Goals and scope

- **Goals:**
  - Gain deeper understanding of single particle dynamics in storage rings.
  - Understand how accelerator modeling codes work.
  - Learn to develop tracking codes to solve realistic problems.

- **Scope:**
  - Assumptions
    - Motion of relativistic particles
    - Reference beam energy is fixed (no acceleration)
    - Large rings
  - Collective effects are not covered.

Outline

- Coordinate system and Hamiltonian
- Magnetic fields in accelerator elements
- Beam motion in accelerator elements
  - Linear model
  - Symplectic integration
  - Fringe field effects
  - Misalignments

Coordinate system for beam motion

Natural coordinate system
\[ \mathbf{r}(\tau) = \mathbf{r}_0(\tau) + x \mathbf{i}(\tau) + y \mathbf{j}(\tau) \]

The Hamiltonian
\[ H = e\Phi + c \sqrt{m^2c^2 + (\mathbf{P} - e\mathbf{A})^2} \]
with canonical momentum \( \mathbf{P} = \mathbf{p} + e\mathbf{A} \) and \( \mathbf{p} \) the mechanical momentum \( \mathbf{A} \) the vector potential.
Lecture Note Part 1

Hamiltonian

After a series of canonical transformations, new Hamiltonian becomes

$$\hat{H} = -\left(1 + \frac{\chi}{\rho}\right) \sqrt{(1 + \delta)^2 - (p_x - a_x)^2 - (p_y - a_y)^2} - \alpha$$

with canonical coordinates \((x, p_x, y, p_y, z, \delta)\) and free variable \(s\), and

$$p_x = \frac{\partial x}{\partial s}, p_y = \frac{\partial y}{\partial s}, a_x = \frac{\partial x}{\partial \delta}, a_y = \frac{\partial y}{\partial \delta},$$

$$a_x = \frac{e A_x}{\rho}, a_y = \frac{e A_y}{\rho},$$

$$a_s = \left(1 + \frac{\chi}{\rho}\right) \frac{e A_s}{\rho},$$

$$z = -\beta c t, \delta = \frac{\omega}{\rho} t,$$

\(P_0\) is the momentum of the reference particle.

Expanding \(\hat{H}\) (small angle approximation, valid for large rings)

$$\hat{H} \approx -\left(1 + \frac{\chi}{\rho}\right) \left(1 + \delta - \frac{1}{2(1 + \delta)} (p_x^2 + p_y^2) + \frac{1}{1 + \delta} (p_x a_x + p_y a_y) \right) - \alpha$$

Knowing the magnetic fields as given by the vector potential, we may solve for the motion of charged particles.

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Vector potential for straight elements

For magnets with straight geometry (curvature of orbit \(h = \frac{1}{\rho} = 0\)), the vector potential is

$$A_x = -B \rho R \sum_{n=1}^{\infty} \frac{ia_n + b_n}{n!} (x + iy)^n$$

where \(B \rho = p/e\) is magnetic rigidity (here as normalization constant), \(a_n\) and \(b_n\) are skew and normal multipole components, respectively. On the middle plane \((y = 0)\):

$$B_x(x, 0) = \frac{\partial A_x}{\partial y} = B \rho \sum_{n=1}^{\infty} a_n x^{n-1}$$

$$B_y(x, 0) = -\frac{\partial A_x}{\partial x} = B \rho \sum_{n=1}^{\infty} b_n x^{n-1}$$

For quadrupole \(n = 2\),

$$a_s = -\frac{1}{2} b_2 (x^2 - y^2)$$

sextupole \(n = 3\),

$$a_s = -\frac{1}{6} b_3 (x^3 - 3x^2y)$$

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Solving beam motion with Hamiltonian

- Approach 1: particle tracking – numerically calculate the canonical coordinates of particles through every accelerator element, for many turns if necessary.
  - To the linear order, tracking can be done with transfer matrix.
  - For nonlinear elements, symplectic integration is needed.
- Approach 2: transfer maps – building Lie maps or Taylor maps for individual accelerator elements and sections of accelerator (or a full ring).
  - Lie map for individual element is derived from the Hamiltonian.
  - Lie map for accelerator section can be obtained with Lie operator concatenation.
  - Taylor maps can be obtained from the Lie maps or numerically with TPSA.

We will discuss the particle tracking approach.

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Tracking through drift space

The equations of motion
\[
\frac{dx}{ds} = \frac{\partial H}{\partial p_x}, \quad \frac{dp_x}{ds} = -\frac{\partial H}{\partial x} = 0
\]
\[
\frac{dy}{ds} = \frac{\partial H}{\partial p_y}, \quad \frac{dp_y}{ds} = -\frac{\partial H}{\partial y} = 0
\]
\[
\frac{dz}{ds} = \frac{\partial H}{\partial z} = 0
\]

The coordinates at the entrance (subscript 0) and exit (subscript 1) faces of a drift space with length \( L \) is related by
\[
x_1 = x_0 + \frac{p_{x1}}{1+\delta} L, \quad p_{x1} = p_{x0},
\]
\[
y_1 = y_0 + \frac{p_{y1}}{1+\delta} L, \quad p_{y1} = p_{y0},
\]
\[
z_1 = z_0 - \frac{p_{x1}^2 + p_{y1}^2}{2(1+\delta)^2} L, \quad \delta_1 = \delta_0.
\]

Tracking through quadrupole – cont’d

For a quadrupole magnet (length \( L \), gradient \( K = b_2 \)) the Hamiltonian is
\[
H = \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{K}{2} (x^2 - y^2)
\]
Since momentum \( \delta \) does not change in this magnet, we first change the canonical momentum coordinates to \( p_{x1} = x' \), \( p_{y1} = y' \), \( p_{z1} = \delta \) by scaling down the
Hamiltonian to
\[
H_1 = \frac{1}{1+\delta} \left( \frac{1}{2} (x'^2 + y'^2) + \frac{K}{2} (x^2 - y^2) \right)
\]
Particle coordinates \( X = (x, x', y, y', z, \delta) \) at entrance and exit faces are connected by \( X_1 = RX_0 \), with transfer matrix \( M \) given by
\[
R = \begin{pmatrix} M_x & 0_2 & 0_2 \\ 0_2 & M_y & 0_2 \\ 0_2 & 0_2 & M_z \end{pmatrix}
\]
With \( M_z = M_{quad}(\frac{K}{1+\delta}) \), \( M_y = M_{quad}(\frac{-K}{1+\delta}) \) and
\[
M_z = \begin{pmatrix} 1 & R_{56} \\ 0 & 1 \end{pmatrix}
\]

Tracking through dipole magnets – linear

Keeping linear optics terms, the Hamiltonian is
\[
H = \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{K}{2} (x^2 - y^2)
\]
Again change the momentum coordinates, the new Hamiltonian is
\[
H_1 = \frac{1}{2} (x'^2 + y'^2) + \frac{K}{2} (x^2 - y^2) - \frac{x^8}{\rho}
\]
Similarly the transfer matrix for \( X = (x, x', y, y', z, \delta) \) is
\[
R = \begin{pmatrix} M_x & 0_2 & E \\ 0_2 & M_y & 0_2 \\ 0_2 & 0_2 & M_z \end{pmatrix}
\]
With \( M_z = M_{quad}(\frac{K+1/\rho^2}{1+\delta}) \), \( M_y = M_{quad}(\frac{-K}{1+\delta}) \) and
\[
E = \begin{pmatrix} M_{z1} & M_{z2} \\ 0_2 & 0_2 \end{pmatrix}, \quad F = \begin{pmatrix} M_{x1} & M_{x2} \\ 0_2 & 0_2 \end{pmatrix}
\]
With (for \( K_x = \frac{K+1/\rho^2}{1+\delta} < 0 \))
\[
R_{16} = -R_{52} = (1 - \cosh(\sqrt{-K_x}L)/\rho K_x)
\]
\[
R_{26} = -R_{51} = \sinh(\sqrt{-K_x}L/\rho \sqrt{-K_x})
\]
Tracking through dipoles – linear (cont’d)

For \( K_x = \frac{k + 1/\rho^2}{1+\delta} > 0 \)
\[
R_{16} = -R_{22} = (1 - \cos \sqrt{K_x L})/\rho K_x ,
\]
\[
R_{26} = -R_{51} = \frac{\sin \sqrt{K_x L}}{\rho \sqrt{K_x}}.
\]

Element \( R_{56} \) can be derived with
\[
\frac{dz}{ds} = -\frac{1}{2} \left( 1 + K \right) x^2 - K y^2 - \frac{x}{\rho}
\]
and \( x(s), y(s) \) as given properly with the transfer matrix (with \( L \to s \)).

The canonical momentum coordinates are restored after applying the transfer matrix.

The above tracking solutions for quadrupole and dipole elements treat momentum deviation properly such that linear optics of off-momentum particles is accurate.

### Symplectic condition

A transformation of canonical coordinates \( X_0 \to X \) (the conjugate coordinates in \( X \) are ordered \( X = (x, p_x, ...) \) ) is symplectic if the Jacobian matrix
\[
R_{\alpha\beta} = \frac{\partial X_\beta}{\partial (X_0)_{\alpha}}
\]
satisfies \( R^T S R = S \), with
\[
S = \begin{pmatrix} S_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_2 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
A symplectic transformation preserves phase space volume. A Hamiltonian system is always symplectic.

### Symplectic integration

A drift-kick-drift approximation for a multipole is symplectic.

An exact solution for a Hamiltonian is always symplectic. If the original Hamiltonian can be split into two solvable parts, one part can be lumped as a “thin element”. The resulting approximate Hamiltonian is solvable (the solution is strictly symplectic, but is not exact for the original Hamiltonian).

