to the space-charge potential can cause post-transition bunch length oscillation.

After passing the nonadiabatic transition-energy region, the particles in the beam bunch start to follow the ellipses of the Hamiltonian torus again. Because the space-charge force above the transition energy is focusing, the bunch ellipse is mismatched to the bucket ellipses. Hence the bunch starts to tumble in the bucket at the rate of synchrotron tune, which causes the rms momentum width of the beam to oscillate at twice the synchrotron tune. Let \(\delta_1\), \(\delta_2\) be the maximum and minimum rms momentum width, which are connected by the matched ellipses [2]

\[
\delta_2 = \frac{r_s \mathcal{A}}{h \eta \delta_1},
\]

(16)

where \(\mathcal{A} = \pi \delta_1 \phi_1\) is the rms phase-space area. The extrema of horizontal beam width are related by

\[
\sigma_{x, \text{max}}^2 - \sigma_{x, \text{min}}^2 = D^2(\delta_1^2 - \delta_2^2).
\]

(17)

We can identify \(\delta\) in Eq. (13) as \(\delta_1\). The phase-space area \(\mathcal{A}\) (in eV·s) in Eq. (13) is related with the phase-space area \(\mathcal{A}\) of Eq. (16) by

\[
\mathcal{A} = \frac{\omega_0}{B^2E} \mathcal{A}.
\]

(18)

Combining Eqs. (13), (16), and (17), we can solve for \(\delta_1\), \(\delta_2\) and the phase-space area \(\mathcal{A}\) from the oscillation magnitude of \(\sigma_x^2\). This is equivalent to the measurement of the quadrupole mode transfer function [17].

The horizontal beam width oscillation can be seen in Fig. 13. We fit oscillation pattern of \(\sigma_x^2\) with

\[
\sigma_x^2(t) = a + bt + ct^2 + A \exp(-\alpha t) \cos(2\pi(f_1 t + f_2 t^2) + \chi),
\]

(19)

where \(t\) is the revolution number. We apply this fitting model to data from revolution 10 501 to revolution 13 500 (transition is at revolution 9500, but we avoid the nonadiabatic motion of the first 1000 revolutions after the transition). Examples of fitting curves are shown in Fig. 21.

The oscillatory part of \(\sigma_x^2\) comes essentially from the longitudinal distribution mismatch with the rf bucket. Figure 22 shows the fitted tune parameters \(f_1\) and \(f_2\) as a function of the injection turn. The fitted oscillation tune is \(f_1 + 2f_2 = 0.0065\) at 10 500 revolution (18.6 ms) for 4-turn injection cycle, which is about twice of the synchrotron tune \(\nu_s = 0.0034\) measured from turn-by-turn data at 18.4 ms with the same intensity [8].

The fitted oscillation amplitude \(A\) and the decoherent coefficient \(\alpha\) are shown in the bottom plots of Fig. 22. The oscillation amplitude

![FIG. 20. The effective space-charge voltage \(V_{\text{spch}}\) (solid) and the \(V_{\text{rf}}\) in the acceleration cycle.](image)

![FIG. 21. (Color) Top: Fit \(\sigma_x^2\) to the model of Eq. (19) for 5-turn injection. Bottom: Fit \(\sigma_x^2\) to the model of Eq. (19) for 10-turn injection.](image)
can be used to solve the longitudinal phase-space area. Employing Eqs. (13), (16), and (17), we can self-consistently solve the mismatched motion. The resulting \( \delta_1 \) and \( \delta_2 \) are shown in Fig. 23. Knowing the maximum \( \delta_1 \) and minimum \( \delta_2 \) of rms momentum width, the average value \( \bar{\delta} \) can be calculated

\[
\bar{\delta} = \sqrt{\frac{\delta_1^2 + \delta_2^2}{2}}.
\]

where subscript “osci” indicates \( \tilde{\delta} \) is derived from the oscillation component of \( \sigma_x^{osc} \).

The nonoscillatory part

\[
\sigma_{x, static}^2 = a + bt + ct^2
\]

is composed of transverse betatron-motion component and the static off-momentum width component. It can be decomposed into the transverse and longitudinal components. Figure 24 shows the fitted parameters for the nonoscillatory part, and the corresponding normalized \( \chi^2 \) normalized to the number of data points for the entire set.

Because the transition energy affects mainly the longitudinal motion, it is reasonable to assume that the transverse emittance will keep growing in the same manner as in the pretransition region. The vertical emittance growth across transition (Fig. 7) suggests the same picture. Thus we have

\[
\sigma_{x, static}^2 = \frac{\beta_x}{\beta_y} \epsilon_{x, rms} (1 + \alpha_x t) + D^2 \tilde{\delta}^2,
\]

where \( \epsilon_{x, rms} \) is normalized rms emittance and \( \alpha_x \) is horizontal emittance growth rate. The scaling rule is \( \tilde{\delta} \sim |\eta|^{-1/4} \gamma^{-3/4} \), neglecting the rf voltage \( V \) factor which is constant in the concerned region. The scaling rule does not include the phase-space dilution from the smearing of the mismatch bunch. By subtracting the predicted transverse component \( \beta_x \epsilon_{x, rms} \) from \( \sigma_{x, static}^2 \), the rms momentum width can also be calculated by

\[
\delta_{static} = D \frac{\sqrt{\sigma_x^2 - \beta_x \epsilon_{x, rms}}}{\beta_x}.
\]
Using the growth rate obtained with the pretransition fitting to predict $\epsilon_{\text{rms}}$, we have calculated $\delta$ for all data sets, which are compared to the results obtained with Eq. (21) in Fig. 25. It is seen that the two methods produce consistent results.

III. MODELING AND EMITTANCE DILUTION MECHANISMS

In the last section, we have carried out phenomenological fit for the IPM measurement data at the Fermilab Booster under various beam intensity levels. Our findings can be summarized as follows:

1. The normalized vertical emittance starts at about $2\pi \text{ mm mrad}$ for all intensity levels, i.e., there is no injection phase-space painting. For intensity less than 10-turn injection, the space charge is less important, and the injection efficiency is nearly independent of the intensity (see Fig. 2). This implies that beam loss essentially arises in the longitudinal phase space. Longitudinal phase-space painting with chopped beam may be needed to minimize the beam loss and satisfy the requirement of fast ramping. When the beam intensity is larger than 10-injection turns, the normalized vertical emittance grows rapidly in the first 4000 revolutions. The space-charge force is believed to be the source of the emittance growth.

2. In the later time of the cycle at $\gamma > 2$, the vertical emittance grows linearly with a growth rate of about $1\pi \text{ mm mrad}$ in $10^4$ revolutions. Both the intrabeam scattering and the beam-gas scattering growth rates are too small to explain this linear emittance growth rate.

3. The horizontal beam width is composed of both the transverse and the longitudinal phase-space distribution of the beam. Making use of the fact that the two components have different momentum scaling rules, we can effectively decompose $\sigma_x^2$ into betatron and off-momentum beam widths. Such fitting works well for data sets with less than 10-turn injection when the IPM profile calibration error due to space charge is mild [10]. The fitting results yield consistent horizontal normalized emittance and the rms momentum width. Note that the horizontal normalized emittance does not exhibit an explosive growth in the first

FIG. 24. The fitting parameters of the post-transition horizontal beam size $\sigma_x^2$. Parameter $a$ (mm$^2$) at top left; parameter $b$ (mm$^2$/revolution) at top right; parameter $c$ (mm$^2$/revolution$^2$) at bottom left; and bottom right: the residual $\chi^2$ normalized by noise sigma and number of data points.

FIG. 25. (Color) The average rms momentum width $\delta$ ($\delta$ in the text) obtained with two methods: Eq. (23) ("static") or Eq. (21) (osci).
V. Resonances in synchrotron motion due to phase modulation

Particle beam in accelerator encounters much time dependent perturbation. If the phase of the rf wave changes by an amount $\phi(\theta)$, where $\theta = \omega_0 t$ is the orbiting angle serving as time coordinate, the synchrotron mapping equation is

$$\begin{align*}
\phi_{n+1} &= \phi_n + 2\pi \hbar \eta \delta_n + \Delta \varphi(\theta), \\
\delta_{n+1} &= \delta_n + \frac{eV}{\beta^2 E} (\sin \phi_{n+1} - \sin \phi_s),
\end{align*}$$

(1)

where $\Delta \varphi(\theta)=\phi(\theta_n+2\pi) - \phi(\theta_n)$ is the difference in rf phase error between successive. We consider only a sinusoidal rf phase modulation with $\varphi=a \sin(\nu_m \theta + \chi_0)$, where $\nu_m$ is the modulation tune, $a$ is the modulation amplitude, and $\chi_0$ is an arbitrary phase factor. The resulting rf phase difference in every revolution is $\Delta \phi = 2\pi \nu_m a \cos(\nu_m \theta + \chi_0)$. This phase modulation can exist in the presence of rf noise, dipole field modulation through power supply ripple or ground motion. The modulation can also be artificially applied to control beam properties. For simplicity, we consider the case with $\phi_s=0$ for storage beam. If we define the normalized phase space coordinate coordinates with $\varphi$ and $\rho=\hbar |\eta|/\nu_s \delta$. Equation (1) can be derived from the Hamiltonian $H=H_0 + H_1$ with

$$H = H_0 + H_1 = \frac{1}{2} \nu_s \alpha \sqrt{\frac{J}{J^2}} \left[ \cos(\psi + \nu_m \theta + \chi_0) + \cos(\psi - \nu_m \theta - \chi_0) \right] + \nu_m a \frac{J^{3/2}}{12} \left[ \cos(3 \psi + \nu_m \theta + \chi_0) + \cos(3 \psi - \nu_m \theta - \chi_0) \right] + \ldots,$$

(2)

Expansion of the perturbing Hamiltonian in action-angle coordinates $(J, \psi)$ gives

$$H_1 = \nu_m a \frac{J^{3/2}}{12} \left[ \cos(3 \psi + \nu_m \theta + \chi_0) + \cos(3 \psi - \nu_m \theta - \chi_0) \right] + \ldots,$$

(3)

The phase modulation produces only odd order synchrotron mode excitation, dipole, sextupole, etc. The perturbation strength decreases with the mode-order. When the resonance condition is occur when the perturbation reaches The Hamiltonian for the dipole-mode excitation is

$$H \approx \nu_s J - \frac{1}{16} \nu_s J^2 + \nu_m a \sqrt{\frac{J}{J^2}} \cos(\psi - \nu_m \theta - \chi_0)$$

$$\approx \nu_s J - \frac{1}{16} \nu_s J^2 + \nu_s a \frac{J^{1/2}}{\sqrt{2}} \cos(\psi - \nu_m \theta - \chi_0).$$

(4)

This is a universal form of dipole mode excitation. The perturbation is important only when $\nu_m \approx \nu_s$, we approximate the perturbation strength with a fixed number for easier analysis. We transform the phase space into the resonant rotating frame by using a generating function:

$$F_2(\psi, I) = (\psi - \nu_m \theta - \chi_0 - \pi) I,$$

$$\chi = \psi - \nu_m \theta - \chi_0 - \pi, \quad I = J;$$

$$\tilde{H} = (\nu_s - \nu_m) I - \frac{1}{16} \nu_s J^2 - \nu_s a \frac{J^{1/2}}{\sqrt{2}} \cos \chi.$$

(5)

where $(\psi, J)$ are transformed to the new phase-space coordinates $(\chi, I)$. Hamilton’s equations of motion are

$$\dot{\chi} = \nu_s - \nu_m - \frac{1}{8} \nu_s I - \nu_s a \frac{J^{1/2}}{2\sqrt{2}} \cos \chi,$$

$$\dot{I} = -\nu_s a \frac{J^{1/2}}{2\sqrt{2}} \sin \chi.$$

(6)
The fixed points of the Hamiltonian, which characterize the structure of resonant islands, are given by the solution of $\dot{I} = 0$, and $\dot{\chi} = 0$. Using $g = \sqrt{2}I \cos \chi$, with $\chi = 0$ or $\pi$, to represent the phase coordinate of a fixed point, we obtain the equation for $g$ as

$$g^3 - 16 \left(1 - \frac{\nu_m}{\nu_s}\right) g + 8a = 0$$

(7)

There are 3 solutions for the cubic equation when the modulation tune is below the bifurcation tune $\nu_{bif}$:

$$\nu_m \leq \nu_{bif} = \nu_s \left[1 - \frac{3}{16}(4a)^{2/3}\right]$$

$$\begin{align*}
g_a(x) &= -\frac{8\sqrt{3}}{\nu_s} x^{3/2} \cos \frac{\xi}{3}, \quad (\psi = \pi) \\
g_b(x) &= \frac{8\sqrt{3}}{\nu_s} x^{3/2} \sin \frac{\pi}{6} - \frac{\xi}{3}, \quad (\psi = 0) \\
g_c(x) &= \frac{8\sqrt{3}}{\nu_s} x^{3/2} \sin \frac{\pi}{6} + \frac{\xi}{3}, \quad (\psi = 0)
\end{align*}$$

(8)

Here $g_a$ and $g_b$ are respectively the outer and the inner stable fixed points (SFPs) and $g_c$ is the unstable fixed point (UFP). Particle motion in the phase space can be described by tori of constant Hamiltonian around SFPs. The lambda-shaped phase amplitudes of the SFPs (solid lines) and UFP (dashed line) shown in the left plot below vs the modulation frequency is a characteristic property of the dipole mode excitation with nonlinear detuning. In the limit $\nu_m \ll \nu_{bif}$, we have $\xi \to \pi/2$, thus $g_a \to -4x^{1/2}$, $g_c \to 4x^{1/2}$, and $g_b \to 0$.

