Solution:

1. In thin length approximation, \(2\pi R = 2NL\), \(L = \rho\theta\), \(\theta = \frac{2\pi}{2N}\)

\[
\alpha_c = \frac{1}{2\pi R} N \cdot \left[ \frac{D_F}{\rho} L + \frac{D_D}{\rho} L \right] = \frac{1}{2\pi R} \frac{2NL^2\theta}{\rho \sin^2 \frac{\theta}{2}} = \frac{\theta^2}{\sin^2 \frac{\theta}{2}} = \left( \frac{2\pi}{2N \sin \frac{\theta}{2}} \right)^2
\]

Similarly, we find \(\alpha_c = \rho\theta^2/6R\) for DBA lattices.

2. The energy gain for a non-synchronous particle is

\[
\Delta E = \frac{e}{\Delta t} \int \frac{\omega_0}{\Delta t} V(t)dt = \frac{eV_g}{\hbar \omega_0 \Delta t} \int \frac{\omega_0}{\Delta t} \sin(h\omega_0 t + \phi) d(h\omega_0 t)
\]

\[
= \frac{eV_g}{\hbar \omega_0 \Delta t} \cdot 2 \sin \phi \sin \left( \frac{h\omega_0 \Delta t}{2} \right) = eV_g \frac{\sin(hg/2R)}{(hg/2R)} \sin \phi = e(V_g T) \sin \phi,
\]

where \(\omega_0 \Delta t = g/R\), \(T\) is the transit time factor, \(V_g\) is the gap voltage, and \(V = V_g T\) is the effective voltage. Thus we find that the effective energy gain is multiplied by a transit time factor as that of a synchronous particle.

3. \(Q_s = \sqrt{\hbar cV|\eta_0 \cos \phi_s|/2\pi \beta^2 E}\)

<table>
<thead>
<tr>
<th>(\nu_s (\times 10^{-5}))</th>
<th>AGS</th>
<th>RHIC</th>
<th>FNAL-MI</th>
<th>FNAL-BST</th>
<th>SSC</th>
<th>Cooler</th>
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<td>0.395</td>
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4. Particle momentum and energy

\(pc = ecB(t)\rho\), \(E = \sqrt{(mc^2)^2 + (pc)^2} = \sqrt{(mc^2)^2 + [ecB(t)\rho]^2}\).

Then

\[
\omega_{rf} = 2\pi h \frac{\beta c}{L} = \frac{hc}{R} \sqrt{1 - \frac{(mc^2)^2}{E^2}} = \frac{hc}{R} \sqrt{\frac{B^2(t)}{[B^2(t) + (mc^2/\rho c)^2]}}
\]

5. \(eV \sin \phi_s = U_0 = C_\gamma E_0^4/\rho\), where \(C_\gamma = 8.85 \times 10^{-5}\) m/(GeV)^3.

<table>
<thead>
<tr>
<th></th>
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</tr>
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<tbody>
<tr>
<td>(C [m])</td>
<td>26658.9</td>
<td>196.8</td>
<td>1060</td>
<td>223</td>
<td>240.4</td>
<td>3018</td>
</tr>
<tr>
<td>Energy [GeV]</td>
<td>50</td>
<td>1.2</td>
<td>7.0</td>
<td>1.98</td>
<td>2.2</td>
<td>30.</td>
</tr>
<tr>
<td>(\rho [m])</td>
<td>3096.2</td>
<td>4.01</td>
<td>38.96</td>
<td>4.35</td>
<td>10.35</td>
<td>246.5</td>
</tr>
<tr>
<td>(V_{rf} [MV])</td>
<td>400</td>
<td>1.5</td>
<td>10</td>
<td>1.0</td>
<td>0.8</td>
<td>400.</td>
</tr>
<tr>
<td>(h)</td>
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<td>328</td>
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<td>531</td>
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<tr>
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<td>50.86</td>
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<td>5.0</td>
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<tr>
<td>(\nu_x)</td>
<td>76.2</td>
<td>14.28</td>
<td>35.22</td>
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<td>6.18</td>
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<tr>
<td>(\nu_z)</td>
<td>70.2</td>
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<td>14.3</td>
<td>8.62</td>
<td>7.12</td>
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</tr>
<tr>
<td>(\phi_s)</td>
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<td>18.0</td>
<td>14.0</td>
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</tr>
<tr>
<td>(Q_s)</td>
<td>0.12</td>
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<td>0.0075</td>
<td>0.0044</td>
<td>0.019</td>
<td>0.107</td>
</tr>
</tbody>
</table>
6. Define $x = \theta$, and $p = |\eta| \delta / \nu_s$ as the normalized phase space coordinates. The tori of linearized synchrotron motion in the normalized phase-space are circles with $x^2 + p^2 = \text{constant}$. The initial rms beam widths in the normalized phase-space coordinates becomes $\sigma_{x0} = \sigma_\theta$ and $\sigma_{p0} = |\eta| \sigma_\delta / \nu_s$. Let $(\sigma_{x0}/\sigma_{p0})^2 = 1 + X$. Then the initial injected beam distribution is

$$\rho(x_0, p_0) = \frac{N_B e}{2\pi \sigma_{x0} \sigma_{p0}} \exp \left\{ -\frac{1}{2 \sigma_{x0}^2} \left[ x_0^2 + (1 + X)p_0^2 \right] \right\}.$$ 

The linear synchrotron motion can be expressed as

$$x = x_0 \cos \omega_s t + p_0 \sin \omega_s t,$$
$$p = -x_0 \sin \omega_s t + p_0 \cos \omega_s t.$$ 