When a single thin element is put in the middle of the original element, the approximation is accurate to the second order of the element length i.e., the accumulated error is of order \( O(L^2) \). This is a step for a second order symplectic integrator. In Lie map language
\[
e^{-LH_1} = e^{-\frac{1}{2}LH_1} e^{-LH_2} e^{-\frac{1}{2}LH_1} + O(L^3)
\]

Nonlinear elements: sextupoles and octupoles

The Hamiltonian for a sextupole is
\[
H = \frac{p_x^2 + p_y^2}{2} + \frac{1}{3} (x^3 - 3xy^2)
\]
and for an octupole
\[
H = \frac{p_x^2 + p_y^2}{2} + \frac{1}{4} (x^4 - 6x^2y^2 + y^4)
\]
with \( S = \frac{b_2}{2} \) and \( \lambda = \frac{b_4}{6} \).

The equations of motion in a sextupole is
\[
x'' = \frac{y}{1+\delta}(x^2 - y^2), \quad y'' = -\frac{2y}{1+\delta}xy.
\]

There is no closed-form solution for these equations, and in general, for beam motion in other nonlinear multipole fields.

The usual numerical solver for differential equations, such as the Runge-Kutta method, does not preserve symplecticity. To study long-term stability of a nonlinear Hamiltonian system, we need symplectic integrator.
Higher order (more accurate) symplectic integrators can be constructed based on the lower order ones. The next order symmetric integrator is fourth order. The coefficients are:

\[ r_1 = 0.6756035959798286386, \quad r_2 = -0.1756035959798286639, \quad s_2 = 1.351207191959657328, \quad s_1 = -1.702414383919314656. \]

Note \( s_2 \) and \( s_1 \) are negative!

Symplectic integration for straight geometry multipoles is simple, just splitting the Hamiltonian into drifts and multipole kicks.

Cont’d

An issue with this approach: the origin of local reference system at \( s = s_0 \) does not transfer to the origin at \( s = s_1 \).

The solution may be to subtract this offset error.

(2) The bend-kick split (E. Forest)
\[
H_1 = -(1 + x) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} + b_1 \left( x + \frac{1}{2} \delta \right), \quad \text{a pure sector dipole}
\]
\[
H_2 = V(x, y), \quad \text{multipole kick}
\]

Solution to the sector dipole Hamiltonian:
\[
x(s) = \frac{1}{b_1} \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} - \rho \frac{dp_x(s)}{ds} - \rho,
\]
\[
p_x(s) = p_x \cos \frac{s}{\rho} + \left( 1 + \frac{(1 + \delta)^2 - p_x^2 - p_y^2}{b_1} \right) \sin \frac{s}{\rho},
\]
\[
y(s) = y + \frac{p_y}{b_1} + \frac{p_y}{b_2} \left( \sin \frac{p_x(s)}{\rho} \right),
\]
\[
z(s) = z + \frac{1}{b_1} \left( \frac{p_x(s)}{\sqrt{(1 + \delta)^2 - p_y^2}} - \frac{p_x(s)}{\sqrt{(1 + \delta)^2 - p_y^2}} \right),
\]
\[
p_y(s) = p_y, \quad \delta(s) = \delta.
\]

(3) Simplified approach: straight drift + bend/multipole kick
\[
H = \frac{p_x^2 + p_y^2}{2(1 + \delta)} + \frac{x^2}{2\rho} - \frac{x\delta}{\rho} + V(x, y)
\]

With
\[
H_1 = \frac{p_x^2 + p_y^2}{2(1 + \delta)} \quad \text{drift on straight reference system.}
\]
\[
H_2 = \frac{x^2}{2\rho} - \frac{x\delta}{\rho} + V(x, y), \quad \text{bend and multipole kick}
\]
Accuracy of symplectic integrator

Symplectic integrator may give different results than the linear transfer method because of integration errors. Slicing the thick element into more equal pieces (in other words, reducing integration step size) help improve accuracy.

Example: errors to kick $\Delta x'$ for a quadrupole for the fourth order integrator.

![Graph showing $\Delta x'$ vs $x_0$ for different number of slices (Nslice=1, 4, 10).](image)

- Quad: $L=0.5\text{ m}, K=1.2\text{ m}^{-2}$
- Nslice=1
- Nslice=4
- Nslice=10

Symplectic integrator may give different results than the linear transfer method because of integration errors. Slicing the thick element into more equal pieces (in other words, reducing integration step size) help improve accuracy.

Example: errors to kick $\Delta x'$ for a quadrupole for the fourth order integrator.

RF cavity

RF cavity is usually modeled as a thin element. The voltage on the RF gap may be given by

$$V(t) = V_0 \sin(\omega t + \phi_0)$$

The reference particle arrives at $t = 0$. The frequency must be a multiple of the revolution frequency $f_0$

$$\omega = 2\pi f = 2\pi\hbar f_0$$

Since $z = -\beta ct$, the change to momentum deviation is

$$\delta = \delta + \frac{eV_0}{E} \sin \left( \frac{\omega z}{\beta c} + \phi_0 \right)$$

Transfer matrix for an RF cavity for $(z, \delta)$ coordinates

$$M_z = \begin{pmatrix} 1 & \frac{e}{\beta c E} \frac{dV}{dt} \bigg|_{t=0} & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{eV_0 \cos \phi_0}{\beta c E} & 1 \end{pmatrix}$$

Orbit correctors

Orbit correctors can be modeled as thin elements (may be sandwiched with drift spaces). Unlike bending magnets, the reference orbit is not changed by the correctors.

The kick angles $\theta_x, \theta_y$ are specified for the reference particle.

$$p_{x1} = p_x + \theta_x$$
$$p_{y1} = p_y + \theta_y$$

(noting that $x'_1 = x' + \frac{\theta_x}{1+\delta}$, and $p_x = x'(1+\delta)$ and similarly for the $y$-plane.)

Orbit correctors cause changes to the closed orbit.

Misalignment

- Complete description of the misalignment of an element include 6 parameters
  - $(\Delta x, \Delta y, \Delta s)$ displacements along axis of local coordinate system.
  - $(\theta, \phi, \psi)$ rotation about the $s$, $x$ and $y$ axis, respectively.

- Misalignment can be modeled as coordinate transformations at the entrance and exit faces of the element.

- Simplified form of these transformations can be used.
  - At entrance, $x = x_0 - \Delta x$, $y = y_0 - \Delta y$, $p_x = p_{x0} - \psi(1+\delta)$, $p_y = p_{y0} - \phi(1+\delta)$ plus rotation about axis $s$ with angle $-\theta$.
  - At exit, rotation about axis $s$ with angle $\theta$, followed by $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, $p_x = p_{x0} + \psi(1+\delta)$, $p_y = p_{y0} + \phi(1+\delta)$. 

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Components of a tracking code

- A data structure that holds the accelerator elements
  - The elements have different parameters
    - Drift: length
    - Dipole: length, bending angle, entrance and exit angles, gradient, …
    - Quadrupole: length, gradient \( K \)
    - Sextupole: length, strength \( S \)
    - Corrector: length, kick angles \( \theta_x, \theta_y \)
    - RF cavity: length, peak voltage, frequency, phase lag, reference energy, …

- Data structure for canonical coordinates
- Functions that transfer phase space coordinates for each type of elements (“pass methods”)
  - An element may use different pass methods for different purposes (e.g., for dipoles one may use the linear pass method or the symplectic integrator)

References

3. F.C. Iselin, Lie transformations and transport equations for combined-function dipoles, Particle accelerators, 17, 143-155 (1985)
6. A. Chao, Lecture notes, chapter 9
Calculation of closed orbit

- Closed orbit is the fixed point of the one-turn map \(M\)

\[ M(X_c) = X_c \]

Here the map is evaluated with tracking.

If the one-turn map is linear, i.e.

\[ X_1 = M(X_0) = R(0)X_0 + \Delta \]

where \(R(0)\) is the transfer matrix on the reference orbit and \(\Delta\) is the accumulated coordinate displacement after one turn, then the closed orbit is

\[ X_c = (I - R(0))^{-1}\Delta \]

For the general nonlinear case, \(X_c\) can be found numerically with iteration.

Closed orbit differs from the design orbit when the guiding magnetic fields deviate from the ideal values (field errors or misalignment) or there is an energy error.
**Lecture Note Part 1**

**Transfer matrix**

- Each element may generate its own transfer matrix internally. The transfer matrix for a section or a full ring is the product of the transfer matrices of the elements.

\[
R_{0\rightarrow N} = R_{0\rightarrow 1} R_{1\rightarrow 2} \cdots R_{N-1\rightarrow N}
\]

- The transfer matrix for an element, a section or a full ring can also be obtained from tracking with numerical differential.

To find the transfer matrix around orbit \( X_0 = (x_0, p_{x0}, y_0, p_{y0}, z_0, \delta_0)^T \) (e.g., the reference orbit \( X_0 = (0,0,0,0,0,0)^T \), tracking particles \( X_1(-\Delta) = (X_0 - \Delta) \), here \( M \) represents the map from location 0 to 1, \( X_1(+\Delta) = M(X_0 + \Delta) \), which is evaluated with tracking. \( \Delta \) is \( (0, 0, 0, 0, 0, 0)^T \), with small quantity \( \epsilon \sim 10^{-6} \sim 10^{-8} \).

Then the column corresponding to coordinate 1 (\( x \)) is

\[
R_{1:6,1} = \frac{X_1(+\Delta) - X_1(-\Delta)}{2\epsilon}
\]

Similarly for the other columns.