The Hamiltonian tori in phase space coordinates $P = -\sqrt{2}I \sin \chi$ vs $X = \sqrt{2}I \cos \chi$ are shown in the right plot. The actual Hamiltonian tori rotate about the center of the phase space at the modulation tune $\nu_m$, i.e. the phase space ellipses return to this structure in $1/\nu_m$ revolutions.

When the modulation frequency approaches the bifurcation frequency from below ($x/x_{bif} > 1$), the UFP and the outer SFP move in and the inner SFP moves out. At the bifurcation frequency, where $x = x_{bif}$ and $\xi = 0$, the UFP collides with the inner SFP with $g_b = g_c = (4a)^{1/2}$; and they disappear together. Beyond the bifurcation frequency, $\nu_m > \nu_{bif}$ ($x < x_{bif}$), Eq. (7) has only one real solution:

$$g_a(x) = -(4a)^{1/3} \left[\left(\sqrt{1 - \frac{x}{x_{bif}}} + 1\right)^{1/3} - \left(\sqrt{1 - \frac{x}{x_{bif}}} - 1\right)^{1/3}\right]$$

(9)
In particular, \( g_a = -(8a)^{1/3} \) at \( x = 0 \) (\( \nu_m = \nu_s \)), and \( g_a = -2(4a)^{1/3} \) at \( x = x_{\text{bif}} \). The characteristics of bifurcation appear in all orders of resonances with nonlinear detuning. When there is a phase space cooling, the stable fixed point becomes an attractor. For accelerators with attractors, the hysteresis phenomenon occurs. As the modulation tune approaches the bifurcation tune, resonance islands can be created or annihilated. This characteristic of \( \lambda \)-shaped bifurcation curve is evidently shown in the SBR of LEP data discussed earlier.

Experiments were carried out in Early 1990’s at IU Cooler Ring. The Cooler Ring was a proton storage ring with kinetic energy from about 45 MeV up to 500 MeV. The circumference was 86.82 m. There were two cavities in this ring with a capability of harmonic number \( h=1 \), up to 6. The ring used electron cooling for proton beam phase space cooling. The damping time was about 0.33 s for the longitudinal phase space and 3 s for the transverse phase space.

The phase modulation can be achieved by either rf phase modulation to rf cavity or dipole field modulation to an external dipole magnet at a dispersive location. Dipole field modulation can arise from power supply ripple or ground motion. For the dipole field modulation, the effective modulation depth (amplitude) is

\[
\Delta C = D_x \theta(t) = D_x \dot{\theta} \sin(\omega_m t + \chi_0), \quad a = \frac{h\omega_0 D_x \dot{\theta}}{\omega_m C}
\]

Experiments were carried out at the IUCF Cooler Ring. For this experiment, the harmonic number was \( h = 1 \), the phase slip factor was \( \eta \approx -0.86 \), the stable phase angle was \( \phi_0 = 0 \), and the revolution frequency was \( f_0 = 1.03168 \text{ MHz} \) at 45 MeV proton kinetic energy. The rf voltage was chosen to be 41 V to obtain a synchrotron frequency of \( f_s = \omega_s/2\pi = 262 \text{ Hz} \) in order to avoid harmonics of the 60 Hz ripple. The synchrotron tune was \( \nu_s = \omega_s/\omega_0 = 2.54 \times 10^{-4} \). We chose betatron tunes at \( \nu_x = 3.828, \nu_z = 4.858 \) to avoid nonlinear betatron resonances. The corresponding smallest horizontal and vertical betatron sideband frequencies were 177 and 146 kHz respectively.

Since the injected beam from the IUCF K200 AVF cyclotron is uniformly distributed in the synchrotron phase space within a momentum spread of about \( (\Delta p/p) \approx \pm 3\times10^{-4} \), all attractors can be populated. The phase coordinates of these attractors could be measured by observing the longitudinal beam profile from BPM sum signals on an oscilloscope. The Figure below shows the longitudinal beam profile accumulated through many synchrotron periods with modulation field \( B_m = 4 \text{ G} \) for modulation frequencies of 210, 220, 230, 240, 250, and 260 Hz from left to right. The rf waveform is also shown for reference.

Using a fast sampling digital oscilloscope for a single trace, we found that the beam profile was not made of particles distributed in a ring of large synchrotron amplitude, but was composed of two beamlets. Both beamlets rotated in the synchrotron phase space at the modulating frequency, as measured from the fast Fourier transform (FFT) of the phase signal. If the equilibrium distribution of the beamlet was elongated, then the sum signal, which measured the peak current of the beam, would show a large signal at both extremes of its phase coordinate, where the peak
current was large. When the beamlet rotated to the central position in the phase coordinate, the beam profile became flat with a smaller peak current. The profile observed with the oscilloscope can be modeled with the equilibrium distribution of charges in these attractors.

The amplitudes of these fixed points are plotted in the graph below and are found to fall on top of the theory for the sinusoidal excitation or square wave excitation. The third harmonic of the square wave can excite synchrotron dipole resonance.

To explore the resonance tori in the island, we also carried out experiments by a phase kick to the beam and observe its phase space coordinates (φ, δ). Transform the phase space coordinates into the resonance rotating frame is shown in the right plot below. The phase space ellipse in the resonance rotating frame obey the Hamiltonian flow of Eq. (5).
If the frequency ramping time is slower than the longitudinal phase space damping time, the measured beam oscillation amplitude will be one of the SFPs of the resonance. If the accelerator parameters are varied adiabatically, particles will stay in the bucket of a resonance island until that the bucket is too small to contain the phase space area so that it will jump from one attractor to the other one. This jump occurs at different parametric space of the tune ramp. This is hysteric phenomenon.

When the modulation tune reached a frequency far below the bifurcation frequency, the phase amplitude jumped from the outer attractor to the inner attractor solution. On the other hand, if the modulation frequency, originally far below the bifurcation frequency, was ramped up toward the bifurcation frequency, the amplitude of the phase oscillations followed the inner attractor solution. At a modulation frequency near the bifurcation frequency, the amplitude of the synchrotron oscillations jumped from the inner to the outer attractor solution.

The hysteresis depended on beam current and modulation amplitude a. Since a large damping parameter could destroy the outer attractor, the hysteresis depended also on the dissipative force. The observed phase amplitudes were found to agree well with the solutions of Eq. (3.106).

Similar hysteretic phenomena have been observed in electron-positron colliders, related to beam-beam interactions, where the amplitudes of the coherent π-mode oscillations showed hysteretic phenomena. At a large beambeam tune shift, the vertical beam size exhibited a flip-flop effect with respect to the relative horizontal displacement of two colliding beams.

**Applications**

RF phase modulation has many applications. One of the applications is the phase space dilution to alleviate space charge effect or to produce beams with uniform distribution. The Hamiltonian for the rf phase modulation in the presence of a second rf system with harmonic number ratio $h$ is

$$H = \frac{1}{2} \nu_s \delta^2 - \nu_s \left[ (1 - \cos \phi) - \frac{r}{h} (1 - \cos [h \phi + \Delta \phi]) \right]$$

(11)

Here, $r=V_2/V_1$, $\Delta \phi$ is the phase modulation to the secondary rf system, and $\Delta \phi_0$ is the phase shift. The bunch dilution can be achieved by RF phase modulation. The plots below show the experimental data (left) and numerical simulations (right).
The bunch length and the momentum spread will reach an equilibrium shape bounded by invariant tori. The final bunch width depends on the modulation frequency and modulation amplitude. The phase space diagram of the final beam width vs modulation frequency and modulation amplitude is shown below [A. Pham, Ph.D. thesis, Indiana University, 2014]. This bunch manipulation scheme can be used to provide bunch dilution for space charge dominated beam, for radiation effects experiments in reducing the peak bunch intensity.
Resonances due to RF Voltage Modulation

The beam lifetime limitation due to rf noise has been observed in many synchrotrons, e.g., the super proton synchrotron (SPS) in CERN. There has been some interest in employing rf voltage modulation to induce super slow extraction through a bent crystal for very high energy beams, rf voltage modulation to stabilize collective beam instabilities, rf voltage modulation for extracting beam with a short bunch length, etc. Since the rf voltage modulation may be used for enhancing a desired beam quality, we will study the physics of synchrotron motion with rf voltage modulation, that may arise from rf noise, power supply ripple, wakefields, etc. Beam response to externally applied rf voltage modulation has been measured at the IUCF Cooler.

In the presence of rf voltage modulation, the synchrotron equations of motion are

\[ \phi_{n+1} = \phi_n - 2\pi \nu_n \frac{P_n}{|P_n|}, \]
\[ P_{n+1} = P_n - 2\pi \nu_m [1 + b \sin(\nu_m \theta_n + \chi)] \sin \phi_{n+1} - \frac{4\pi \alpha}{\omega_0} P_n, \]

where \( P = -h |\eta| \delta/\nu_s \) is the normalized off-momentum coordinate conjugate to \( \phi \); \( \delta = \Delta p/p_0 \) is the fractional momentum deviation from the synchronous particle; \( \eta \) is the phase slip factor; \( \nu_s = \sqrt{h |\eta| eV / 2|\beta^2|E_0} \) is the synchrotron tune at zero amplitude; \( E_0 \) is the beam energy; \( b = \Delta V/V \) is the fractional rf voltage modulation strength \( (b > 0) \); \( \nu_m \) is the rf voltage modulation tune; \( \chi \) is a phase factor; \( \theta \) is the orbital angle used as time variable; \( \omega_0 = 2\pi f_0 \) is the angular revolution frequency; and \( \alpha \) is the phase-space damping factor resulting from phase-space cooling.

At the IUCF Cooler, the phase-space damping rate was measured to be about \( \alpha \approx 3.0 \pm 1.0 \text{ s}^{-1} \), which is much smaller than \( \omega_0 \nu_s \), typically about 1500 s\(^{-1}\) for the \( h = 1 \) harmonic system.

Without loss of generality, we discuss the case for a particle energy below the transition energy, i.e. \( \eta < 0 \). Neglecting the damping term, i.e. \( \alpha = 0 \), the equation of motion for phase variable \( \phi \) is

\[ \ddot{\phi} + \nu_s^2 [1 + l \sin(\nu_m \theta + \chi)] \sin \phi = 0, \]
\[ \frac{d^2 \phi}{dz^2} + (p - 2q \cos 2z) \phi = 0. \]

where the overdot indicates the time derivative with respect to \( \theta \). In linear approximation with \( \sin \phi \approx \phi \), it reduces to Mathieu’s equation by choosing \( \chi = -\pi/2 \) and \( z = \nu_m \theta, \quad p = 4\nu_s^2 /\nu_m^2, \quad q = 2b \nu_s^2 /\nu_m^2. \)

In accelerator physics applications, \( p \) and \( q \) are real with \( q \ll 1 \). The stable solutions of Mathieu’s equation are obtained with the condition that the parameter \( p \) is bounded by the characteristic roots \( r \) and \( r + 1 \), where \( r = 0, 1, 2, \cdots \). In other words, unstable solutions are in the region \( br(q) \leq p \leq ar(q) \), where \( r = 1, 2, \cdots \). The first order unstable region is

\[ 2\nu_s(1 - \frac{1}{4}b) \leq \nu_m \leq 2\nu_s(1 + \frac{1}{4}b) \]

(3)

What does unstable region mean? The Mathieu instability is a linear stability, it means that there is no stable motion, i.e. particle cannot exist in this condition for finite amplitude motion.

Synchrotron motion is nonlinear, what happens when the Mathieu instability condition of Eq. (3) occurs? To solve this problem, we need to understand the Hamiltonian harmonic!

Equation (2) can be derived from the Hamiltonian:

\[ H = H_0 + H_1; \quad H_0 = \frac{1}{2} \nu_s P^2 + \nu_s (1 - \cos \phi), \quad H_1 = \nu_s b \sin(\nu_m \theta + \chi) [1 - \cos \phi], \] (4)
where $H_0$ is the unperturbed Hamiltonian and $H_1$ the perturbation. For a weakly perturbed Hamiltonian system, we expand $H_1$ in action-angle coordinates of the unperturbed Hamiltonian

$$H_1 = \nu_s b \sum_{n=-\infty}^{\infty} |G_n(J)| \sin(\nu_m \theta - n \psi - \gamma_n)$$

where we choose $\chi = 0$ for simplicity, and $|G_n(J)|$ is the Fourier amplitude of the factor $(1-\cos \phi)$ with $\gamma_n$ its phase. Since $(1-\cos \phi)$ is an even function of $\phi$, there are only even order resonances. Most important resonance is $n=2$, the quadrupole mode. When the modulation tune is near $2\nu_s$, the Hamiltonian can be approximated by

$$H = H_0(J) + b \frac{\nu_s}{4} J \sin(\nu_m \theta - 2\psi - \gamma_n) + \cdots$$  \hspace{1cm} (5)

One can transform the Hamiltonian to the resonance rotating frame, and obtain

$$H = (\nu_s - \frac{\nu_m}{2}) J - \frac{\nu_s}{16} J^2 + \frac{\nu_s}{4} J \cos 2\psi$$

Hamilton’s equations of motion are

$$\dot{J} = \frac{\nu_s}{2} b J \sin 2\psi,$$

$$\dot{\psi} = \nu_s - \frac{\nu_m}{2} - \frac{\nu_s}{8} J + \frac{\nu_s}{4} b \cos 2\psi.$$  \hspace{1cm} (6)

The fixed points that determine the locations of islands and separatrix of the Hamiltonian are obtained from $\dot{J} = 0$, $\dot{\psi} = 0$. The stable fixed points (SFPs) ($\psi = 0$ and $\pi$) and the unstable fixed points (UFPs) ($\psi = \pi/2$ and $3\pi/2$) are

$$J_{\text{SFP}} = \begin{cases} 8(1 - \frac{\nu_m}{2\nu_s}) + 2b, & \text{if } \nu_m \leq 2\nu_s + \frac{1}{2}b\nu_s \\ 0, & \text{if } \nu_m > 2\nu_s + \frac{1}{2}b\nu_s \end{cases}$$

$$J_{\text{UFP}} = \begin{cases} 8(1 - \frac{\nu_m}{2\nu_s}) - 2b, & \text{if } \nu_m \leq 2\nu_s - \frac{1}{2}b\nu_s \\ 0, & \text{if } 2\nu_s - \frac{1}{2}b\nu_s \leq \nu_m \leq 2\nu_s + \frac{1}{2}b\nu_s \end{cases}$$  \hspace{1cm} (7)

Note that the $J_{\text{UFP}}$ in Eq. (8) is exactly the Mathieu instability of Eq. (3). The nonlinearity of the synchrotron motion gives us a stable fixed point within the Mathieu instability region. The fixed points of Eq. (8) are shown in the left plot below. The phase space ellipses and separatrix of the Hamiltonian are shown in the right plot below.

![Graphs](image-url)
Experimental measurements of the quadrupole mode fixed points are also shown in the Figure above. We also note that the Mathieu instability is clearly experimentally verified. Experimental measurement of the invariant tori has also been carried out to verify the ellipses shown above.

**Applications of voltage modulations**

The voltage modulation can be used to compensate injection mismatch, bunch compression at the extraction, or voltage mis-match after the passage of the transition energy. The data below shows the bunch quadrupole mode oscillation after passage of the transition energy. The left plot shows the effect of space charge induced voltage calculated based on the measured rf parameters. The bunch quadrupole mode oscillation after the transition energy is reflected from the measurement of the bunch width oscillation from the Ionization profile monitor (IPM).

RF voltage modulation can be used to control the space charge induced voltage mis-match during the transition crossing.
Spin Dynamics

Spin and the associated magnetic moment are fundamental properties of elementary particles. The spin quantum number has been employed to understand many phenomena in atomic, nuclear, elementary particle, solid state, and statistical physics on its importance in high energy physics. The spin orbit interaction has been important in understanding atomic and nuclear physics. The magnetic moments of baryons also offer new insights into the study of the constituent quark model. High energy polarized beam collisions of elementary particles may also provide information necessary for a better understanding of the quantum chromodynamics.

The magnetic moment of a charged particle moving in a circular orbit is given by

$$\vec{\mu}_{\text{orbital}} = \frac{q}{2m} \vec{L}, \quad \vec{L} = \vec{r} \times \vec{p},$$

(1)

where q and m are the charge and the mass of the particle respectively, and L is the orbital angular momentum. However, the magnetic moment of elementary particles and nuclei may not be related to their spin in such a simple minded magneto-mechanical relation. In fact, two fundamental difficulties in atomic spectra during 1920–1925, i.e., the anomalous Zeeman effect and the multiplets, led Uhlenbeck and Goudsmit in 1925 to postulate the concept of electron spin with the intrinsic magnetic moment

$$\vec{\mu}_e = g_e \frac{e}{2m_e} \vec{S},$$

(1)

where an empirical value of $g_e = 2$ greatly simplified the interpretation of the atomic spectra. Nevertheless, the magneto-mechanical ratio $g_e = 2$ caused problems in the spin–orbit interaction term. In 1926, L.H. Thomas [13] showed that the spin orbit coupling was consistent with $g_e = 2$ provided that the “Thomas precession” correction is included, where the Lorentz transformation in a circular orbit can introduce a correction term to the spin precessing frequency. In 1927, P.A.M. Dirac showed that $g_e = 2$ arose solely from charged particles satisfying the relativistic Dirac equation, which led to new concepts in physics such as anti-particles and the vacuum state. Since then, the spin quantum number has been conclusively established. The magnetic moments of elementary particles and nuclei are

$$\vec{\mu}_e = g_e \frac{e}{2m_e} \vec{S}, \quad \vec{\mu}_\mu = g_\mu \frac{e}{2m_\mu} \vec{S}, \quad \vec{\mu}_{\text{hadron}} = g_{\text{hadron}} \frac{e}{2m_{\text{hadron}}} \vec{S}, \quad \vec{\mu}_{\text{nuclei}} = g_{\text{nuclei}} \frac{q}{2m_{\text{nuclei}}} \vec{J},$$

where $e$ and $q$ are the charges of elementary particles and nuclei; $m_i$ are the masses of these particles; $\vec{S}$ is the spin of an elementary particle; $\vec{J}$ represents the spin of the nuclei; and the g factors are determined from experimental measurements.

In general, the magnetic moment of a composite system is a superposition of magnetic moments of its constituents. Since the magnetic moment of an electron is 2000 times larger than that of a proton, the magnetic moments of atoms are mainly determined by the magnetic moment due to valence electrons in atomic shell orbits. Similarly, the magnetic moments of nuclei are determined by nucleons. Although nuclei, in a first order approximation, can be represented by an independent particle model with neutrons and protons occupying shell model orbits, the magnetic moments of nuclei differ substantially from the prediction of the single particle shell model due to strong interaction. On the other hand, it is a pleasant surprise that the constituent
quark model “accurately” predicts magnetic moments of nucleons and hyperons to within ±0.2 nuclear magnetons. The unit of the magnetic moments for electron, muon and proton are

\[ \mu_B = \frac{e\hbar}{2m_e} = 5.788382 \times 10^{-11} \text{ MeV/T} \]
\[ \mu_\mu = \frac{e\hbar}{2m_\mu} = 2.790154 \times 10^{-13} \text{ MeV/T} \]
\[ \mu_N = \frac{e\hbar}{2m_p} = 3.152452 \times 10^{-14} \text{ MeV/T} \]

The anomalous \( g \)-factor are defined as \( a = G = \frac{g}{2} - 1 \).

<table>
<thead>
<tr>
<th>Particle</th>
<th>( J^\pi )</th>
<th>M(MeV)</th>
<th>( g/2 )</th>
<th>( G ) or ( a )</th>
<th>( \mu (\mu_B, \mu_\mu, \mu_N) )</th>
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<td>0.001159652</td>
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<tr>
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<td>( \frac{1}{2}^+ )</td>
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<td>1.00</td>
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<tr>
<td>( n )</td>
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</tr>
<tr>
<td>( \Lambda )</td>
<td>( \frac{1}{2}^+ )</td>
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<td>-0.729</td>
<td>-0.613 ± 0.004</td>
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<tr>
<td>( \Sigma^- )</td>
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<td>2.42 ± 0.05</td>
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<td>( {}^{23}_{\text{Na}} )</td>
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<td>21409.21</td>
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<td>0.533</td>
<td>2.2175</td>
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For a beam of particles, the polarization vector is defined as the ensemble average of spin vectors. Parameters needed in specifying the alignment of spin vectors of particles in the beam bunch are illustrated in the following examples for the spin- \( \frac{1}{2} \) and spin-1 cases.

**Spin-1/2 Particles:** For a system of spin- \( \frac{1}{2} \) particles, the degree of polarization is

\[ P = \frac{N_+ - N_-}{N_+ + N_-} \]

(3)

Here \( N_\pm \) are the numbers of particles in two-spin states \( |\frac{1}{2}, \pm\frac{1}{2}\rangle \) along a quantization axis. For a 100% polarized beam, the spin states of all particles are quantized along a polarization axis. By varying the relative numbers of two spin states, a beam of arbitrary polarization may be obtained. In general, three parameters are needed to specify the polarization of spin- \( \frac{1}{2} \) particles, i.e., two parameters for the direction of the quantization axis and one parameter for the \( N_+/N_- \) ratio. Thus the polarization of the spin- \( \frac{1}{2} \) system is a vector characterized by a direction and a magnitude. It
is usually called vector polarization, which is the ensemble average of the spin vector \( \vec{P} = \langle \vec{\sigma} \rangle \), where \( \vec{\sigma} \) are Pauli spin matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\]  
(4)

Any coherent linear combination of pure states of spin- \( \frac{1}{2} \) particles results in a 100\% polarization with the polarization vector defined by a proper quantization axis. Any spin- \( \frac{1}{2} \) system is cylindrically symmetric about the direction of polarization.

Spin-1 System: For a spin-1 system, there are three \( m \)-states along the quantization axis. If all particles in the bunch occupy a pure \( m = 0 \) state, the vector polarization is zero, i.e., \( \langle \vec{S} \rangle = 0 \), yet the beam is polarized. This is called an aligned state. States of this type are characterized by the magnitude of the spin component \( S_z \) along the alignment axis. Thus the polarization of spin-1 particles is characterized by the vector polarization and the alignment.

We consider a beam composed of \( N_+ \), \( N_0 \), and \( N_- \) particles along the quantization axis with \( m = +1 \), 0, and \( -1 \) respectively. The vector polarization is given by \( P = (N_+ - N_-) / (N_+ + N_0 + N_-) \). An unpolarized non-aligned beam corresponds to equal mixing in all three states, i.e., \( N_+ = N_0 = N_- \). Since \( \langle S^2 \rangle = S(S + 1) = 2 \), the alignment can be defined as
\[
A = 3\langle S_z^2 \rangle - 2 = \frac{3(N_+ - N_-)}{N_+ + N_0 + N_-} - 2 = \frac{N_+ - 2N_0}{N_+ + N_0 + N_-} .
\]  
(5)

An unpolarized beam has \( P = 0 \) and \( A = 0 \) as expected. A zero vector polarization does not imply a zero alignment. Some examples: (1) \( N_+ = N_0 = N_- \), we have \( P = 0, A = 0 \); (2) \( N_+ = N_- = 0 \), we have \( P = 0, A = -2 \); (3) \( N_+ = N_- \) and \( N_0 = 0 \), we have \( P = 0, A = 1 \); etc.

Since the alignment of the spin-1 system depends on the ensemble average of the square of the spin vector, we need five of the following quadratic spin functions, \( \langle S_x^2 \rangle, \langle S_y^2 \rangle, \langle S_z^2 \rangle, \langle S_xS_y \rangle, \langle S_xS_z \rangle \), and \( \langle S_xS_z \rangle \) with \( \langle S_x^2 + S_y^2 + S_z^2 \rangle = 2 \) to define the tensor polarization. Including the vector polarization, eight parameters are needed to specify a polarized spin-1 system. There are two conventions being used:

1. \( P_i = \langle S_i \rangle \quad P_{ij} = \frac{3}{2} \langle S_i S_j + S_j S_i \rangle - 2\delta_{ij} \), \( i, j = 1, 2, 3 \) \( P_{11} + P_{22} + P_{33} = 0 \).

2. \( T_{kq} \quad 0 \leq k \leq 2S, \quad -k \leq q \leq k \)
   \[
   \langle T_{00} \rangle = 1, \quad T_{10} = \sqrt{\frac{3}{2}} \langle S_z \rangle, \quad T_{1,\pm 1} = \mp \sqrt{\frac{3}{2}} \langle S_x \pm S_y \rangle .
   \]
   \[
   T_{20} = \sqrt{\frac{1}{2}} \langle 3S_z^2 - 2 \rangle, \quad T_{2,\pm 1} = \pm \sqrt{\frac{3}{2}} \langle S_x S_z + S_y S_z \rangle, \quad T_{2,\pm 2} = \frac{3}{2} \langle S_z^2 \rangle .
   \]  
(6)

A beam with a nonzero \( P_i \) or \( \langle T_{10} \rangle \) is called vector polarized. A beam with nonvanishing \( P_{ij} \) or \( \langle T_{2q} \rangle \) is either called aligned, tensor polarized or having a rank-2 polarization. An analogy to help visualize these polarization parameters is the multipole moment expansion of static charge or mass distribution. A charged distribution can be completely specified by multipole moments, i.e., the monopole (or intensity), dipole, quadrupole, etc. A spin system can be specified
completely with moments up to $2^N$-poles, e.g. up to vector polarization for the spin- $\frac{1}{2}$ particle and up to rank 2 tensor polarization for the spin-1 particle.