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**Calculating TRANSPORT map with numerical differential**

For \( \beta = \gamma \), tracking particles (from location 0 to 1)

\[
X_1(0) = M(X_0),
X_1(-\Delta) = M(X_0 - \Delta),
X_1(+\Delta) = M(X_0 + \Delta),
\]

Example: \( \Delta_{\beta \gamma} = (\epsilon, 0, 0, 0, 0, 0)^T \), with small quantity \( \epsilon \sim 10^{-3} \sim 10^{-4} \).

Then the elements corresponding to coordinate 1 (\( x \))

\[
T_{i:11} = \frac{X_1(+\Delta) + X_1(-\Delta) - 2X_1(0)}{2\epsilon^2}
\]

For \( \beta \neq \gamma \), tracking particles (assuming \( \beta = 1, \gamma = 2 \))

\[
X_1(-, -) = M(X_0 + \Delta_{-, -}),
\Delta_{-, -} = (-\epsilon, -\epsilon, 0, 0, 0, 0)^T
X_1(-, +) = M(X_0 + \Delta_{-, +}),
\Delta_{-, +} = (-\epsilon, +\epsilon, 0, 0, 0, 0)^T
X_1(+, -) = M(X_0 + \Delta_{+, -}),
\Delta_{+, -} = (+\epsilon, -\epsilon, 0, 0, 0, 0)^T
X_1(+, +) = M(X_0 + \Delta_{+, +}),
\Delta_{+, +} = (+\epsilon, +\epsilon, 0, 0, 0, 0)^T
\]

Then

\[
T_{\beta \gamma} = T_{\gamma \beta} = \frac{1}{8 \epsilon^2} (X_1(+, +) + X_1(-, -) - X_1(+, -) - X_1(-, +))
\]

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**Transport map**

- TRANSPORT map is an extension of transfer matrix to second order (Taylor map).

\[
Y_\alpha = R_{\alpha \beta} X_\beta + T_{\alpha \beta \gamma} X_\gamma X_\gamma, \quad \alpha, \beta, \gamma = 1 \ldots 6,
\]

Summation of identical indices is assumed. Note by convention (making \( T \) symmetric), \( T_{\alpha \beta \gamma} = T_{\gamma \beta \alpha} \).

- For an element with a constant Hamiltonian (no \( s \)-dependence), the second order TRANSPORT map \( T \) can be derived from second and third order polynomials in the Hamiltonian.

- TRANSPORT map is generally not symplectic. But it can be useful for one-pass system.

- Combination of TRANSPORT maps (\( R = R^{0\rightarrow 2}, T = T^{0\rightarrow 2} \)).

\[
R_{11} = R_{12} R_{21} \beta, \quad T_{11} = T_{12} T_{21} \gamma.
\]

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**Linear optics for a storage ring (uncoupled)**

- Assuming there is no \( x - y \) coupling, the one-turn transfer matrix looks like

\[
R = \begin{pmatrix}
M_x & 0 & E \\
0 & M_y & 0 \\
F & 0 & M_z
\end{pmatrix}
\]

where \( M_x, M_y \) and \( M_z \) are \( 2 \times 2 \) matrices. The horizontal and vertical optics functions can be determined from \( M_x \) and \( M_y \), respectively.

Caution: When using tracking to compute linear optics, elements with time dependence or elements that change momentum deviation (like RF cavity) should be turned off.

- Courant-Snyder parameterization of the one turn matrix

\[
M = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
= \begin{pmatrix}
\cos \Phi + \alpha \sin \Phi & -\beta \sin \Phi \\
\beta \sin \Phi & \cos \Phi - \alpha \sin \Phi
\end{pmatrix}
\]

With \( \Phi = 2 \pi \nu \), and \( \nu \) the tune.

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Lecture Note Part 1

Tune and Twiss functions

- The tune can be determined from
  \[ \cos \Phi = \frac{M_{11} + M_{22}}{2}, \quad \sin \Phi = \text{sign}(M_{12}) \sqrt{|M_{12}M_{21} - \frac{1}{4}(M_{11} - M_{22})^2|} \]
  \[ v = \frac{1}{2\pi} \tan^{-1} \frac{2 \sin \Phi}{\cos \Phi}. \]

- Twiss functions at the location of the one-turn matrix
  \[ \beta_0 = \frac{M_{11}}{2 \sin 2\psi}, \quad \alpha_0 = \frac{(M_{11} - M_{22})}{2 \sin 2\psi} \]

- Propagation of Twiss functions to another location (knowing transfer matrix \( M \rightarrow M_{0 \rightarrow 1} \))
  \[ M = M_{0 \rightarrow 1} \]

- Phase advance
  \[ \Delta \Phi_{01} = \Phi_1 - \Phi_0 = \tan^{-1} \frac{M_{12}}{M_{11}\beta_0 - M_{12}\alpha_0} \]

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Linear optics (coupled)

- With \( x - y \) coupling, the \( 4 \times 4 \) matrix of the transverse phase space is
  \[ T_{4 \times 4} = \begin{pmatrix} M & m \\ n & N \end{pmatrix} \]
  where \( M, N, m, n \) are all \( 2 \times 2 \) matrices. The \( x, y \) motion are coupled when \( m, n \neq 0 \). In this case, the tunes derived from \( M \) and \( N \) are inaccurate.

- Decoupling: with a coordinate transformation \( Y = V^{-1}X \), such that the transfer matrix for \( Y \) is block diagonal (decoupled), i.e., for \( Y_1 = UY_0 \), with
  \[ U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \text{and} \quad T_{4 \times 4} = VUV^{-1} \]
  The transformation matrix can be computed.
  \[ V = \begin{pmatrix} y & 0 \\ -C^+ & y \end{pmatrix} \]

  where \( C^+ \) is the symplectic conjugate of \( C \) (required such that \( V \) is symplectic)

  \[ C^+ = \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \]

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Dispersion function and chromaticity

- By definition dispersion is the differential closed-orbit w.r.t. momentum deviation.
  \[ D = \frac{\partial X_c^{(4)}}{\partial \delta} \]
  The first order and second order dispersion functions are defined through
  \[ X_c(\delta) = X_c(0) + D_1 \delta + \frac{1}{2} D_2 \delta^2 + O(\delta^3) \]
  where \( D_1 = (D_{1x}, D'_{1x}, D_{1y}, D'_{1y})^T \) and similarly for \( D_2 \).

- Numerical calculation by off-momentum closed orbit (fixed momentum).
- Chromaticities are calculated with numerical differential of the tunes w.r.t. momentum deviation.

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Lecture Note Part 1

Dispersion function from transfer matrix

- The one-turn transfer matrix is

$$ R = \begin{pmatrix} M_{4 \times 4} & E_{4 \times 2} \\ F_{2 \times 4} & M_2 \end{pmatrix} $$

With

$$ E_{4 \times 2} = \begin{pmatrix} 0 & R_{16} \\ 0 & R_{26} \\ 0 & R_{36} \\ 0 & R_{46} \end{pmatrix} $$

The closed orbit with an infinitesimal momentum deviation $\delta$ would be $X_c(\delta) = (D, 0, 1)\delta$. From

$$ RX_c = X_c $$

One gets

$$ M_{4 \times 4}D + R_\delta = D $$

where $R_\delta = (R_{16}, R_{26}, R_{36}, R_{46})^T$. Therefore

$$ D = (I - M_{4 \times 4})^{-1}R_\delta $$

- Propagation of dispersion

$$ D(s) = M_{4 \times 4}(s_0 \rightarrow s)D(s_0) + R_\delta (s_0 \rightarrow s) $$

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011

Momentum compaction factor

- Momentum compaction factor (MCF) is defined

$$ a_c = \frac{1}{\rho} \frac{d\Delta \mathcal{C}(\delta)}{d\delta} = \frac{1}{\rho} \left[ H(s) \right]_{s=-\rho}^{s=+\rho} $$

Higher order momentum compaction factors

$$ \Delta \mathcal{C}(\delta) = a_1 \delta + \frac{1}{2} a_2 \delta^2 + \cdots $$

- MCF is related to one-turn transfer matrix element $R_{56}$

$$ -a_c C = R_{56} - \mathcal{H} \sin 2\pi x $$

where $\mathcal{H} = (D_z^2 + (\alpha D_x + \beta_x D_y)^2)/\beta_x$ is the local dispersion invariant. Usually $a_c = -R_{56}/C$ is a good approximation.

- Calculating MCF with tracking.

1. Calculate the closed orbit for $\delta = -\epsilon$ and $\delta = \epsilon$ (using $\epsilon \sim 10^{-8} - 10^{-6}$) with fixed momentum deviation, to get $X_c(-\epsilon)$ and $X_c(+\epsilon)$.

2. Set the $z$ coordinate in $X_c(\pm \epsilon)$ to 0. Then track the closed orbits for one turn $X_1(-\epsilon) = M(X_c(-\epsilon))$, $X_1(+\epsilon) = M(X_c(+\epsilon))$.

3. The MCF is $2^n$ order MCF can be computed similarly.

$$ a_c = \frac{(X_1(+\epsilon) - X_1(-\epsilon))/\delta}{2^n} $$

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Radiation integrals.

- Radiation integrals are related to important lattice parameters for electron storage rings.