A special class of mixed states corresponds to a system exhibiting symmetry about an axis, e.g. deuterons produced in a polarized ion source, where certain substates in an external magnetic field are selected. Because of the axial symmetry about the quantization axis, such a system can be specified by four parameters; two for the direction of quantization axis, and two for the relative populations in three states. Once the axis is specified, two parameters, $P_3$ (or $T_{10}$) and $P_{33}$ (or $T_{20}$), are sufficient to describe the system.

**Interaction of magnetic moment with the magnetic field**

The Hamiltonian and equation of motion for the spin vector for the magnetic interaction of a charged particle with the magnetic field are

$$H_{int} = -\vec{\mu} \cdot \vec{B}, \quad \vec{\mu} = g \frac{e}{2m} \vec{S},$$

$$\frac{d\vec{S}}{dt} = \vec{\mu} \times \vec{B}, \quad \frac{d\vec{p}}{dt} = \nabla (\vec{\mu} \cdot \vec{B}),$$

(7) (8)

where the first equation describes the spin precession around the magnetic field and the second equation describes the force on particles at different spin states. Many nuclei possess magnetic moments. These nuclei in a strong magnetic field occupy one of two states: the state parallel and the state anti-parallel to the magnetic field. For $ge > 0$, the state parallel to the magnetic field is lower than the state antiparallel to the magnetic field in energy by $2\mu B$ where $\mu$ is the magnetic dipole moment of the nuclei. The spin of these nuclei will precess about the magnetic field direction at the Larmor precessing frequency $\omega = geB/2m$.

<table>
<thead>
<tr>
<th>particles</th>
<th>electron</th>
<th>muon</th>
<th>proton</th>
<th>deuteron</th>
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</thead>
<tbody>
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<td>freq.</td>
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<td>135 MHz/Tesla</td>
<td>42.6 MHz/Tesla</td>
<td>6.5 MHz/Tesla</td>
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</table>

In 1921, O. Stern and W. Gerchach proved the existence of the magnetic moment in atoms through a classic experiment in which they passed a beam of slowly moving neutral silver atoms through a region of non-uniform magnetic field. They observed that the atoms were separated into two bands governed by Eq. (8). We choose $(\hat{x}, \hat{y}, \hat{z})$ as respectively unit vectors for the horizontal transverse axis, the horizontal longitudinal axis, and the transverse vertical axis. If the vertical field $B_z$ is symmetric with respect to $x$, and independent of the longitudinal coordinate $s$, the force on the atom is in the vertical direction, i.e. $F_z = \mu_z \partial B_z/\partial z$.

Given a beam of silver atoms (A=109, $m = 108.9$ amu, $\mu = 1$ Bohr magneton) emitted from an oven with temperature 1300$^\circ$K with velocity of $(3kT/m)^{1/2} \approx 545$ m/s, the magnetic force for a 10 Tesla/meter magnetic field gradient is about $9.3 \times 10^{-23}$ N. This force is much larger than the gravitational force. Given a 10 Tesla/meter magnetic gradient, the beam will split into two lines at a spacing of $\Delta z = 0.0034(\Delta t) \cdot L$, where $\Delta t$ is the length of the magnet and $L$ is the distance between the magnet and the beam measuring instrument.

The Schrodinger equation is given by $i\hbar \partial \psi /\partial t = H_{int}$. The equation for the spin wave function for a spin- $\frac{1}{2}$ particle with magnetic moment $\vec{\mu} = (ge/2m) \vec{S}$ in the magnetic field $\vec{B}$ is
\[
\frac{\partial \Psi}{\partial t} = \frac{i}{2} \frac{g}{2m} \sigma \cdot \vec{B} \Psi
\]  
(9)

We choose the quantization axis along the magnetic field \( \vec{B} = B_z \hat{z} \), and use the upper (\( \psi_+ \)) and lower (\( \psi_- \)) components of the spinor wave function for the \( \frac{1}{2} \) and \( -\frac{1}{2} \) eigenstates. Now we add a sinusoidal perturbing field \( \Delta B = B_{1z} (\hat{x} \cos \omega t - \hat{y} \sin \omega t) \), the equation for the spin wave function becomes

\[
\frac{\partial \Psi}{\partial t} = \frac{i}{4m} \left( \begin{array}{cc} B_z & B_{1z} e^{i\omega t} \\ B_{1z} e^{-i\omega t} & -B_z \end{array} \right) \Psi.
\]  
(10)

Transform the Schrödinger equation into the resonance precessing frame and show that the solution of the Schroedinger equation is

\[
\Psi(t) = e^{\frac{i}{2} \omega t \sigma_3} e^{\frac{i}{2} \chi(t-t_0) \sigma \cdot \hat{n}} e^{-\frac{i}{2} \omega t_0 \sigma_3} \Psi(t_0),
\]  
(11)

where \( \psi(t_0) \) is the initial spinor at time \( t_0 \), and the unit vector \( \hat{n} \) along which the spin vector is stationary is given by

\[
\hat{n} = \frac{\delta}{\lambda} \hat{z} + \frac{\epsilon}{\lambda} \hat{x}; \quad \lambda = \sqrt{\delta^2 + |\epsilon|^2}, \quad \delta = \omega_L - \omega, \quad \epsilon = \omega_L B_{1z}.
\]  
(12)

We consider an initial spin up state at \( t_0 = 0 \), i.e., \( \psi_+(t_0) = 1, \psi_-(t_0) = 0 \), the probability for finding the particle in the spin up and down states are

\[
|\Psi_+(t)|^2 = \cos^2 \frac{\lambda t}{2} + \frac{\delta^2}{\lambda^2} \sin^2 \frac{\lambda t}{2}, \quad |\Psi_-(t)|^2 = \frac{\epsilon^2}{\lambda^2} \sin^2 \frac{\lambda t}{2}.
\]  
(13)

In quantum mechanics, we define the polarization as \( P(t) = |\Delta \psi_+|^2 - |\Delta \psi_-|^2 \). In beam physics, we take ensemble average of all particles.

**The Thomas–BMT equation**

Equation (8) for the spin motion is in the rest frame of the particle. The equation of motion for the spin vector defined in accelerator need to transform the EM field to the rest frame of the particle. The resulting equation of spin motion is the Thomas–BMT equation:

\[
\frac{d\vec{S}}{dt} = \frac{e}{\gamma mc} \vec{S} \times \left[ (1 + G\gamma) \vec{B}_\perp + (1 + G) \vec{B}_\parallel + (G\gamma + \frac{\gamma}{\gamma + 1}) \frac{\vec{E} \times \vec{B}}{c} \right]
\]  
(14)

where \( \vec{S} \) is the spin vector of a particle in the particle rest frame, \( B_\perp \) and \( B_\parallel \) are the transverse and longitudinal components of the magnetic fields in the laboratory frame with respect to the velocity \( \beta c \) of the particle. The vector \( \vec{E} \) stands for the electric field, \( G \) is the anomalous gyromagnetic \( g \)-factor, and \( \gamma mc^2 \) is the energy of the moving particle. This peculiar expression arises from the transformation of the electric and magnetic fields of the curvilinear laboratory
reference frame to the particle rest frame. This is done in order to obtain a proper description of the spin–magnetic interaction. We choose the coordinate system shown below:

For a planar circular accelerator, we have

\[ \vec{r} = \vec{r}_0(s) + x\hat{x} + z\hat{z}, \quad \hat{s} = d\vec{r}_0/ds, \quad \frac{d\hat{x}}{ds} = \frac{\hat{s}}{\rho}, \quad \frac{d\hat{s}}{ds} = -\frac{\hat{x}}{\rho}, \quad \frac{d\hat{z}}{ds} = 0 \]

Here \( \rho \) is the bending radius of dipole, and \((x, z)\) are betatron coordinates. Instead of using the time \( t \), we use the orbiting angle \( \theta \) as the independent coordinate with \( \frac{d}{dt} = \left[ \frac{v}{\rho x} \right] \frac{d}{d\theta} \). The Thomas-BMT equation becomes

\[ \frac{d\hat{S}}{d\theta} = \hat{S} \times \vec{F}, \quad \vec{F} = F_1\hat{x} + F_2\hat{y} + F_3\hat{z}, \]

\[ F_1 = -\rho z'' (1 + G\gamma), \quad F_2 = (1 + G\gamma) z' - \rho (1 + G) \left( \frac{z}{\rho} \right)', \quad F_3 = -(1 + G\gamma) + (1 + G\gamma) \rho^2 x'' \]

Expanding the spin vector as \( \hat{S} = S_1\hat{x} + S_2\hat{y} + S_3\hat{z} \), we find

\[ \frac{dS_1}{d\theta} = (1 + F_3) S_2 - F_2 S_3, \]

\[ \frac{dS_2}{d\theta} = -(1 + F_3) S_1 + F_1 S_3, \]

\[ \frac{dS_3}{d\theta} = F_2 S_1 - F_1 S_2. \]
We define \( S_\pm = S_1 \pm iS_2 \) and \( F_\pm = F_1 \pm iF_2 \). With \((1 + F_3) = G\gamma\), the equation of motion becomes

\[
\frac{dS_\pm}{d\theta} = \pm iG\gamma S_\pm \pm iF_\pm S_3, \\
\frac{dS_3}{d\theta} = \frac{i}{2}(F_- S_+ - F_+ S_-),
\]

(17)

When \( F_1 = 0 \) and \( F_2 = 0 \), the spin vector becomes

\[
S_\pm = e^{\pm iG\gamma \theta} S_{\pm 0}, \quad S_3 = S_{30}.
\]

(18)

Here, the spin components \( S_\pm \) precess about the vertical axis at \( G\gamma \) precession turns per orbital revolution. Thus \( G\gamma \) is called the spin tune, which is defined to be the number of spin precession around a spin closed orbit per orbital revolution. Using three \( \theta \)-independent unit vectors \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) to coincide with \((\hat{x}, \hat{y}, \hat{z})\) at any azimuth in the ring. Equation (16) becomes

\[
\frac{d\vec{S}}{d\theta} = \vec{n} \times \vec{S}, \quad \vec{n} = G\gamma \hat{e}_3 - F_1 \hat{e}_1 - F_2 \hat{e}_2.
\]

(19)

Defining a two-component spinor \( \psi \) such that the \( i \)-th component of the spin vector is given by

\[
S_{\psi i} \equiv \langle \Psi | \sigma_i | \Psi \rangle = \Psi^\dagger \sigma_i \Psi
\]

We can transform the spin equation onto the spinor equation of motion:

\[
\frac{d\Psi}{d\theta} = -\frac{i}{2} \left( \begin{array}{c} \hat{\sigma} \cdot \vec{n} \\ 0 \end{array} \right) \Psi = -\frac{i}{2} H \Psi = -\frac{i}{2} \begin{pmatrix} G\gamma & -\xi \\ -\xi^* & -G\gamma \end{pmatrix} \Psi
\]

(20)

where \( \xi(\theta) = F_1 - iF_2 \) characterizes the spin depolarization kick by coupling the up and down components of the spinor wave function. Since the betatron coordinate of the particle is a periodic or quasi-periodic function of the dipole bend angle \( \theta \), we can expand the depolarization driving term \( \xi \) in Fourier series

\[
\xi(\theta) = F_1 - iF_2 = \sum_K \epsilon_K e^{-ik\theta}
\]

(21)

The Fourier amplitude \( \epsilon_k \) is called the resonance strength and the corresponding mode number \( K \) is called the spin resonance tune. If the spinor is decomposed into two components, we find

\[
\Psi = \begin{pmatrix} u \\ d \end{pmatrix}, \quad S_1 = u^* d + ud^*, \quad S_2 = -i (u^* d - ud^*), \quad S_3 = |u|^2 - |d|^2.
\]

(22)

Since the spin precessing kernel \( H \) is hermitian, we get \(|\vec{S}| = |u|^2 + |d|^2 = \langle \psi | \psi \rangle \) and \( d|\vec{S}|/d\theta = 0 \). Thus we choose the normalization of the spinor wave function as \( 1 = |u|^2 + |d|^2 = \langle \psi | \psi \rangle \).
Using the Thomas–BMT equation, the spin resonance strength, or the Fourier amplitude of the spin perturbing fields, is given by

\[
\epsilon_K = \frac{1}{2\pi} \int \left[ (1 + G\gamma) \frac{\Delta B_x}{B\rho} + (1 + G) \frac{\Delta B_{\parallel}}{B\rho} \right] e^{iK\theta} dS,
\]

where \(\Delta B_x\) is the radial perturbing field, and \(\Delta B_{\parallel}\) is the solenoidal perturbing field. In many synchrotrons, there is little or no solenoidal field. The transverse radial field arises mainly from the dipole roll, and/or the vertical displacement with respect to the center of quadrupoles. The error magnetic field is also expressed in terms of the particle orbit. The resulting spin resonance tunes of Eq. (23) are classified according to their sources:

<table>
<thead>
<tr>
<th>(k)</th>
<th>Classification of resonances</th>
</tr>
</thead>
<tbody>
<tr>
<td>(kP \pm \nu_z)</td>
<td>intrinsic resonances</td>
</tr>
<tr>
<td>(kP \pm \nu_z \pm m\nu_{syn})</td>
<td>intrinsic synchrotron sideband resonances</td>
</tr>
<tr>
<td>integer</td>
<td>imperfection resonances</td>
</tr>
<tr>
<td>integer</td>
<td>imperfection synchrotron sideband resonances</td>
</tr>
<tr>
<td>(j + kP \pm m_z\nu_z \pm m_x\nu_x \pm m_y\nu_y)</td>
<td>higher-order spin resonances</td>
</tr>
</tbody>
</table>

The dominant term in Eq. (3.3) is given by the \(\rho \nu_\gamma''\) term in the integrand, i.e.

\[
\epsilon_K \approx -\frac{1 + G\gamma}{2\pi} \int \nu_\gamma'' e^{iK\theta} dS = +\frac{1 + G\gamma}{2\pi} \int \frac{\partial B_z}{\partial x} \rho e^{iK\theta} dS.
\]

The vertical displacement of the beam from the center of a quadrupole can be decomposed into two parts: \(z = zco - z_{offset} + z_{\beta}\), where \((zco - z_{offset})\) is the closed orbit displacement from the center of a quadrupole resulting from dipole rolls and/or quadrupole misalignments, \(z_{offset}\) is the offset of the quadrupole alignment from the beam closed orbit, and \(z_{\beta}\) describes the betatron motion of the orbiting particle. Basic beam dynamics gives

\[
zco(s) = \beta_z^{1/2}(s) \sum_{k=-\infty}^{\infty} \nu_k^2 f_k e^{i\phi_k(s)} \frac{\nu_z}{\nu_z^2 - k^2}, \quad z_{\beta}(s) = \left(\frac{\beta_z}{\pi\gamma}\right)^{1/2} \cos(\nu_z\phi_z + \chi).
\]

where \(\nu_z\) is the vertical betatron tune, \(\epsilon_N\) is the normalized emittance, \(\phi_z(s)\) is the betatron phase, and \(f_k\) is the Fourier amplitude of the error harmonic \(k\) given by

\[
f_k = \frac{1}{2\pi\nu_z} \int \frac{\beta_z^{1/2}}{\nu_z^2} \frac{\Delta B_x}{B\rho} e^{-i\phi_z} dS, \quad \phi_z(s) = \frac{1}{\nu_z} \int_{s_0}^{s} \frac{dS}{\beta_z}.
\]

Spin resonances due to the closed orbit errors \(zco\) are called the imperfection spin resonances. The imperfection resonance strength depends on quadrupole misalignment errors and/or dipole rolls. For a perfect machine with a zero closed orbit error, the imperfection resonance strength is zero. The imperfection resonance strength is approximately \(\epsilon_{\text{imperfection}} \approx 2 \times 10^{-3} G\gamma\) for most
hadron accelerators with a random misalignment error of 0.1 mm in quadrupoles without orbit correction.

The resonances arising from the betatron motion $z_\beta$ are called the intrinsic spin resonances. The intrinsic resonance strength is nearly independent of the machine alignment. Due to the adiabatic damping of the betatron motion for proton synchrotrons, the intrinsic resonance strength is proportional to $\sqrt{\gamma \varepsilon_N}$, i.e.

$$
\varepsilon_{\text{intrinsic}} \approx 2 \times 10^{-2} G \gamma \sqrt{\frac{\varepsilon_N/\gamma}{10 \pi \text{mm} - \text{mrad}}}.
$$

The graph below shows the scaling property of the imperfection and intrinsic depolarization resonances for some proton synchrotrons.

![Graph showing scaling property of imperfection and intrinsic depolarization resonances](image)

<table>
<thead>
<tr>
<th>$K$</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$nP \pm [\nu_B]$</td>
<td>super strong imperfection resonances</td>
</tr>
<tr>
<td>$nP \pm [\nu_z]$</td>
<td>very strong imperfection resonances</td>
</tr>
<tr>
<td>integer</td>
<td>regular imperfection resonances</td>
</tr>
<tr>
<td>$K_{\text{imp}} \pm m \nu_{\text{sym}}$</td>
<td>imperfection synchrotron sideband resonances</td>
</tr>
</tbody>
</table>

There are other different types of resonances: synchrotron-sideband resonances; nonlinear coupling resonances;
**Effect of spin resonances:**

Spin resonances give rise to depolarization. The graph below shows the electron polarization measured around 3.7 GeV in SPEAR where beam depolarization has been observed at the imperfection resonance $K = 8$, the linear betatron coupling resonance and its synchrotron sidebands at $3 + Q_x + mQ_{syn}$, the intrinsic resonance $3 + Q_x$ and its synchrotron sidebands, and higher-order resonances at $8 + Q_x - Q_z$ and $-2 + Q_x + Q_z$. The data can easily be fitted by using the parameters: $v_x = 5.279662$, $v_z = 5.182604$, $v_s = 0.04276$, and $\sigma_3 = 8.7 \times 10^{-4}$, shown at the right plot below.

![Graph showing electron polarization](image)

So the spin resonance can depolarize beam polarization in accelerators. How?

**Spinor at a Constant Acceleration Rate**

The effect of spin resonance can be obtained by solving Equation (20). We consider linear acceleration rate passing through a single-resonance, i.e. $G \gamma = \kappa_0 + \alpha \theta$ and $\xi = e^{e^{iK0}}$. To solve the spinor equation for this simple model, we transform Eq. (20) onto the spin precession frame (interaction picture) and the spinor equation of motion becomes

$$
\Psi(\theta) = e^{-\frac{i}{2} \int_0^\theta G \cdot d\theta \sigma_3 \Psi_I(\theta)}, \quad \frac{d\Psi_I}{d\theta} = \frac{i}{2} \left( \begin{array}{cc} 0 & \tilde{\xi} \\ \tilde{\xi}^* & 0 \end{array} \right) \Psi_I, \quad \tilde{\xi} = \epsilon \cdot e^{i[(\kappa_0 - K) \theta + \frac{1}{2} \alpha \theta^2]}.
$$

(27)

The spinor equation of motion has analytic solution in terms of the confluent hypergeometric functions. The final polarization is given by the celebrated Froissart–Stora formula:

$$
\langle S_3 \rangle = \langle \Psi_I | \sigma_3 | \Psi_I \rangle = 2e^{-4\pi q} - 1 = 2e^{-\pi \frac{|\kappa_0|}{2\alpha}} - 1.
$$

(28)

The Froissart-Stora formula has been experimentally verified in many accelerators. This formula has also been used to flip the spin by adiabatically crossing a resonance.

**Spin Motion at Constant $G \gamma$ or Slow Acceleration**

Another interesting analytically solvable model corresponds to a slow or zero acceleration rate. At a constant spin tune $\tilde{G} \gamma$, we can transform the spinor equation into the resonance precessing frame and obtain the resulting spinor equation of motion:

$$
\Psi_K(\theta) = e^{i \frac{\delta}{2} K \theta \sigma_3} \Psi(\theta), \quad \frac{d\Psi_K}{d\theta} = \frac{i}{2} \left( \begin{array}{cc} \delta & \epsilon \\ \epsilon^* & -\delta \end{array} \right) \Psi_K, \quad \Psi_K = \frac{i}{2} [\delta \sigma_3 + \epsilon R \sigma_1 - \epsilon I \sigma_2] \Psi_K
$$

(29)

where $\delta = K - G \gamma$ is the resonance proximity parameter. The solution is
where $\epsilon_R$ and $\epsilon_I$ are the real and the imaginary parts of the resonance strength, $\lambda = (\delta^2 + |\epsilon|^2)^{1/2}$ is the spin tune, and the spin-closed orbit in the resonance precessing frame is

$$\hat{n}_{co} = \frac{\delta}{\lambda} \hat{e}_3 + \frac{\epsilon_R}{\lambda} \hat{e}_1 - \frac{\epsilon_I}{\lambda} \hat{e}_2$$

The spin vector, which follows adiabatically the spin closed orbit, will have the average polarization value $S_3 = \delta/\lambda$. At $\delta = \pm |\epsilon|$, the spin closed orbit vector tilts 45° away from the vertical axis. The system has three eigenvalues, 0 and $\pm i\lambda$, which correspond to three eigensolutions describing the spin vector along the spin closed orbit and the spin vectors precessing right/left with respect to $\hat{n}_{co}$. Figure below shows schematically the evolution of the spin closed orbit in passing through a spin resonance.

From these two analytically soluble model, we know that depolarization occurs when the spin tune is near a resonance within the width of the resonance. Spin preservation in accelerator is to avoid spin resonance, reduce resonance strength!

**Spin Transfer Matrix in the Particle Rest Frame**

Transforming the spinor wave function of Eq. (30) back to the particle rest frame, one finds

$$\Psi(\theta_f) = e^{-\frac{i}{2}K\theta_f}e^{\frac{i}{2}[\delta_3 + \epsilon_R\sigma_1 - \epsilon_I\sigma_2]}(\theta_f - \theta_i)\Psi(\theta_i) = e^{\frac{i}{2}K\theta_f}e^{\frac{i}{2}K\theta_i}\Psi(\theta_i)$$

The components of the spin transfer matrix $t(\theta_f, \theta_i)$ are

$$\begin{align*}
  t_{11}(\theta_f, \theta_i) &= a e^{i(\theta_f - \theta_i)/2}, & t_{12}(\theta_f, \theta_i) &= i b e^{-i(\theta_f + \theta_i)/2}, \\
  t_{21}(\theta_f, \theta_i) &= -t_{12}^{*}(\theta_f, \theta_i), & t_{22}(\theta_f, \theta_i) &= t_{11}^{*}(\theta_f, \theta_i),
\end{align*}$$

$$\begin{align*}
  b &= \frac{|\epsilon|}{\lambda} \sin \left[ \frac{\lambda(\theta_f - \theta_i)}{2} = \left( 1 - a^2 \right)^{1/2},
\right. \\
  c &= \arctan \left[ \frac{d}{\lambda} \tan \left( \frac{\lambda(\theta_f - \theta_i)}{2} \right) \right], & d &= \arg(\epsilon^*).
\end{align*}$$

The off-diagonal matrix elements $t_{12}$ and $t_{21}$ are the depolarization driving terms, where the effective strength parameter $b$ oscillates with an amplitude $|\epsilon|/\lambda$. The effect of the depolarization kicks gives rise to an average polarization given by $S_3 = \delta/\lambda$. 

---

**Fig.4**
In the laboratory frame the spin closed orbit \( \hat{n}_{co} \) of Eq. (4.21) is also precessing at a frequency \( K \) about the vertical axis. If the polarized protons were injected initially along the vertical polarization direction, the final spin vector would precess around the spin closed orbit \( \hat{n}_{co} \) at a precession frequency \( \lambda \) and the net vertical spin vector would become

\[
\langle S_3 \rangle = |t_{11}|^2 - |t_{12}|^2 = 1 - 2\beta^2 = 1 - 2|\epsilon|^2 \frac{\lambda^2}{\lambda^2} \sin^2 \left( \frac{\lambda}{2}(\theta_f - \theta_i) \right).
\]

(35)

Here the time average becomes \( \langle S_3 \rangle = \delta^2/\lambda^2 \). The spin vector precesses about the \( \hat{n}_{co} \) at a rate of \( 1/|\epsilon| \) per revolution on resonance with \( \lambda = |\epsilon| \). Here the inverse of the resonance strength \( 1/|\epsilon| \) plays the role of spin precessing tune about the spin closed orbit. At the same time, the spin closed orbit precesses at a spin tune \( K \) about the vertical axis. If all particles in the bunch had identical spin tunes \( G_\gamma \), then the polarization vector would coherently precess around the spin closed orbit \( \hat{n}_{co} \) without depolarization.

**Spin Tune and Spin Closed Orbit**

The one-turn-map (OTM) can generally be expressed in \( \exp[-i\nu_s \hat{n}_{co} \hat{\sigma}] \), where \( \nu_s \) is the spin tune and \( \hat{n}_{co} = (\cos \phi_1, \cos \phi_2, \cos \phi_3) \) is the directional cosine of the spin closed orbit. Identifying the matrix elements of Eq. (33) with the OTM for a single spin resonance, the spin tune is given by

\[
\cos \pi \nu_s = a \cos (\gamma - K \pi), \quad a = (1 - \beta^2)^{1/2}, \quad b = \frac{|\epsilon|}{\lambda} \sin \pi \lambda, \quad c = \arctan \left( \frac{\delta}{\lambda} \tan \pi \lambda \right),
\]

\[
\lambda = \sqrt{\delta^2 + |\epsilon|^2}, \quad \delta = K - G_\gamma,
\]

(36)

Summary: The spin tune in the resonance rotating frame is \( \lambda \). This is the spin precession frequency around the spin closed orbit. At an imperfection resonance where \( K \) is an integer, the spin tune is shifted away from the resonance by a magnitude equal to \( |\epsilon|_K \). The \( \hat{n}_{co} \) vector is stationary at a location in the accelerator, where for every orbital revolution \( \theta_i \) advances \( 2\pi \). Thus the spin closed orbit exists for imperfection resonance. When the spin tune is exactly on resonance, where \( \delta = K - G_\gamma = 0 \), the projection of \( \hat{n}_{co} \) on the \( \hat{z} \) axis vanishes. The vertical projection of the polarization vector changes sign as the spin tune passes through the resonance.