- Definition

$$ I_1 = \int \frac{D}{\rho} ds, \quad I_2 = \int \frac{1}{\rho^2} ds, \quad I_3 = \int \frac{1}{|\rho^3|} ds,$$

$$ I_4 = \int \frac{1}{\rho^3} (1 + 2K) ds, \quad I_5 = \frac{1}{2\pi |\rho|} ds. $$

- For $I_1$, $I_4$ and $I_5$, integration needs to consider variation of dispersion and Twiss functions (for $I_5$) inside the magnets. This can be done analytically, with

$$ D(s) = D_0 \cos \sqrt{K_x}s + \frac{1}{\sqrt{K_x}} \sin \sqrt{K_x}s + \frac{1 - \cos \sqrt{K_x}s}{\rho K_x}, \quad \text{for} \quad K_x = K + \frac{1}{\rho^2} > 0 $$

$$ D(s) = D_0 \cosh \sqrt{-K_x}s + \frac{1}{\sqrt{-K_x}} \sinh \sqrt{-K_x}s + \frac{1 - \cos \sqrt{-K_x}s}{\rho K_x}, \quad \text{for} \quad K_x < 0 $$

and formulas for $\beta_x(s)$ and $\alpha_x(s)$.

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Lattice parameters

Note: $\alpha_x$ and $D_y'$ change at the dipole edges due to edge focusing!

$$ D_y' = D_y'_{0} + \frac{D_y}{\rho} \tan \psi, \quad \alpha_{x0} = \alpha_{x0} - \frac{\beta_{x0}}{\rho} \tan \psi $$

where $\psi_1$ is the entrance angle and subscript "-" indicate values before the edge ($D_y$ and $\beta_x$ are continuous at the edge).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy loss per turn</td>
<td>$U_0[\text{keV}] = 14.085 E^4 [\text{GeV}] I_z \tau^{-1}$</td>
</tr>
<tr>
<td>Momentum spread</td>
<td>$\sigma_z^2 = C_y^2 \left( \frac{I_3}{I_2} + \frac{I_4}{I_2} \right)$</td>
</tr>
<tr>
<td>Horizontal emittance</td>
<td>$\epsilon_x = C_y^2 \left( \frac{I_5}{I_2} - I_4 \right)$</td>
</tr>
<tr>
<td>Damping partition</td>
<td>$J_x = 1 - \frac{I_4}{I_2}, J_y = 1, J_z = 2 + \frac{I_4}{I_2}$</td>
</tr>
<tr>
<td>Damping time</td>
<td>$\tau_{x,y,z} = \frac{2E}{J_x y z U_0 T_0}$</td>
</tr>
</tbody>
</table>

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Precise tune determination from tracking data

• To characterize nonlinear dynamics of a storage ring, it is often useful to determine tunes from tracking or measured turn-by-turn data for beam with various energy error or oscillation amplitude.
• Simple FFT is not accurate (precision \( \sim 1/N \))
• NAFF (numerical analysis of fundamental frequency) is a method to accurately determine the tune from turn-by-turn data.

Assume a discrete sampling of a quasi-periodic signal

\[ s(t) = \cos(2\pi ft + \phi_0) \]

\( f \) may have small secular drifting) is

\[ s_n = s(nh) = \cos(2\pi vn + \phi_0), \quad n = 1, 2, ..., N \]

with sampling frequency \( f_0 = 1/h \) and \( v = f/f_0 \). How to determine \( f \) from the data?

NAFF: find the frequency \( \bar{f} = \bar{v} f_0 \) that maximizes the overlap between the power contents of \( s_n \) and \( \exp(-i2\pi fn) \), i.e.

\[
F(\bar{v}) = \left| \sum_{n=1}^{N} s_n e^{-i2\pi v_n W(n)} \right|
\]

Then \( \bar{f} \) is a precise approximation to \( f \).

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Characterization of nonlinear dynamics

• The motion of particles with large oscillation amplitude (nonlinear dynamics) is critical to storage ring performance.
• Common indicators of beam nonlinear dynamics
  – Nonlinear chromaticity (tune shifts with momentum deviation)
  – Tune shifts with amplitude
  – Tune diffusion
  – Frequency map
  – Resonance driving terms
• Direct measures of nonlinear dynamics performance
  – Dynamic aperture
  – Momentum aperture

Nonlinear chromaticity

• First order chromaticities are usually corrected to a small positive number (with sextupoles). The higher order chromaticities may dominate the \( \nu - \delta \) behavior.
• Tracking setup: correct linear chromaticity to zero, turn off cavity (fixed momentum deviation), small initial \( x, y \) offsets, track 256 to 1024 turns.
• Example: SPEAR3
Amplitude-dependent tune shifts

Nonlinear elements cause dependence of betatron tunes over oscillation amplitude.

\[ v_x(l_x, l_y) = v_{x0} + \frac{\partial v_x}{\partial (2l_x)} |_{0} 2l_x + \frac{\partial v_x}{\partial (2l_y)} |_{0} 2l_y + 0(l^2) \]

\[ v_y(l_x, l_y) = v_{y0} + \frac{\partial v_y}{\partial (2l_x)} |_{0} 2l_x + \frac{\partial v_y}{\partial (2l_y)} |_{0} 2l_y + 0(l^2) \]

where \( l_x = (x^2 + (\alpha x + \beta x^2)^2)/\beta \), and likewise for \( l_y \).

In general, \( \frac{\partial v_x}{\partial (2l_y)} = \frac{\partial v_y}{\partial (2l_x)} \).

The coefficients can be computed with formulas.

Numerical calculation of amplitude-dependent tune shifts.

Track particles with small initial \( y \) offset and a series of \( x \) offsets, obtain tunes for each initial \( x \) offset, fit \( v_x \) and \( v_y \) vs. \( l_x \) with linear model.

Repeat for \( l_y \) dependence.

An example: SPEAR3

The coefficients are

\[
\begin{pmatrix}
\frac{\partial v_x}{\partial (2l_x)} & \frac{\partial v_x}{\partial (2l_y)} \\
\frac{\partial v_y}{\partial (2l_x)} & \frac{\partial v_y}{\partial (2l_y)}
\end{pmatrix} = \begin{pmatrix} 1900 & 2130 \\ 2110 & 1760 \end{pmatrix}
\]

Frequency map

A more complete revelation nonlinear behavior is the frequency map analysis (FMA).

With initial transverse position coordinates \( x \) and \( y \) distributed on a grid (all other coordinates set to zero), find the corresponding betatron tunes and tune diffusion.

\( (x, y) \rightarrow (v_x, v_y) \)

Frequency map for the ALS ideal lattice.

Evaluation of dynamic aperture

Dynamic aperture (DA) is the (single particle dynamics) stability region around the reference orbit. Large DA is critical for storage ring performance (injection efficiency and lifetime).

DA is evaluated in simulation with tracking.

- Long term tracking is needed (one damping time or more for electron storage rings).
- 6D phase space tracking with radiation damping.
- To be realistic, model may include insertion device effects, systematic errors and random errors of magnets.
- Multiple random error seeds are used.
- Physical apertures may also be included in the model.
- Tracking may start at the injection point (septum magnet).
Initial particle coordinates
Initial particle distribution
\( p_x = p_y = 0, \delta = 0 \) for on-momentum case.
\( x, y \) distributed on grid or rays

Search for the first lost particle along each ray from the inner side outward.

Evaluation of momentum aperture

- Momentum aperture (MA): the largest initial momentum deviation error for a particle that will survive in the ring.
- MA is the most important factor that determines the Touschek lifetime of a beam.
  - Touschek lifetime, \( \tau \sim \sigma_x \sigma_y \delta^3 / \langle \xi \rangle \)
- MA may vary with location.
  - RF bucket height is an overall limit for the ring. Transverse motion and aperture may place a tighter limit.
  - Estimate with a linear model (with local equivalent aperture \( A_x \))
    - \( -A_x < (\delta^2 + \sqrt{\beta_x \gamma}) \delta < A_x \)
    - Higher order dispersion, coupling and nonlinear dynamics complicate the calculation.
- Particle tracking is the usual way of determining MA in simulation.

Tracking for momentum aperture

- Lattice model settings are the same as DA tracking
- MA is evaluated at locations all around the ring, or locations of a period.
- Particles are launched with a range of energy errors.

References

Lecture 3: Modeling Accelerators – Fringe fields and Insertion devices

X. Huang
USPAS, January 2015
Hampton, Virginia

Outline

• Fringe field effects
  – Dipole
  – Quadrupole

• Modeling of insertion devices
  • Radiation damping and quantum excitation
    – Damping, excitation, Ohmi envelope
  • Longitudinal tracking with acceleration

Magnetic field profile and the hard-edge model

• Most accelerator modeling codes use the hard-edge model for magnet – constant Hamiltonian.
• Real magnets always have a smooth transition at the edges – fringe fields.

- Hard edge model
  – Field or gradient is a constant equal to the average value inside the magnet body.
  – The effective length is the integrated strength divided by the field or gradient.

The hard-edge model is non-maxwellian!

Vector potential with fringe field

• With cylindrical symmetry, the vector potential of a normal multipole (n = 1 for dipole, etc.) with fringe field (with Coulomb gauge $V \cdot A = 0$) on a straight geometry is

\[
A_r = \frac{\cos n\theta}{2n!} \sum_{p=0}^{\infty} \frac{1}{n+p+1} G_{n,2p+1}(s)r^{2p+n+1},
\]
\[
A_\theta = \sin n\theta \sum_{p=0}^{\infty} \frac{1}{n+p+1} G_{n,2p+1}(s)r^{2p+n+1},
\]
\[
A_z = -\cos n\theta \sum_{p=0}^{\infty} G_{n,2p}(s)r^{2p+n}.
\]

With $x = r \cos \theta$ and $y = r \sin \theta$ and

\[
G_{n,2p}(s) = (-1)^p \frac{m!}{4^n (n+p)!} \frac{d^p G_m(s)}{ds^p}, \text{ and } G_{n,2p+1}(s) = \frac{d G_{n,2p}(s)}{ds}.
\]

Example: for dipole $n = 1$, assuming $G_{1,0} = B_0 \Theta(x)$, then

\[
A_x = \frac{x^2 - y^2}{4} B_0 \Theta'(x) + O(4), \quad A_y = 0
\]
\[
A_z = -B_0 \Theta(x) x + \frac{1}{2} B_0 \Theta''(x)(x^2 + y^2) + O(5).
\]

Correspondingly, the magnetic field is

\[
B_x = -B_0 \Theta''(x) \frac{x y}{4}, \quad B_y = B_0 \Theta(x) - \frac{1}{8} B_0 \Theta''(x)(x^2 + 3y^2)
\]
\[
B_z = B_0 \Theta'(x)y.
\]
Approximate field profile with Enge function

• The longitudinal profile function can be approximated well with the Enge function

\[ \theta(z) = \frac{1}{1 + e^{\sum a_i \theta(x_i)}} \]

Where \( s_0 \) is at the effective edge, \( G \) is the full gap.