At an intrinsic resonance where \( K \) is not an integer, the spin precessing phase is not a constant (modulo \( 2\pi \)) as \( \theta_i \) advances \( 2\pi \). Only the spin tune and the vertical component \( \hat{n}_{co} \), \( \hat{e}_3 \) of the \( \hat{n}_{co} \) do not change with \( \theta_i \). The spin closed orbit vector \( \hat{n}_{co} \) precesses about the vertical axis with a precessing tune \( K \). Since \( K \) is not an integer, the spin closed orbit will not return to the same direction in successive revolutions. Thus the spin closed orbit does not exist in the laboratory frame. Any polarization vector which deviates from \( \hat{n}_{co} \) will precess about the \( \hat{n}_{co} \) with a precessing tune \( \lambda \). At the same time the \( \hat{n}_{co} \) vector also precesses about the vertical axis at a tune \( K \).
Effect of Spin Rotator on Spin Motion

We consider a spin perturbing kick about the \( \hat{e}_2 \) direction with a kick angle \( \chi \), which may arise from a solenoid, in an otherwise perfect planar circular accelerator. The one turn map (OTM) can be used to find the spin-closed orbit:

\[
T = e^{-i \frac{1}{2} G \gamma (2 \pi - \theta)} \sigma_3 e^{-i \frac{1}{2} \sigma_2} e^{-i \frac{1}{2} G \gamma \theta} \sigma_3 = e^{-i \pi \nu_s n_{\text{co}} \cdot \vec{\sigma}}
\]  

(37)

Here \( \theta \) is the orbital angle between the measurement location and the spin kick location, \( \nu_s \) is the spin tune and \( \hat{n}_{\text{co}} \) is the spin closed orbit given by

\[
\hat{n}_{\text{co}} = n_x \hat{e}_3 + n_y \hat{e}_1 + n_z \hat{e}_2 = \cos \alpha_3 \hat{e}_3 + \cos \alpha_1 \hat{e}_1 + \cos \alpha_2 \hat{e}_2,
\]

(38)

where \( \cos \alpha_1, \cos \alpha_2, \cos \alpha_3 \) are directional cosines. A spin vector lying along the \( \hat{n}_{\text{co}} \) direction is invariant under the transformation of the spin transfer matrix. Similarly, any spin vector not lying along \( \hat{n}_{\text{co}} \) will precess about \( \hat{n}_{\text{co}} \) at a rate of \( \nu_s \) precessing turns per revolution around the accelerator. The projection of the spin vector onto the spin closed orbit \( \hat{n}_{\text{co}} \) is invariant under the transformation of the OTM, i.e.

\[
\hat{n}_{\text{co}} \cdot \vec{S} = \langle \Psi | [\hat{n}_{\text{co}} \cdot \vec{\sigma}] | \Psi \rangle = \langle \Psi | T^\dagger [\hat{n}_{\text{co}} \cdot \vec{\sigma}] T | \Psi \rangle = \hat{n}_{\text{co}} \cdot \langle \Psi | T^\dagger \vec{\sigma} T | \Psi \rangle.
\]

(39)

Identifying matrix elements of OTM, we obtain

\[
\cos \pi \nu_s = \cos [\pi G \gamma] \cos \chi = \frac{1}{2}, \quad \cos \alpha_1 = \frac{1}{\sin \pi \nu_s} \sin [G \gamma (\pi - \theta)] \sin \chi = \frac{1}{2},
\]

\[
\cos \alpha_2 = \frac{1}{\sin \pi \nu_s} \cos [G \gamma (\pi - \theta)] \sin \chi = \frac{1}{2}, \quad \cos \alpha_3 = \frac{1}{\sin \pi \nu_s} \sin [\pi G \gamma] \cos \chi = \frac{1}{2}
\]

(40)

The resonance strength arising from the spin rotator is \( \epsilon_K = \chi/2\pi \) for all \( K = \text{integer} \), i.e., a spin rotator generates imperfection resonances at all integer harmonics with equal resonance strength. When a spin perturbing kick is included, the spin tune \( \nu_s \) is not equal to \( G \gamma \). At \( G \gamma = m = \text{integer} \), the spin tune \( \nu_s = m \pm \chi/2\pi = m \pm \epsilon \), i.e., the spin tune is shifted away from an integer by \( \pm |\epsilon| \).

At the symmetry point \( \theta = \pi \) from the spin rotator, the spin closed orbit \( \hat{n}_{\text{co}} \) is located on the \((\hat{e}_2, \hat{e}_3)\) plane. The tilt angle from the vertical axis depends on \( G \gamma \) value and the spin rotation angle \( \chi \). In particular, at an integer \( G \gamma \) value, the spin closed orbit lies on the axis parallel to the spin rotator, i.e., \( \cos \alpha_2 = \pm 1 \). If the spin vector is lying along the stable spin direction, where the magnetic interaction is zero with \( \vec{S} \times \vec{B} = 0 \) at the perturbing kick location, the spin vector will not be perturbed.

Any spin vector which is not lying along \( \hat{n}_{\text{co}} \) will precess about \( \hat{n}_{\text{co}} \) at a spin tune \( \nu_s \). For a beam bunch, depolarization may occur due to spin decoherence, i.e., when the strengths and the tunes of precessing kicks are different for different particles in the bunch, the ensemble average of the spin states is the projection of spin states onto the stable spin closed orbit \( \hat{n}_{\text{co}} \). When \( |G \gamma - m| \leq \chi/2\pi \), the spin closed orbit vector lies near the horizontal plane.

At \( \chi = \pi \), the spin tune and the spin closed orbit become energy independent. Furthermore, the spin closed orbit at the symmetry point \( \theta = \pi \) lies along the axis of the spin rotator. Derbenev and Kondratenko proposed to use the localized spin rotator for resonance correction because of the spin tune is \( \frac{1}{2} \) and independent of energy.
The left Figure below shows the measured vertical and radial polarizations vs the longitudinal field error at the IUCF Cooler Ring. The theoretical prediction using Eq. (40) is shown as solid lines. The equivalent spin precessing angle is given by \( \chi = (1+G) \int \mathbf{B}dl/Bp \). The 120 MeV vertically polarized protons with 77\% polarization were injected into the Cooler Ring. The radial and the vertical components of the beam polarization were measured as a function of the integrated compensating solenoidal field of the electron cooling section. The polarimeter was located at \( \theta = 60^\circ \) bend angle from the compensating solenoids. Since the stable spin direction is given by Eq. (40), it is easy to see that the vertical and the radial polarizations are given by

\[
P_v = P_{\text{inj}} \cos^2 \alpha_3, \quad P_t = P_{\text{inj}} \cos \alpha_3 \cos \alpha_1.
\]  

(40)

The spin closed orbit and the spin tune in Eq. (40) are continuous functions of the energy \( G\gamma \). The Figure (right plot above) shows the spin tune and spin closed orbit vector at a location opposite to the 40° (upper plots) and 120° (lower plots) partial solenoidal snakes as a function of \( G\gamma \). Note that \( \cos \alpha_1 = 0 \) for the solenoidal snake. The spin closed orbit vector oscillates around the snake axis on a plane defined by \( \hat{\mathbf{e}}_3 \) and \( \hat{\mathbf{e}}_2 \) (for a solenoidal snake). When the partial snake is turned on/off adiabatically, e.g., 100 revolutions with an acceleration rate of \( \alpha = 5 \times 10^{-5} \) at a \( G\gamma \) value not too close to an integer, the spin will adiabatically follow the spin closed orbit. We also note that the spin tune does not jump from \( n - \chi/2\pi \) to \( n + \chi/2\pi \) at the integer \( G\gamma \).
Spin Dynamics with Snakes

A solution, proposed by Derbenev and Kondratenko, to overcome spin resonances during the polarized beam acceleration is to use local spin rotators, or snakes. The local spin rotators that precess the particle spin by $180^\circ$ about an axis on the horizontal plane. In the presence of snakes, the spin tune may become $\frac{1}{2}$. This particular spin tune can avoid all of the spin resonance conditions. Thus the polarization can be maintained. Since the spin tune is independent of the energy of the particle, the spin chromaticity becomes zero. This would greatly help the polarized beam acceleration through spin resonances. The spinor wave function at a snake is transformed locally according to

$$\Psi(\theta^+) = e^{-i\frac{\pi}{2}n_s\hat{\sigma}}\Psi(\theta^-) = -i\hat{n}_s \cdot \hat{\sigma}\Psi(\theta^-). \quad (41)$$

Accelerators with one snake

Without loss of generality, we choose snake axis $\hat{n}_s = \cos \phi_s \hat{e}_1 + \sin \phi_s \hat{e}_2$. The one turn map (OTM) of the spin transfer matrix at the symmetry point ($\pi$ orbital angle from the snake) is given by

$$\tau(\theta_0 + 2\pi, \theta_0) = t(\theta_0 + 2\pi, \theta_0 + \pi) e^{-i\frac{\pi}{2}n_s\hat{\sigma}t(\theta_0 + \pi, \theta_0)}, \quad (42)$$

where the parameter $\theta_0$ is the initial orbital angle, and $t(\theta, \theta)$ is the spin transfer matrix (STM) from an initial orbital angle $\theta$ to a final orbital angle $\theta$. For an isolated spin resonance, the STM is given by $\tau$. The stable spin direction at the opposite point of a single snake is along the snake axis. To visualize the one turn spin transfer matrix more clearly, we make a unitary transformation on the spinor wave function by, $\Psi = A \tilde{\Psi}$, or $\tilde{\Psi} = A^\dagger \Psi$, with

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\frac{1}{2}\phi_s} & \frac{1}{\sqrt{2}} e^{-i\frac{3}{2}\phi_s} \\ -\frac{1}{\sqrt{2}} e^{i\frac{1}{2}\phi_s} & \frac{1}{\sqrt{2}} e^{i\frac{3}{2}\phi_s} \end{pmatrix} \equiv A^\dagger \begin{pmatrix} e^{-i\frac{\pi}{2}G\gamma\sigma_2} & e^{-i\frac{\pi}{2}n_s\hat{\sigma}} \\ e^{i\frac{\pi}{2}G\gamma\sigma_2} & e^{i\frac{\pi}{2}n_s\hat{\sigma}} \end{pmatrix} A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (43)$$

In this reference frame, the spin vector lying along the spin closed orbit is described by the spinor $\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Including an isolated spin resonance, the matrix elements of the OTM in the new reference frame $\tau = A^\dagger \tilde{\tau}$ ($\theta_0 + 2\pi, \theta_0$)A are given by

$$\tau_{11} = i + 2abc \cos \Phi \cos (c - K\pi) - 2i\delta \cos^2 \Phi, \quad \tau_{12} = 2iab \sin (c - K\pi) \cos \Phi + b^2 \sin 2\Phi, \quad \tau_{21} = -\tau^*_{12}, \quad \tau_{22} = \tau^*_{11}. \quad (44)$$

$$b = \frac{|\epsilon|}{\lambda} \sin \left(\frac{\lambda \pi}{2}\right) = (1 - a^2)^{1/2}, \quad c = \arctan \left[\frac{\delta}{\lambda} \tan \left(\frac{\lambda \pi}{2}\right)\right], \quad d = \arg (\epsilon^*) \quad \Phi = d + K\pi + K\theta_0 - \phi_s, \quad \lambda = \sqrt{\delta^2 + |\epsilon|^2} \quad \delta = K - G\gamma \quad (45)$$

In a perfect accelerator without spin depolarization resonances, we have $b = 0$ and the OTM is diagonal with $\tau_{11} = i, \tau_{12} = 0$, where the corresponding spin tune is $\frac{1}{2}$. For an accelerator with $b \neq 0$, the perturbed spin tune and the spin component along the spin closed orbit direction are

$$\cos \pi Q_s = 2abc \cos \Phi \cos (c - K\pi), \quad P = |\tau_{11}|^2 - |\tau_{12}|^2 = 1 - 2|\tau_{12}|^2. \quad (46)$$

The deviation of the spin tune from $1/2$ varies linearly with the resonance strength. The maximum spin tune deviation from $1/2$ is given by $|Q_s - \frac{1}{2}| \leq |\epsilon|$. 
The spin transfer matrix can be tracked along the accelerator with the spin tracking equation
\[ T(\theta_{n+1}) = \tau(\theta_{n+1}, \theta_n)T(\theta_n) \] with \( \theta_{n+1} = \theta_n + 2\pi \). Solving the spin tracking equation perturbatively, one obtains
\[
T_{12}(\theta_{n+1}) = (i)^{n+1}2eb\sin(c-K\pi)\cos(\Phi + n(K - \frac{1}{2})\pi)\zeta_{n+1}(K - \frac{1}{2}) + (i)^nb^2\sin(2\Phi + n(2K - \frac{1}{2})\pi)\zeta_{n+1}(2K - \frac{1}{2}) + \cdots,
\]
(47)
where \( \zeta_n(x) = \sin(n\pi x)/\sin(\pi x) \) is the enhancement function. The spin perturbation is coherently added up when the spin resonance \( K \) is located at either \( K = \frac{1}{2} \) = integer, or \( 2K - \frac{1}{2} = \) integer.