Evaluation of fringe field effect for dipoles

• At the entrance edge:

- \( y \) and \( Y \) axes pointing out of page.

- Effective edge at \( s_0 = 0.7473286 \) m
- Gap \( G = 0.05 \) m.
- Coefficients:
  - \( \alpha = [0.0053, -0.0856, 0.4409, -0.9460, 2.0744, 0.2416] \)

Fringe field effect of quadrupoles

• The Hamiltonian for a quadrupole with fringe field

From \( H \approx -\frac{1}{2(1+\delta)}(p_x^2 + p_y^2) + \frac{1}{1+\delta}(p_x a_x + p_y a_y) \) - \( a_v \), ignore \( \delta \), apply vector potential for quadrupole, get

\[ H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}K(s)(x^2 - y^2) - \frac{1}{2}K'(s)(x^2 - y^2)(x p_x + y p_y) - \frac{1}{12}K''(s)(x^4 - y^4) + O(x^6) \]

• Linear effect (for the entrance edge)

\[ H = H_0 + \tilde{H}(s) \]

With the hard-edge model

\[ H_0 = \begin{cases} \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}K_0(x^2 - y^2) & s \geq s_0 \\ \frac{1}{2}(p_x^2 + p_y^2) & \frac{1}{2}K_0(x^2 - y^2) & s < s_0 \end{cases} \]

and the perturbation term

\[ \tilde{H}(s) = \frac{1}{2}\tilde{K}(s)(x^2 - y^2) \]

with

\[ \tilde{K}(s) = \begin{cases} K(s) - K_0 & s \geq s_0 \\ K(s) & s < s_0 \end{cases} \]
Lecture Note Part 1

- Linear effect as a thin element $F_0$

  Transfer matrix
  
  \[ M(1 \rightarrow 2) = M_D(1 \rightarrow 0)F_0M_Q(0 \rightarrow 2) \]

  The generating function for map* $F_0 = e^{i\phi}$ is calculated to be
  
  \[ f_z = \frac{I_1}{2}(xp_x - yp_y) \]

  With integral $I_1$ defined as
  
  \[ I_1 = \int_{s_0}^{s} K(s)(s - s_0) \, ds \]

  The transfer matrix is $\text{diag}(e^{i\phi}, e^{-i\phi}, e^{-i\phi}, e^{i\phi})$.

At the exit edge, integral $I_1$ has opposite sign. So the two edges tend to cancel. The cancellation is not complete for quadrupoles with finite length. The net effect includes a tune shift (always negative)

\[ \Delta \nu_x = -\frac{K^2_lB_\perp}{2\pi} |I_1|, \quad \Delta \nu_y = -\frac{K^2_lB_\parallel}{2\pi} |I_1| \]

*The Lie map $e^{i\phi}$: $\text{diag}(e^{i\phi}, e^{-i\phi}, e^{-i\phi}, e^{i\phi})$ is an operator, where $f_0: g = [f_0, g]$. Poisson bracket. The Lie map for a constant Hamiltonian is $e^{-i\Delta H}$.

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Example

Magnetic field

\[ B_z = B_0[y\Theta(z) - \frac{1}{12} \Theta''(z)(3x^2y + y^3)] \]
\[ B_y = B_0[x\Theta(z) - \frac{1}{12} \Theta''(z)(x^3 + 3x^2y)] \]
\[ B_x = \text{sgn}(z)B_0\Theta'(z)xy \]

SPEAR3 quadrupole fringe field

\[ f_z = -\frac{1}{12}K_0(x^3p_x + 3xy^2p_y - y^3p_y - 3x^3yp_x) \]

This cannot be symplectically integrated. However, the generating function for a skew quadrupole is integrable.

\[ f_{s,kew} = \frac{a_2}{6}(x^3p_y + y^3p_x) \]
\[ \exp(\alpha y^3p_x) \]

Similarly for the $x^3p_y$ term. These are two kicks!

Therefore, we can model the nonlinear effects of quadrupole fringe field with a symplectic map by rotating $\frac{\pi}{4}$, applying the skew quad fringe kick, and rotating back.

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Insertion device modeling

- Wiggler and undulator

Wiggler parameter $K = \frac{eB_0}{m_e} = 0.934B_0[T]\lambda_0[\text{cm}]$.

With $k = \frac{2\pi}{\lambda_0}$.

On the mid-plane, the magnetic field

\[ B_y = B_0 \cos kx \]

Trajectory

\[ x(z) = x_0 - A \cos kx \]

with amplitude $A = \frac{1}{k^2p} = \frac{K_2}{y} \frac{2\pi}{\lambda_0}$
Lecture Note Part 1

Magnetic field in an ID

- The field is periodic in the longitudinal direction.
  
  Suppose the vertical field is
  \[ B_y = B(x, y) \cos kz \]
  
  with symmetry \( B(x, y) = B(-x, y) \) and \( B(x, y) = B(x, -y) \).
  
  It can be shown that to satisfy \( \nabla \cdot B = 0 \) and \( \nabla \times B = 0 \), we need
  \[
  \frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} - k^2 B = 0
  \]

  The other field components
  \[ B_x = \frac{\partial}{\partial y} B(x, y) \cos k z, \]
  \[ B_z = (-k) \int B(x, y) dy \cos k z. \]
  
  Symmetry leads to \( B_x(x = 0) = 0, B_x(y = 0) = 0 \) and \( B_z(y = 0) = 0 \).

  The Halbach wiggler field model
  \[ B_y = B_0 \cosh k z \cosh k_y y \cos k z, \]
  \[ B_z = \frac{k_y}{k} B_0 \sinh k z \sinh k_y y \cos k z, \]
  \[ B_z = -\frac{k}{k_y} B_0 \cosh k z \sinh k_y y \cos k z, \]
  
  with \( k_x^2 + k_y^2 = k^2 \).

Effects of ideal ID on the beam

- The Hamiltonian for the Halbach model (to 4th order)
  \[
  H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + \frac{1}{4k^2p^2} \left( k_x^2 x^2 + k_y^2 y^2 \right) + \frac{1}{12k^4p^4} \left( k_x^4 + k_y^4 + 3k_x^2k_y^2y^2 - \sin k z \left( p_x (k_x^2 x^2 + k_y^2 y^2) - 2k_y p_y y \right) \right).
  \]

  Assume \( k_x = 0 \) (ideal planar wiggler with wide poles)
  \[
  H = \frac{1}{2} \left( p_x^2 + p_y^2 \right) + \frac{y^2}{4p^2} - k \sin k z \left( \frac{p_x}{2p^2} p_y y + \frac{k^2}{12p^4} y^4 + O(X^6) \right)
  \]

  For the ideal planar wiggler, the effect on beam is on the vertical plane.

- Linear effect
  
  Tune shift:
  \[
  \Delta v_y = \frac{\beta^* l}{8\pi \rho^2} + \frac{L^3}{96\pi \rho^2 \beta^*}
  \]
  
  where \( L \) is wiggler length, \( \beta^* \) is beta function at ID center (minimum)

- Nonlinear effects
  
  - Octupole-like effect causing amplitude-dependent tune shift, \( \approx \frac{\beta^*}{2k^2} \)

Effects of an imperfect wiggler

Kick to the beam (by field integral on trajectory)
\[
\Delta x' = -\frac{1}{B_p} \frac{\partial B_y}{\partial z} \Delta x(z) dz
\]
\[
= -\frac{1}{B_p} \int B_y(x, y, z) dz + \frac{1}{B_p} \int \frac{\partial B_y}{\partial y} \Delta x(z) dz
\]

Static field integrals (on straight path):

First integral
\[ I_{1y}(x) = \int B_y(x, 0, z) dz, \quad \Delta x' = I_{1y}/B_p \]

Second integral
\[ I_{2y}(x) = \int dz \int B_y(x, 0, z) dz, \quad \Delta x = I_{2y}/B_p \]

(similarly for \( I_{1x} \) and \( I_{2x} \))

Quadrupole int.
\[ I_q = \int \frac{\partial B_y}{\partial x} dz, \quad \Delta \psi_x = \frac{\beta y_i q}{4\pi}, \quad \Delta \psi_y = -\frac{\beta y_i q}{4\pi} \]

Skew quad int.
\[ I_{sq} = \int \frac{\partial B_y}{\partial z} dz, \quad \text{coupling} \]

Sextupole int.
\[ I_{s6} = \int \frac{\partial^2 B_y}{\partial x^2} dz, \quad \text{nonlinear dynamics} \]

... The field integrals can be obtained from measurements. These static effects can be modeled as multipole kicks.

Dynamic effects from field roll-off

Kick from the dynamic effect
\[
\Delta x' = -\frac{1}{B_p^{\ast}} \frac{\partial B_y}{\partial x} \Delta x(x) dx
\]

with
\[ \Delta x(z) = -\frac{1}{k^2 p^2} \cos k z, \quad \frac{\partial B_y}{\partial x} = \frac{\partial B_y(x, 0)}{\partial x} \cos k z, \quad \text{(on the mid-plane)} \]

So the kick is
\[
\Delta x' = -\frac{1}{2(B_p)^2 k^2} B_y \partial B_y(x, 0) \frac{L}{dx}
\]

The dynamic kick effect is particularly severe for elliptically-polarized undulator (EPU)
There are methods of symplectic integration for \( s \)-dependent Hamiltonian when analytic forms of vector potential is available (not discussed here).

A commonly used method for ID modeling is the kick map method:

\[
(x, y) \rightarrow (\Delta x', \Delta y')
\]

The kicks can be derived from the potential (P. Elleaume)

\[
\Psi(x, y) = \int d\zeta (\int_{-\infty}^{\infty} B_x(x, y, \zeta) d\zeta)^2 + (\int_{-\infty}^{\infty} B_y(x, y, \zeta) d\zeta)^2
\]

with the kicks given by

\[
\Delta x' = -\frac{1}{2(Bp)^2} \frac{\partial \Psi(x, y)}{\partial x}, \quad \Delta y' = -\frac{1}{2(Bp)^2} \frac{\partial \Psi(x, y)}{\partial y},
\]

Comparison of the Elleaume kick map with numerical integration (Runge-Kutta)

References

4. E. Forest and Milutinovic, NIMA 269, 474-482 (1989)
6. X. Huang, et al, Lattice modeling for SPEAR3, IPAC 2010
8. P. Elleaume, A new approach to the electron beam dynamics in undulators and wigglers, EPAC 92, p. 681 (1992)
Outline

- Motivation
  - Error sources in real machines.
  - Correcting the errors improves machine performance.
- Orbit correction
- Model calibration with orbit response matrix
  - LOCO
- Optics measurement and lattice calibration with turn-by-turn BPM data
  - Traditional method
  - MIA and ICA.

Lattice errors in machine

- An actual machine always deviates from the ideal model.
  - The model makes simplifying assumptions, e.g., hard-edge magnet model.
  - Systematic and random field errors in actual magnets.
  - Magnet power supply setpoint errors (e.g., gradient to current conversion)
  - Errors in power supply regulation.
  - Magnet hysteresis.
  - Misalignments.
  - Malfunctions, human errors, etc.
- Symptoms of errors in the machine
  - Closed orbit different from design orbit (side effects include optics and coupling errors due to magnet “feed-down”).
  - Linear optics and coupling errors.
  - Degraded nonlinear dynamic performance (poor injection efficiency and lifetime)

Orbit errors and correction

- Sources of orbit errors
  - Misalignments
  - Steering errors from bending magnets

Closed orbit errors (simulated) from a SPEAR3 QF magnet with alignment errors of 100 μm for both planes (kick angles $\theta = K L A \approx 64 \text{ urad}$).

Orbit errors from one kick:

$$\Delta(s) = \frac{\sqrt{N} \beta(s)}{2 \sin \nu} \cos([\psi(s) - \psi(s_0)] - \nu)$$

from N random kicks:

$$< \Delta(s) >_{\text{rms}} = \sqrt{\beta(s)} \frac{\sqrt{N \beta}}{2 \sqrt{2 \sin \nu}} < \theta >_{\text{rms}}$$
Beam-based alignment

- Target of orbit correction: centers of quadrupole magnets.
- Determination of quadrupole centers with beam-based alignment (BBA) measurement.
  - Principle: when there is an orbit offset through a quadrupole, a change to quadrupole strength causes orbit shifts.
  - Method (model independent): For a quadrupole, steer the beam orbit at the quadrupole with a corrector, at each point change quadrupole strength and measure orbit shift. Fit orbit shifts vs. readings of the nearest BPM to find the position corresponding to quad center.

Orbit response matrix and orbit correction

- Orbit response: closed orbit shifts due to a kick by an orbit corrector.

\[
R_{xx,ij} = \frac{\Delta x_j}{\theta_{x,j}} \quad \text{and} \quad R_{yy,ij} = \frac{\Delta y_i}{\theta_{y,j}}
\]

Similarly on the vertical plane (correctors may be different from \( x \)-plane.

Cross-plane responses (caused by coupling)

\[
R_{xy,ij} = \frac{\Delta x_j}{\theta_{y,j}} \quad \text{and} \quad R_{yx,ij} = \frac{\Delta y_i}{\theta_{x,j}}
\]

Overall response matrix

\[
\begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix} \begin{pmatrix} \theta_x \\ \theta_y \end{pmatrix} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}
\]

Or simply \( \mathbf{R} \theta = \Delta \mathbf{X} \).

- Orbit correction

\( \theta = R^{-1} \Delta \mathbf{X} \)

Singular value decomposition may be used to invert the response matrix with selection or weighting of the singular values.

Optics errors

- Error sources
  - Quadrupole components (e.g., random errors) from magnets
  - Horizontal orbit offsets in sextupoles
  - Effects not accounted for in ideal model (e.g., quadrupole fringe field, errors in edge focusing model in dipole.)

- Small quadrupole errors accumulate to large optics distortions.

Beta-beat from one quadrupole error

\[
\frac{\Delta \beta(s)}{\beta(s)} = \frac{\Delta K_1}{2 \sin 2\pi v} \cos(2|\psi(s) - \psi_0| - 2\pi v)
\]

From \( N \) random (phase uncorrelated) quadrupole errors

\[
< \frac{\Delta \beta}{\beta} >_{rms} = \frac{\sqrt{N} \beta}{2 \sqrt{2} \sin 2\pi v} < \Delta K_1 >_{rms}
\]

Betatron phase error

\[
\Delta \psi = \Delta \int \frac{ds}{\beta} = -\int \frac{\Delta \beta}{\beta^2} ds = -\int \frac{\Delta \beta}{\beta} d\psi
\]

Therefore

\[
< \Delta \psi >_{rms} = \frac{1}{\beta} < \Delta \beta >_{rms}
\]

Optics errors - example

SPEAR, 99 quads, \( \sigma_{kl} = 0.001 \) m

Estimate w/ formula: [0.036, 0.037] mean \( \beta_x, \beta_y \sim 7 \) m

APS, 400 quads, \( \sigma_{kl} = 0.001 \) m

Estimate w/ formula: [0.135, 0.117] mean \( \beta_x, \beta_y \sim 15.5, 17.9 \) m
Measurement of optics errors

- Direct beta function measurement at a quadrupole location
  
  Measure tune shift induced by a small quadrupole strength change:
  \[ \Delta v = -\frac{1}{4\pi} [\Delta K L] \beta \rightarrow \beta = -\frac{4\pi \Delta v}{[\Delta K L]} \]

  Difficulty: the actual \( \Delta K \) by a quadrupole current change cannot be accurately determined because of magnet hysteresis.

- Measurement with turn-by-turn BPM data with beam in oscillation.
  - Betatron oscillation amplitude \( \beta \)
  - Betatron phase advances

- Indirect measurement – fit lattice model to data that represent linear optics
  - Orbit response matrix

Measure tune shift induced by a small quadrupole strength change:
\[ \Delta v \equiv -\frac{1}{2} \frac{\Delta K}{K} \]

Dispersion measurement is very reliable and accurate. The main uncertainty is from BPM calibration.

Dispersion measurement is very reliable and accurate. The main uncertainty is from BPM calibration.

Linear optics from closed orbit (LOCO)

- Principle: adjust quadrupole strengths in the lattice model so that the calculated orbit response matrix agrees with the measured orbit response matrix, i.e., to minimize function

  \[ f(\Delta K) = \chi^2 = \sum_{ij} \frac{(R_{ij}^{\text{meas}} - R_{ij}^{\text{model}}(\Delta K))^2}{\sigma_i^2} \]

- Effects of BPM gains and roll: what BPMs report is not exactly where the beam is.

  \( \begin{pmatrix} x_{\text{meas}} \\ y_{\text{meas}} \end{pmatrix} = \begin{pmatrix} x \\ c_x \\ c_y \\ y_{\text{beam}} \end{pmatrix} \begin{pmatrix} \theta_{x,\text{beam}} \\ \theta_{y,\text{beam}} \end{pmatrix} \]

  Note the difference between \( c_x \) and \( c_y \) accounts for BPM "crunch" (deformation from ideal configuration).

- Effects of corrector gain and roll: kicks beam gets are different from what corrector readback report.

  \( \begin{pmatrix} \theta_{x,\text{beam}} \\ \theta_{y,\text{beam}} \end{pmatrix} = \begin{pmatrix} \theta_{x,\text{meas}} \\ \theta_{y,\text{meas}} \end{pmatrix} \)

  Dispersion functions are included in fitting (additional terms in \( \chi^2 \) definition).

  - This helps decouple the BPM and corrector gain as dispersion function measurements do not involve correctors.

  The minimization problem is

  \[ f(p) = \chi^2 = \sum_{ij} \frac{(R_{ij}^{\text{beam}} - R_{ij}^{\text{model}})^2}{\sigma_i^2} + \sum_{ji} \frac{(P_{ij}^{\text{beam}} - P_{ij}^{\text{model}})^2}{\sigma_{ij}^2} + \sum_{ji} \frac{(P_{ji}^{\text{beam}} - P_{ji}^{\text{model}})^2}{\sigma_{ji}^2} \]

  where \( p \) includes all fitting parameters.
Solving a least square problem

Problem: find lattice parameters that minimize the objection function.