Expanding the spin tracking equation to higher order in resonance strength parameters, we find that the spin perturbing kicks are enhanced at the following “snake resonance” condition:
\[ \frac{1}{2} - \ell K = \text{integer}, \quad \ell = \text{integer}. \]
(48)

A snake resonance is called odd-order if \( \ell = \text{odd} \), and even order if \( \ell = \text{even} \).

The left plot above shows the projection of the final spin vector onto the spin closed orbit after passing through a spin resonance as a function of the resonance tune. The left plot shows the result of an isolated intrinsic resonance with \( q_{\text{int}} = 0.2 \). We observe that depolarization occurs when the spin resonance tune \( K \) satisfies Eq. (48). The right plot above shows the final polarization vs the spin resonance tune \( v_z \) when an imperfection resonance at \( \epsilon_{\text{imp}} = 0.03 \) is included. We observe that each snake resonance splits into two and the width of the snake resonance becomes larger.
A Model with Two Snakes and a Local Spin Kick

We consider a perfect circular accelerator with two snakes $-\sigma_1$ and $-\sigma_2$ separated by an orbital angle $\pi$. The OTM is given by

$$\begin{bmatrix} -i\sigma_2 \end{bmatrix} e^{-\frac{G\gamma}{2} \sigma_3} \begin{bmatrix} -i\sigma_1 \end{bmatrix} e^{-i\frac{G\gamma}{2} \sigma_3} = i\sigma_3.$$  \hspace{1cm} (49)

The trace of the spin transfer matrix is zero, thus the spin tune is $\frac{1}{2}$ and the stable spin closed orbit is vertical. Now we introduce a small constant local spin angular precessing kick $\chi$ about an axis $\hat{n}_k$ in the horizontal plane, the spin transfer matrix becomes

$$T_1 = e^{-\frac{i}{2} \hat{n}_k \cdot \hat{\sigma} i\sigma_3}.$$  \hspace{1cm} (50)

Because $\hat{n}_k$ is in the horizontal plane, the evolution of the spin transfer matrix at the $n$th revolution becomes

$$T^{(n)} = [T_1]^n = \begin{cases} [i\sigma_3]^n & \text{if } n = \text{even}, \\ T_1[i\sigma_3]^{n-1} & \text{if } n = \text{odd}. \end{cases}$$  \hspace{1cm} (51)

This means that the spin-perturbing kicks cancel each other every two revolutions. Thus the snake is very effective in correcting the imperfection resonances. Extending the model a step further by assuming that the kick strength is different in each revolution, the spin transfer matrix becomes

$$T^{(n)} = \prod_{m=1}^n T_m = e^{-i\frac{1}{2}[\sum_{m=1}^n (-1)^{n-m} m \hat{n}_k \cdot \hat{\sigma} [i\sigma_3]^n]}.$$  \hspace{1cm} (52)

The vertical spin vector at the $n$th turn becomes

$$S_3^{(n)} = 1 - 2\sin^2 \left[ \frac{1}{2} \sum_{m=1}^n (-1)^{n-m} m \chi_m \right].$$  \hspace{1cm} (53)

Now if these spin perturbation kicks were due to a betatron motion, they would be correlated with $\chi_m = \chi_0 \cos 2m\pi \nu_z$, where $\nu_z$ is the vertical betatron tune. When the vertical betatron tune is $\nu_z = \frac{1}{2} + \text{integer}$, each kick adds up coherently. The spin vector will precess around the $\hat{n}_k$ axis at a precessing tune of $\chi_0/2 \pi$, or it takes $2\pi/\chi_0$ revolutions to complete one turn about the $\hat{n}_k$ axis. The snake is ineffective in dealing with this type of spin perturbation called the first order snake resonance, where the betatron tune is equal to the spin tune. On the other hand, this spin perturbing kicker can be used to precess the spin direction and invert the polarization direction.

Basic Requirements of Snake Configurations

Let ($\phi_1$, $\phi_2$, \ldots, $\phi_{N_s}$) be the snake axes of $N_s$ snakes in an accelerator and let $\theta_{i,i+1}$ be the azimuthal orbit rotation angle between the $i$th, and $(i+1)$th snakes. The one turn spin transfer matrix for a perfect planar synchrotron is

$$e^{-i\frac{G\gamma}{2} [\theta_{0,1}-\theta] \sigma_3} e^{-i\frac{\pi}{2} \hat{n}_{N_s} \cdot \hat{\sigma}} \prod_{k=1}^{N_s-1} e^{-i\frac{G\gamma}{2} \theta_{k,k+1} \sigma_3} e^{-i\frac{\pi}{2} \hat{n}_k \cdot \hat{\sigma}} e^{-i\frac{G\gamma}{2} \theta \sigma_3} = e^{-i\pi \nu_z \hat{n} \cdot \hat{\sigma}},$$  \hspace{1cm} (54)
where the spin tune \( \nu_s \) and the spin closed orbit vector \( \hat{\mathbf{r}}_{co} \) can be obtained by identifying the matrix elements of Eq. (54). To ensure that the spin tune is energy independent, the distribution of snakes should satisfy the following condition

\[
\theta_{\text{odd}} = \theta_{\text{even}} = \pi, \quad \theta_{\text{odd}} = \sum_{k=\text{odd}}^{N_s} \theta_{k,k+1}, \quad \theta_{\text{even}} = \sum_{k=\text{even}}^{N_s} \theta_{k,k+1},
\]

where \( \theta_{\text{odd}} + \theta_{\text{even}} = 2\pi \) is the total orbital angle for a circular path. This condition precludes a configuration with an odd number of snakes in a synchrotron except in the case of 1 snake. If the odd (or even) orbital angle deviates from \( \pi \), the spin tune becomes \( \frac{1}{2} + G\gamma(1 - \theta_{\text{odd}}/\pi) \), i.e., the spin tune is shifted away from \( \frac{1}{2} \) by an amount \( \Delta \nu_s = G\gamma \Delta \theta/\pi \), where \( \Delta \theta = \pi - \theta_{\text{odd}} \). For high energy storage rings, \( G\gamma \) is a large number, e.g., \( G\gamma = 478 \) for RHIC at 250 GeV beam energy and \( G\gamma \approx 38000 \) at 20 TeV, accurate placement of snakes becomes a very important issue. In fact, the horizontal closed orbit error can also affect the spin tune in high energy accelerators. The spin tune is the trace of the one-turn transfer map, or \( \nu_s = 1/\pi \sum_{k=1}^{N_s} (-1)^k \phi_k \).

### The Perturbed Spin Tune

Without loss of generality, we consider two snakes with snake axis angles \( \phi_1 \) and \( \phi_2 \) located at an orbital angle of \( \pi \) from each other. The spin transfer matrix for passing through two snakes is

\[
\begin{align*}
\tau_{11}(\theta_0 + 2\pi, \theta_0) &= -e^{-i\nu_s}(1 - 2b^2e^{i\phi_1} \cos \Phi), \\
\tau_{12}(\theta_0 + 2\pi, \theta_0) &= -2i\nu_s e^{-i(c-K\pi+\phi_2)} \cos \Phi, \\
\tau_{21} &= -\tau_{12}^*, \quad \tau_{22} = \tau_{11}^*, \quad \nu_s = \phi_2 - \phi_1, \quad \Phi = K\theta_0 + K\pi + d - \phi_1
\end{align*}
\]

Note that the \( \tau_{12} \) for two or more snakes differs from that of one-snake in Eq. (44) by the absence of \( b^2 \) term. The accelerator will not have even order snake resonances. The perturbed spin tune \( Q_s \), defined as the trace of the OTM, is

\[
\cos \pi Q_s = -\cos \pi \nu_s + 2b^2 \cos \Phi \cos(\Phi - \pi \nu_s) = b^2 \sin(2\Phi).
\]

Because of the betatron phase, the perturbed spin tune for an intrinsic resonance \( Q_s \) oscillates around \( \frac{1}{2} \) up to a maximum and minimum given by

\[
Q_{s \text{max/min}} = \frac{1}{2} \pm \frac{1}{\pi} \arcsin |\sin \frac{\pi \epsilon}{N_s}|.
\]

If the spin resonance strength is \( |\epsilon| \approx mN_s/2, m = 1, 2, \ldots \), the perturbed spin tune \( Q_s \) will oscillate in a whole integer unit at two times the spin resonance tune during the acceleration across the intrinsic resonance. Since the spin tune crosses the spin resonance many times, the polarization may be lost. One can surmise that the critical spin resonance strength \( \epsilon_c \) is bounded by the condition that the perturbed spin tune does not cross the spin resonance tune \( K \) during the acceleration cycle, i.e.

\[
\langle \epsilon_c \rangle \leq \frac{\arcsin(|\cos \pi K|^{1/2})}{\pi} N_s.
\]

One might argue that depolarization occurs when the perturbed spin tune \( Q_s \) satisfies \( Q_s \pm \nu_s \) = integer to obtain the critical resonance strength. The critical resonance strength of Eq. (59) is only an upper limit, or optimistic estimate of the tolerable resonance strength.
Odd Order Snake Resonances

With two or more snakes, the accelerator will not have even order snake resonance. A plot below shows an example of beam polarization after passing through a resonance vs the betatron tune. We note that the depolarization occurs at $\frac{1}{2} - \ell v_z = 0$ with odd integer $\ell$ as predicted from the spin tracking Eq. (56).

![Fig. 7](image)

Even Order Snake Resonances

In the presence of overlapping intrinsic and imperfection resonances, the depolarization mechanism becomes complicated. However some basic rules persist. The plot below shows the polarization after passing through the resonance with an additional small imperfection resonance.

![Fig. 8](image)

The snake resonance condition becomes $\frac{1}{2} - \ell K = \text{integer}$, where $\ell = \text{integer}$. Here $\ell$ includes both odd the even integers. The presence of even integer $\ell$ means a drastic reduction of available tune space. Even more important is that the imperfection spin resonance produce splitting of each snake resonance. Figure 9 below shows the beam polarization after passing through an
intrinsic and an imperfection resonance. We note that each snake resonance split into two. The split is proportional to $\epsilon_{\text{imp}}^2$. What mechanism induces even order snake resonances?

A Model for Even Order Snake Resonances

To understand the essential mechanism of even order snake resonances in the presence of overlapping spin resonances, we consider the model of a localized spin kick added to the OTM of an intrinsic resonance. We assume that the imperfection resonance is caused by a small spin precessing kick $\chi$ about the $e1$ axis. The OTM of this overlapping resonances model is

$$\tilde{\tau} = e^{i\frac{\chi}{2} \tau}(\theta_0 + 2\pi, \theta_0),$$

where $\tau (\theta_0 + 2\pi, \theta_0)$ is given by Eq. (56). The imperfection resonance strength is $\epsilon_{\text{imp}} = \chi/2\pi$ at all integer harmonics. The matrix elements of the OTM are

$$\tilde{\tau}_{11} = -e^{-i\pi s_1} \left[ 1 - 2b^2 e^{i\phi} \cos \Phi \cos \frac{\lambda}{2} - 2abe^{i(\pi - K + \phi_2)} \cos \Phi \sin \frac{\lambda}{2} \right],$$

$$\tilde{\tau}_{12} = -2iabe^{-i(\pi - K + \phi_2)} \cos \Phi \cos \frac{\lambda}{2} - ie^{i\pi s_2} (1 - 2e^{\phi} \cos \Phi) \sin \frac{\lambda}{2},$$

$$\tilde{\tau}_{21} = -\tilde{\tau}_{12}^{*}, \quad \tilde{\tau}_{22} = \tilde{\tau}_{11}^{*}.$$  

Due to the imperfection resonance, the off-diagonal matrix elements now contain a term oscillating at two times the betatron frequency with amplitude proportional to $b^2 \sin \chi/2$. This spin perturbing term will produce even order snake resonances. The spin tune shift due to the imperfection resonance will produce splitting of the snake resonance. RHIC polarized proton collider, equipped with 2 snakes, has observed even and odd order snake resonances shown below.
Electron Polarization

Electrons differ from protons in two aspects. First of all, the electron g-factor is nearly equal to 2 and the anomalous g-factor is small with \( a = G = g/2 - 1 = 0.00115965 \). Secondly, electrons emit synchrotron radiation in dipoles and replenish their energy in rf cavities. The classical radiation power spectrum is continuous, extending up to the critical frequency \( \omega_c = (3/2) \gamma^3 c / \rho \), where \( c \) is the speed of light, \( \gamma \) is the relativistic factor of electrons, and \( \rho \) is the bending radius of dipoles. The radiation spectrum falls off exponentially beyond \( \omega_c \) as \( \exp[-\omega/\omega_c] \) with the total integrated radiation power given by

\[
P_{\text{classical}} = \frac{e C \gamma E^4}{2 \pi \rho^2}, \quad C_\gamma = \frac{4 \pi r_e}{3(m c^2)^3} = 8.85 \times 10^{-5} \frac{m}{(\text{GeV})^3},
\]

\[
P_{\text{av}} = \langle P_{\text{classical}} \rangle = \frac{1}{2 \pi R} \int P_{\text{classical}} ds = \frac{e C_\gamma E^4}{2 \pi R \rho} = 4.2 \times 10^3 \frac{E^4}{R \rho} \text{GeV/s}
\]

The synchrotron radiation is basically a quantum mechanical process, where the electromagnetic radiation is emitted in quanta of energy \( u = h \omega \).