General nonlinear least square problem:

\[ f(p) = \chi^2 = \sum_i [y_i - y(x_i; p)]^2 = r^T r \]

Residual vector \( r_i = y_i - y(x_i; p) \)

Jacobian matrix \( J^T = \frac{\partial f}{\partial p} \)

Usually it is solved iteratively, at each step, solve for \( \Delta p \) by minimizing

\[ f(p) = f(p_0) + \frac{\partial f}{\partial p}(p_0) \cdot \Delta p + \frac{1}{2} \Delta p \cdot H(p_0) \cdot \Delta p \]

At the minimum we have \( \frac{\partial f}{\partial p}(p_0) = 0 \)

Leading to condition \( H(p_0) \Delta p = -\frac{\partial f}{\partial p}(p_0) \)

Challenges due to cross-talk between \( \Delta K \) parameters

For LOCO, the Jacobian matrix is often near degenerate, resulting in large prediction of \( \Delta K \), which is unrealistic and potentially impossible to use for optics correction or even updating the model. The near-degeneracy is caused by correlation of the fitting parameters.

Fitting LOCO with constraints

A common approach is to cut off the singular values with a threshold. This prohibits the solution to have any component in the sub-space of the parameter space removed by the cut-off and thus limits the accuracy of the solution.

Another common approach is to use a reduced set of fitting parameters (quads). This also will remove a sub-space. However, it would work if the effects of the left-out quads are represented by the fitting quads. (In other words, each left-out quad is highly correlated with some or a combination of fitting quads).

Yet another way to adding constraints or cost functions to the fitting parameters:

This approach limits the stray of the solution in the under-constrained sub-space (formed by directions with small SVs) and allows the solution to have components in it if that is what data demands.

\[ \chi^2 = \chi^2 + \frac{1}{\sigma_{\Delta K}} \sum_i w_i \Delta K_i^2 \]

Equivalently, we are requiring \( \Delta K_{\perp} = 0 \) with some weight relative to other terms.
Effect of the constraints

An illustration of the changes to the convergence path with or without constraints. Solid: no constraints; Dashed: with constraints.

The rms relative change of gradients vs. the residual $\chi^2$ for real SPEAR3 data set. Green: no constraints; Blue: with constraints. Point 0 is located at $(2 \times 10^6, 0)$.

Optics correction with LOCO results

• When $\Delta K$ is found with fitting, make correction to quadrupole setpoint accordingly to compensate these errors. High precision of optics correction can be achieved.

Correction results at ALS

• LOCO fit indicated gradient errors in ALS QD magnets making $\beta$ distortion.
• Gradient errors subsequently confirmed with current measurements.
• LOCO used to fix $\beta$ periodicity.
• Operational improvement.

Coupling correction

• The off-diagonal blocks of the response matrix $R_{xy}$ and $R_{yx}$ are due to rolls of BPMs and correctors and $x-y$ coupling in the lattice.
• Fitting skew quadrupole parameters can uncover equivalent sources of coupling. Compensating these sources (changing skew quad setpoints) can reduce coupling.
• At SPEAR3 coupling can be reduced to 0.05% (vertical emittance 5 pm).
Optics measurement and correction with TBT BPM data

• Example of turn by turn data. For TBT BPM data to contain optics information, beam needs to be excited (kicked or resonantly driven) so that the beam centroid undergoes coherent oscillation.

• Decoupled betatron oscillation

At BPM \(i\), on the \(n\)-th turn
\[
x_i(n) = \sqrt{2\beta_x J_x} \sin(2\pi \nu_x n + \psi_x),
\]
\[
y_i(n) = \sqrt{2\beta_y J_y} \sin(2\pi \nu_y n + \psi_y),
\]
Where \(J_{x,y}\) are action variables, \(\psi_{x,y}\) are phase advances.
TBT BPM data contain information of beta function and phase advance that are not difficult to extract.

How to extract such information consistently, accurately and efficiently?

A method of processing TBT data

1. Obtain the tunes with NAFF or interpolated FFT.
2. Calculate amplitude and phase for each BPM

\[
C = \sum_n x(n) \cos 2\pi \nu v, \quad S = \sum_n x(n) \sin 2\pi \nu v,
\]
Then amplitude and phase are
\[
A = \frac{2\sqrt{C^2 + S^2}}{N}, \quad \psi = -\cot^{-1} \frac{S}{C},
\]
where \(N\) is the number of turns. Error of phase
\[
\sigma_\psi = \frac{1}{A} \sqrt{\frac{2}{N} \sigma_x^2}
\]

3. The measured phase can be used to derive beta function, using model values of beta function and phase advances.

\[
\beta_{1\text{meas}} = \frac{\beta_{1\text{model}} \cot \psi_{12\text{meas}} - \cot \psi_{13\text{meas}}}{\cot \psi_{12\text{model}} - \cot \psi_{13\text{model}}}
\]

Disadvantages

– Not suited if \(x\) or \(y\) data are not a single sinusoidal signal (e.g., with linear coupling, synchrotron motion or contaminating signals).
– Each BPM is treated independently, not benefiting from the fact all BPMs observe the same oscillations.
– Finite number of turns introduces errors to phases since \(\sum \cos 2\pi \nu \sin 2\pi \nu \neq 0\)
– Need the ideal model to calculate beta function.

• Model independent analysis (MIA) and Independent component analysis for TBT data processing

– Treat all BPM data consistently.
– De-couple individual signals from observations.
– Reduce noise.

A model of BPM turn-by-turn data

• The turn-by-turn beam position signal is a combination of various source signals.

\[
x_i(t) = \sum_j d_{ij} s_j(t) + n_i(t) \quad \text{For the } i\text{'th BPM}
\]

or
\[
x(t) = As(t) + n(t) \quad A \text{ is the mixing matrix}
\]

There are only a few meaningful source signals, such as betatron oscillation and synchrotron oscillation.

Form a matrix of the BPM data

\[
x = \begin{pmatrix}
x_1(1) & x_1(2) & \cdots & x_1(T) \\
x_2(1) & x_2(2) & \cdots & x_2(T) \\
\vdots & \ddots & \ddots & \vdots \\
x_m(1) & x_m(2) & \cdots & x_m(T)
\end{pmatrix}
\]
m BPMs and T turns

1/20/2015 X. Huang, USPAS Jan 2015
Betatron modes via singular value decomposition

It has been proven that when the BPM reading contains only one betatron mode, i.e.
\[ x(t) = \sqrt{2J(t)} \beta \cos(\phi + \psi) \]
then there are only two non-trivial SVD eigen-modes
\[ x = USV^T = s_1 u_1 v_1^T + s_2 u_2 v_2^T \]
\[ u_1 = \frac{1}{x} \sqrt{J - \beta \cos(\phi + \psi)} \]
\[ u_2 = \frac{1}{x} \sqrt{J - \beta \sin(\phi + \psi)} \]
\[ v_1 = \frac{1}{\sqrt{2J}} \cos(\phi - \psi) \]
\[ v_2 = \frac{1}{\sqrt{2J}} \sin(\phi - \psi) \]
\[ \mathbf{u} \text{: spatial vector} \]
\[ \mathbf{v} \text{: temporal vector} \]

Beta function and betatron phase advance can be calculated from the spatial vector.

\[ \psi = \tan^{-1}\left(\frac{x}{u_{1,\psi}}\right) \]
\[ \beta = \frac{1}{\sqrt{J} - s_{1,\psi}} (x_{1,\beta}^2 + x_{2,\beta}^2) \]

As the random noises are distributed in all eigen-modes while the signals are concentrated in the leading eigen-modes, noise can be reduced by re-constructing the data after removing the noise-only (with small singular values) modes.

\[ \sigma_x = \sigma \sqrt{\frac{p}{2m}} \]

Note the constant orbit offsets are always removed for each BPM. This is called “centering.”
The independent component analysis (ICA)

- The source signals are assumed statistically independent.
  \[ p(x_1, x_2) = p(x_1) p(x_2) \]

This is a strong condition that the PCA analysis does not make full use of.

\[ E[h_1(x_1)h_2(x_2)] = E[h_1(x_1)]E[h_2(x_2)] \quad \text{for any function } h_1, h_2. \]

PCA only requires the components to be linearly uncorrelated, i.e., the covariance between two variables is zero.

\[ E[x_1x_2] - E[x_1]E[x_2] = 0 \]

For two Gaussian variables, uncorrelatedness is equivalent to independence. Many ICA algorithms exploit the non-gaussianity of the signals, such as fastICA.

It is possible to use non-gaussianity based methods for BPM data analysis. But we will focus on an algorithm that relies on the time-spectrum of the source signals.

The Principle

- The source signals are assumed to be narrow-band with non-overlapping spectra, so their un-equal time covariance matrices are diagonal.

\[ \langle s(t)s(t + \tau)^T \rangle = \text{diag}[\rho_1(\tau), \rho_2(\tau), \ldots, \rho_n(\tau)] \]

Since

\[ x(t) = As(t) + n(t) \]

\[ C_x(\tau) \equiv \langle x(t)x(t + \tau)^T \rangle = AC_x(\tau)A^T + \sigma^2 I \]

The mixing matrix A diagonalizes the un-equal time sample covariance matrices simultaneously.

The Algorithm* - 1

- Diagonalize the equal-time covariance matrix (data whitening)

  \[ C_x(0) = [U_1, U_2] \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} [U_1, U_2]^T \quad \text{with} \quad 0 = \max(D_1) < \lambda_i < \min(D_1) \]

  Construct an intermediate "whitened" data matrix

  \[ z = D_1^{-1}U_1^T x = Vx \quad \text{which satisfies} \quad <zz^T> = I \]

  This pre-processing step is just PCA. Matrix z contains the temporal vectors.


The Algorithm - 2

- Jointly diagonalize* the un-equal time covariance matrices of matrix z of selected time-lag constants.

  \[ C_x(\tau) = WC_x(\tau)W^T \quad \text{for} \quad \tau = \{\tau_i \mid i = 1, 2, \ldots, k\} \]

  Then

  \[ s = W^T Vx \quad \text{and} \quad A = (U_i D_i^T)^T W \]

  The columns of A (spatial vectors) and corresponding rows (temporal vectors) of s are the resulting modes.

**Linear Lattice Functions Measurements**

- There are two betatron modes because each BPM sees different phase.

  The betatron component
  \[ x = A_1 s_1 + A_2 s_2 \]

  Beta function and phase advance
  \[ \beta = a (A_1^2 + A_2^2) \quad \psi = \tan^{-1} \left( \frac{A_1}{A_2} \right) \]

- There is one synchrotron mode.

  The synchrotron component
  \[ x = A_i s_i \]

  Dispersion function and momentum deviation
  \[ D = b A_i \quad \delta = \frac{s_i}{b} \]

---

**Example: de-coupling**

The ICA method can de-couple the normal modes in presence of linear coupling.

FFT spectra of raw horizontal and vertical BPM signals at section L1. Both BPMs see a mixture of the "plus" mode and "minus" mode.

---

**Comparison of phase measurement results**

3. Actual phase error due to a quadrupole error.
5. ICA with noise. 6. ICA w quad error and noise.

---

**Lattice calibration with ICA**

- The measured beta functions, phase advances and dispersion functions can be used to fit lattice model by comparing them to calculated values.

\[
\chi^2 = \sum_i \left( \frac{(\beta_{x_i}^{\text{meas}} - \beta_{x_i}^{\text{model}})^2}{\sigma_{\beta_x}^2} + \frac{(\beta_{y_i}^{\text{meas}} - \beta_{y_i}^{\text{model}})^2}{\sigma_{\beta_y}^2} + \frac{(\psi_{x_i}^{\text{meas}} - \psi_{x_i}^{\text{model}})^2}{\sigma_{\psi_x}^2} + \frac{(\psi_{y_i}^{\text{meas}} - \psi_{y_i}^{\text{model}})^2}{\sigma_{\psi_y}^2} + \frac{(D_{x_i}^{\text{meas}} - D_{x_i}^{\text{model}})^2}{\sigma_{D_x}^2} + \frac{(D_{y_i}^{\text{meas}} - D_{y_i}^{\text{model}})^2}{\sigma_{D_y}^2} \right)
\]

- This method can also be extended to fit coupling.
References

Lecture 5: Optimization of accelerators in simulation and experiments

X. Huang
USPAS, Jan 2015

Considerations of optimization in simulation

• Single objective or multiple objectives?
  – When there are multiple conflicting objectives, consider using multi-objective algorithms.

• The decision variables (knobs)
  – The number of decision variables – size of the problem

• Evaluation of objective function(s)
  – Slow or fast?
  – Numerical noise

• The constraints
  – Parameter range
  – Cost functions

• Complexity of parameter space
  – Smoothness of function
  – Local minima

Choice of algorithms

• Traditional algorithms
  – Gradient based algorithms: Gauss-Newton, quasi-GN, Levenberg-Marquadt
  – Iterative line search: Powell’s method
  – Downhill simplex method
  – More …
  – comments: single objective function; smooth function; result may depend on initial condition (local minima); small scale problems; converge fast.

• Stochastic algorithms
  – Simulated annealing
  – Evolutionary algorithms
  – Particle swarm algorithms
  – …
  – Features: complex parameter space; can work with multi-objective problems; large scale problems; less efficient.
Comparison of solutions for multi-objective optimization

Problem: minimize $f_m(x)$, $m = 1, 2, \ldots, M$ with parameter ranges $x_i \in [x_i^L, x_i^U]$

Comparison of two solutions (definition of domination): Solution $x_a$ dominates solution $x_b$ if for all $m = 1, 2, \ldots, M$, we have $f_m(x_a) \leq f_m(x_b)$ and for at least one objective $m'$, $f_{m'}(x_a) < f_{m'}(x_b)$.

Pareto front: the set of all solutions in the search space that are non-dominated by any solutions.

Goal of multi-objective optimization is obtain the Pareto front for further analysis.

Application of NSGA-II: injector optimization (Cornell)

Decision variables:

- Fields:
  - DC Gun Voltage (300-900 kV)
  - 2 Solenoids
  - Buncher
  - SRF Cavities Gradient (5-13 MV/m)
  - SRF Cavities Phase
- Buncher & Photocathode:
  - $E_0$ (kV)
  - Charge

Examples of application to nonlinear dynamics optimization

Examples of application to nonlinear dynamics optimization

- M. Borland
- L. Yang

I. Bazarov, PRSTAB 8, 034202, (2005)

K. Deb, IEEE Transitions On Evolutionary Computation Vol 6, No 2, April 2002
Multi-objective particle swarm optimization

MOPSO also manipulates a population of solutions over many iterations with random operations.

Updating particle population in an iteration

\[ x_{i}^{t+1} = x_{i}^{t} + v_{i}^{t+1} \]
\[ v_{i}^{t+1} = wv_{i}^{t} + c_{1}r_{1}(p_{i}^{t} - x_{i}^{t}) + c_{2}r_{2}(g^{t} - x_{i}^{t}) \]

\( x_{i}^{t} \) position (parameter vector) of particle \( i \) at iteration \( t \).

\( v_{i}^{t} \) velocity (increment) of particle \( i \) at iteration \( t \).

Control parameters: \( w = 0.4, c_{1} = c_{2} = 1 \).

\( r_{1} \) and \( r_{2} \) are random within \([0,1]\) or fixed values.

MOPSO can also include mutation operation.

X. Pang, L.J. Rybarcyk, NIMA 741 (2014)

The need of online optimization

- **The general need of online optimization.**
  - Lack of diagnostics (that monitor the sub-systems)
    - Injection steering and transport line optics.
  - Target values of monitors not established (or drifting)
    - Initial commissioning.
  - Lack of deterministic procedure to go to target values.
    - Nonlinear beam dynamics in storage rings.
  - **Manual tuning – online optimization of a complex system.**
    - We want automate the tuning process with efficient algorithms.
  - **Difficulty in automated tuning**
    - Automated tuning is basically optimization of **noisy** functions of multiple variables.
    - Traditional optimization algorithms are usually for smooth functions.

Considerations of online optimization

- **High efficiency – get to the optimum fast**
  - Online evaluation of the objective is usually slow.
  - Machine study time is usually limited (and expensive).
  - Efficiency may be measured by the number of function evaluations.
- **Robustness – surviving noise, outliers and machine failures**
- **Live status reporting during optimization.**
- **Traditional algorithms are usually not suitable for noisy online problems.**
- **An algorithm designed for online optimization**
  - Robust conjugate direction method (RCDs) – A combination of Powell’s method and a new noise resistant line optimizer.
Powell’s method

Powell’s method has two components:
1. A procedure to update the direction set to make it a conjugate set.
2. A line optimizer that looks for the minimum along each direction.

Directions \( u, v \) are conjugate if:
\[
0 = \mathbf{u}^T \mathbf{H} \mathbf{v}
\]
where the Hessian matrix is defined
\[
\mathbf{H} = \frac{\partial^2 f}{\partial x_i \partial x_j}.
\]
Searching along mutually conjugate directions is more efficient since a scan along one direction doesn’t ruin previous results on other directions.

Line optimizer:

Another choice is golden section search (bisection).

Both line optimization approaches are sensitive to noise – one wrong decision (due to noise) leads the search away from the real minimum.

*M.J.D. Powell, Computer Journal 7 (2) 1965 155

A simulation study: coupling correction for SPEAR3

– Using calculated beam loss rate as the objective function.
– Noise is generated in the objective function by adding random noise to beam current values (used for loss rate calculation).
– There are 13 coupling correction skew quads in SPEAR3.
– Initial conjugate direction set is from SVD of the Jacobian matrix of orbit response matrix w.r.t. skew quads
– With
  • 500 mA beam current with 1% random variation. On top of that a DCCT noise with sigma = 0.003 mA. The beam loss rate noise evaluated from 6-s duration is 0.06 mA/min.
  • 40 hour gas lifetime; 10 hour Touschek lifetime with 0.2% coupling.
  • The coupling ratio with all 13 skew quads off is 0.9% (with simulated error), corresponding to a loss rate of 0.6 mA/min.

The IMAT method uses the same RCDS line optimizer, but keep the direction set of unit vectors (not conjugate).

Clearly,
1. the line optimizer is robust against noise.
2. Searching with a conjugate direction set is much more efficient.

Note that the direction set has been updated only about 8 times after 500 evaluations (out of 13 directions). So the high efficiency of RCDS is mostly from the original direction set.
Experiment: coupling correction with loss rate

Beam loss rate is measured by monitoring the beam current change on 6-second interval (no fitting). Noise sigma 0.04 mA/min. Data were taken at 500 mA with 5-min top-off. Initially setting all 13 skew quads off. Loss rate at about 0.4 mA/min. Final loss rate at about 1.75 mA/min. At 500 mA, the best solution had a lifetime of 4.6 hrs. This was better than the LOCO correction (5.2 hrs).

Best result with RCDS is loss rate >2.0 mA/min and 500 mA lifetime 4.2 hrs.

References:
(2). K. Deb, IEEE Transitions On Evolutionary Computation Vol 6, No 2, April 2002
(3). I. Bazarov, PRSTAB 8, 034202, (2005)
(4). X. Pang, L. J. Rybacyk, NIMA 741 (2014)