The total number of photons \( N \) emitted per second and the moments of energy distribution are given respectively by

\[
N = \frac{15 \sqrt{3}}{8} \frac{P_{\text{av}}}{u_c}, \quad \langle u \rangle = \frac{8}{15 \sqrt{3}} u_c, \quad \langle u^2 \rangle = \frac{11}{27} u_c^2, \quad N' \langle u^2 / E_0^2 \rangle = \frac{55 \sqrt{3} u_c}{72} \frac{E^2}{P_{\text{av}}}
\]

When photons are emitted, the energy of the electron will decrease by the same discrete amount. Thus the corresponding instantaneous radiation power will be lessened. To the first order in \( \hbar \), the quantum correction to the radiation power is

\[
P_{\text{qm}} = P_{\text{classical}} \left( 1 - \frac{55}{8 \sqrt{3}} \frac{h \omega_c}{E} \right)
\]

where \( E \) is the electron energy. The quantum mechanical correction factor is of the order of \( 10^{-5} \) and cannot easily be measured. However, the quantum effect is evidently observable in the equilibrium phase space distribution due to radiation damping (a classical phenomena) and radiation excitation (a quantum mechanical effect). The equilibrium momentum width and emittance for an isomagnetic ring are

\[
\left( \frac{\sigma_p}{E} \right)^2 = \frac{55}{48 \sqrt{3}} \frac{h \omega_c}{J_E E} = C_q \frac{\gamma^2}{J_E \rho}, \quad \epsilon_x = C_q \frac{\gamma^2 J_{\text{dipole}}}{J_x \rho}
\]

where \( C_q = 3.84 \times 10^{-13} \) m and \( J_E \) is the damping partition number, \( H \) is the dispersion action and \( J_x \) is the horizontal damping partition number.

In addition to the discreteness of photon emission, an electron has an intrinsic spin quantum number. The angular momentum carried by the spin is \( \vec{S} = \frac{1}{2} \hbar \vec{\sigma} \), where \( \vec{\sigma} \) are Pauli matrices. Including the spin correction, the radiation power is

\[
P_{\text{qme}} = P_{\text{classical}} \left[ 1 - \left( \frac{55}{8 \sqrt{3}} + \frac{\hbar \omega_c}{E} \right) \right]
\]
where \( \hat{n} \) be the spin orientation in the electron’s rest frame before photon emission, and \( \hat{z} \) is the direction of the magnetic field that bends electrons. Averaging overall spin orientations \( \hat{n} \), Eq. (65) reduces to Eq. (63). The spin correction is also very small. The important effect is the disparity in the transition rate. The instantaneous spin flip transition rate is

\[
W = \frac{1}{2} W_0 \left[ 1 - \frac{2}{9} (\hat{n} \cdot \hat{s})^2 + \frac{8}{5\sqrt{3}} (\hat{n} \cdot \hat{z}) \right], \quad W_0 = \frac{5\sqrt{3}}{8} \frac{v_c}{m|\rho|^3} = \frac{5\sqrt{3}}{8} \frac{u_c}{E^2} \rho_{av}
\]

(66)

where \( \hat{s} \) is the direction of particle motion, \( \hat{z} \) is the quantization axis, and \( W_0 \) is the average spin flip transition rate. The instantaneous spin flip transition power is the product of the transition rate and the energy carried by each photon, i.e., \( P_{\text{transition}} = W_0 \hbar \). The spin flip transition power spectrum is much smaller than the classical power by a factor of \( (\hbar c/E)^2 \sim 10^{-11} \). Even if the power of radiation is small, the disparity in the transition rate may eventually populate more electrons (positrons) with spin antiparallel (parallel) to the magnetic field because electrons (positrons) with spin parallel (anti-parallel) to the magnetic field have a larger spin flip transition rate.

Let \( ^z \) be the vertical direction along the guide field. Let the quantum states of electrons be either parallel (\( \uparrow \)) or anti-parallel (\( \downarrow \)) to the guide field \( ^z \). The spin flip transition rates become

\[
W_- = \frac{1}{2} W_0 (1 + \frac{8}{5\sqrt{3}}), \quad W_+ = \frac{1}{2} W_0 (1 - \frac{8}{5\sqrt{3}}),
\]

(67)

where \( W_- \) is the transition rate from the parallel to the anti-parallel state while \( W_+ \) is the transition rate from the anti-parallel to the parallel state. The transition probability is larger for electrons from the up to the down states. We assume that there are \( N_+ \) electrons with spin parallel and \( N_- \) electrons anti-parallel to the guide field with \( N_+ + N_- = N_0 \). The Boltzmann equation can then be expressed as

\[
\frac{dN_+}{dt} = W_+ N_- - W_- N_+, \quad \frac{dN_-}{dt} = -W_+ N_- + W_- N_+,
\]

(68)

\[
\langle S \rangle = \frac{N_+ - N_-}{N_+ + N_-} = \frac{W_+ - W_-}{W_+ + W_-} = -\frac{8}{5\sqrt{3}} = P_{\text{SST}}
\]

(69)

where \( P_{\text{SST}} = -8/5\sqrt{3} \) is the Sokolov–Ternov radiative polarization limit. The time constant in reaching equilibrium is

\[
\tau_{\text{ST}} = (W_+ + W_-)^{-1} = W_0^{-1} = \frac{4}{5\sqrt{3}} \frac{E}{\hbar \omega_c} \tau_0 \approx \frac{99}{E^3} \frac{\rho^2 R^3}{[\text{GeV}]} [\text{sec}],
\]

\[
\tau_0 = \frac{2E}{P_{\text{av}}} = \frac{4\pi \rho R}{c \gamma E^3}
\]

(70)

Here \( \tau_0 \) is the damping time factor for the betatron and synchrotron motions. The polarization time is of the order \( E/\hbar \omega_c \) longer than the phase space damping time. Typically, the polarization time is of the order of a few minutes to hours. Table below lists the polarization time of some electron storage rings.
The Spin Equation of Motion

Including the radiative spin transition term in the Thomas–BMT equation, we obtain

\[
\frac{d\vec{S}}{dt} = -\frac{e}{\gamma m} \vec{S} \times \left( (1 + a\gamma) \vec{B}_\perp + (1 + a) \vec{B}_\parallel \right) - \frac{1}{\tau_{ST}} \left( \vec{S} - \frac{2}{9} \vec{S} \cdot \hat{z} \hat{z} + \frac{8}{5\sqrt{3}} \hat{z}_* \right) \tag{71}
\]

We transform the time coordinate to the orbiting angle coordinate with \(d\theta/dt = \omega_0 = \beta c/\rho\). Using the relation \(d\vec{x}/d\theta = \hat{x}\), \(d\vec{y}/d\theta = -\hat{y}\), and defining \(\hat{e}_1\), \(\hat{e}_2\), and \(\hat{e}_3\) as time independent unit vectors that coincide with \(\hat{x}\), \(\hat{y}\), and \(\hat{z}\) at all locations in a storage ring, i.e. \(\vec{S} = S_1 \hat{e}_1 + S_2 \hat{e}_2 + S_3 \hat{e}_3\), Eq. (71) becomes

\[
\frac{dS_1}{d\theta} = -\alpha S_2 - F_2 S_3 - \frac{1}{\tau_{\theta}} S_1,
\]
\[
\frac{dS_2}{d\theta} = \alpha S_1 + F_1 S_3 - \frac{7}{9\tau_{\theta}} S_2,
\]
\[
\frac{dS_3}{d\theta} = F_2 S_1 - F_1 S_2 - \frac{1}{\tau_{\theta}} \left( S_3 + \frac{8}{5\sqrt{3}} \right).
\]

(72)

Here \(\tau_{\theta} = \tau_{ST} d\theta/dt = \beta c \tau_{ST}/\rho\) is the polarization time expressed in the orbiting angle, and \(F_1\) and \(F_2\) are functions of particle coordinates given by

\[
F_1 = -(1 + \alpha\gamma) \rho \dot{z}'' - \rho \dot{z}(1 + a)(\dot{\rho})',
\]
\[
F_2 = (1 + \alpha\gamma) \dot{z}' - \rho(1 + a)(\ddot{\rho})/\rho.
\]

(73)

which depend on the lattice design and particle coordinates. In a simplified ideal accelerator with vertical guide field only, the spin equation of motion becomes
\[ \frac{dS_1}{d\theta} = -\alpha S_2 - \frac{1}{\tau_0} S_1, \quad \frac{dS_2}{d\theta} = \alpha S_1 - \frac{7}{9\tau_0} S_2, \quad \frac{dS_3}{d\theta} = -\frac{1}{\tau_0} (S_3 + \frac{8}{5\sqrt{3}}). \] (74)

The vertical polarization vector \( S_3 \) will damp to the Sokolov–Ternov limit with a damping time \( \tau_0 \).

\[ S_3 = -\frac{8}{5\sqrt{3}} \left[ 1 - \left( 1 + \frac{5\sqrt{3}}{8} S_3^0 e^{-\theta/\tau_0} \right) \right]. \] (75)

The vertical polarization will slowly reach its equilibrium Sokolov–Ternov value of \(-92.4\%\). At the same time, the radial and longitudinal components of the polarization vector \( S_1 \) and \( S_2 \) will damp to zero. To simplify our discussion, we replace the different damping times of \( S_1 \) and \( S_2 \) by a uniform damping time \( \tau_0 \). The spin equation of motion becomes

\[ \frac{dS_{\pm}}{d\theta} = \left( \pm i\alpha - \frac{1}{\tau_0} \right) S_{\pm} \pm iF_{\pm} S_3, \]
\[ \frac{dS_3}{d\theta} = \frac{i}{2} \left[ (S_+ F_- - S_- F_+) - \frac{1}{\tau_0} \left( S_3 + \frac{8}{5\sqrt{3}} \right) \right], \] (76)

where \( S_{\pm} = S_1 \pm iS_2, \ F_{\pm} = F_1 \pm iF_2 \). We consider a single resonance with \( F_{\pm} = |e|e^{\pm i(K0+d)} \), and transform the equation of motion to the resonance rotating frame with \( S_{\pm} = e^{\pm i(K0+d)} \tilde{S}_{\pm}, \ S_3 = \tilde{S}_3 \). The spin equation of motion becomes

\[ \frac{d\tilde{S}_{\pm}}{d\theta} = \left( \pm i(\alpha - K) - \frac{1}{\tau_0} \right) \tilde{S}_{\pm} \pm i|e|\tilde{S}_3, \]
\[ \frac{d\tilde{S}_3}{d\theta} = \frac{i}{2} [\tilde{S}_+ - \tilde{S}_-] - \frac{1}{\tau_0} \left( \tilde{S}_3 + \frac{8}{5\sqrt{3}} \right), \] (77)

The equilibrium spin vectors are

\[ S_3^{eq} = -\frac{8}{5\sqrt{3}} \left[ 1 + \frac{\delta^2}{\tau_0} \right]^2 \approx -\frac{8}{5\sqrt{3}} \left[ \frac{\delta^2}{\tau_0} \right], \]
\[ \tilde{S}_3^{eq} = \frac{\pm i|e|\tau_0}{1 \pm i\delta/\tau_0} S_3^{eq}, \quad \delta = K - \alpha \gamma \] (78)

The magnitude of the resulting polarization vector is

\[ \langle S \rangle = \langle S_3^2 + \frac{1}{2}(S_+ S_- + S_- S_+) \rangle^{1/2} = \frac{\delta}{\sqrt{\delta^2 + |e|^2}} P_{ST}. \] (79)

Polarized beam experiments have been successfully carried out in HERA at DESY. Using harmonic correction with eight vertical closed orbit bumps similar to those of PETRA, a polarization of about 65–70% was obtained. Figure 11 below shows the measured polarization vs the equilibrium polarization derived from the polarization time measurements. The longitudinal polarization at the interaction region is obtained by a system of mini-spin rotators designed by J. Buon and K. Steffen for the energy range of 29 GeV to 35 GeV.
The equilibrium polarization of electron storage rings was compiled in the following graph by Buon and Koutchouk, where they tried to fit the data by

\[ P_\infty = \frac{8}{5\sqrt{3}} \frac{1}{1 + (\alpha E)^2}, \]

(80)

Figure 12 shows very pessimistic electron beam polarization. The question is how to achieve reasonable beam polarization for electron beam at the Higgs factory. The electron polarization can be valuable for energy calibration for electron beam colliders.