Thus we find

$$\frac{1}{2\sigma_{x0}^2} [x_0^2 + (1 + X)p_0^2] = \frac{x^2}{2\sigma_x(t)} + \frac{1 + X \cos^2 \omega_s t}{2\sigma_{x0}} \left[ p + \frac{X \sin \omega_s t \cos \omega_s t}{1 + X \cos^2 \omega_s t} x \right]^2,$$

where

$$\sigma_x(t) = \sigma_{x0}^2 \cos^2 \omega_s t + \sigma_{p0}^2 \sin^2 \omega_s t.$$ 

(a) The beam distribution $\rho(\theta)$ can be obtained by integrating over the coordinate $p$, i.e.

$$\rho(x) = \int \rho(x, p) dxdp = \frac{N_B e}{\sqrt{2\pi \sigma_x(t)}} e^{-\frac{x^2}{2\sigma_x(t)}}.$$ 

(b) The phase-space area of the injected beam is given by $\pi \sigma_{x0} \sigma_{p0}$. Let the rms beam widths of a matched beam with an identical phase-space area be $\sigma_{x m}$ and $\sigma_{p m}$, i.e.

$$\pi \sigma_{x0} \sigma_{p0} = \pi \sigma_{x m} \sigma_{p m},$$

or

$$\sigma_{x0}^2 = \frac{\nu_{s,0}}{\nu_{s,m}} \sigma_{x m}^2 \approx \left( 1 - \frac{\Delta V}{2V} \right) \sigma_{x m}^2,$$

where $V$ is the rf voltage of the matched phase-space ellipse, and the rf voltage of the injected machine is $V - \Delta V$.

7. Expanding the phase coordinate around the SFP with $\phi = \phi_s + \varphi$, the synchrotron Hamiltonian becomes

$$H = \frac{1}{2} \hbar \omega_0 \eta \delta^2 + \frac{1}{2\hbar \eta} \omega_0 Q_s^2 \left[ \varphi^2 - \frac{1}{3} \tan \phi_s \varphi^3 - \frac{1}{12} \varphi^4 + \cdots \right],$$

where

$$Q_s = \sqrt{\frac{\hbar e V |\eta \cos \phi_s|}{2\pi \beta^2 E}} = \nu_s \sqrt{\cos \phi_s}.$$ 

For simplicity, we assume $\eta > 0$ in this exercise.
(a) We expand the Hamiltonian in the action-angle variables of the linear synchrotron motion with the generating function

\[ F_1(\phi, \psi) = -\frac{Q_s}{2\hbar\eta} \phi^2 \tan \psi. \]

Thus

\[ \delta = \frac{\partial F_1}{\partial \phi} = -\frac{Q_s}{\hbar\eta} \phi \tan \psi, \quad J = -\frac{\partial F_1}{\partial \psi} = \frac{Q_s}{2\hbar\eta} \phi^2 \sec^2 \psi, \]

or

\[ \phi = \sqrt{\frac{2\hbar J}{Q_s}} \cos \psi \quad \delta = -\sqrt{\frac{2Q_s J}{\hbar\eta}} \sin \psi. \]

The Hamiltonian in action-angle coordinates becomes

\[ H = \omega_0 Q_s J - \frac{\omega_0 \sqrt{2\hbar Q_s}}{12} \tan \phi \, J^{3/2} \left[ \cos 3\psi + 3 \cos \psi \right] - \frac{\omega_0 \hbar \eta}{6} \, J^2 \cos^4 \psi. \]

(b) Let \( F_2 \) be the generating function that transforms \((J, \psi)\) to \((I, \chi)\) with

\[ F_2(\psi, I) = \psi I + G_3(I) \sin 3\psi + G_1(I) \sin \psi, \]

where \( G_1 \) and \( G_3 \) are determined by canceling low order terms in the action variable. The coordinate transformation is given by

\[ J = \frac{\partial F_2}{\partial \psi} = I + 3G_3 \cos 3\psi + G_1 \cos \psi, \]

\[ \chi = \frac{\partial F_2}{\partial I} = \psi + G'_3(I) \sin 3\psi + G'_1(I) \sin \psi \approx \psi. \]

Inserting the above relation into the Hamiltonian in (b) and using of the expansion

\[ J^{3/2} = (I + 3G_3 \cos 3\psi + G_1 \cos \psi)^{3/2} \approx I^{3/2} + \frac{3}{2} I^{1/2} (3G_3 \cos 3\psi + G_1 \cos \psi) \]

we obtain

\[ H = \omega_0 Q_s (I + 3G_3 \cos 3\psi + G_1 \cos \psi) - \frac{\omega_0 \hbar \eta}{6} I^2 \cos^4 \psi \]

\[ - \frac{\omega_0}{12} \sqrt{2\hbar Q_s} \tan \phi \, I^{3/2} \left( \cos 3\psi + 3 \cos \psi \right) \]

\[ - \frac{\omega_0}{8} \sqrt{2\hbar Q_s} \, I^{1/2} \tan \phi \left( 3G_3 \cos 3\psi + G_1 \cos \psi \right) \left( \cos 3\psi + 3 \cos \psi \right). \]

We choose \( G_1 \) and \( G_3 \) to cancel the term with \( I^{3/2} \) and get

\[ G_3 = \left( \frac{\sqrt{2\hbar \eta}}{36 \sqrt{Q_s}} \tan \phi \right) I^{3/2} \quad G_1 = \left( \frac{\sqrt{2\hbar \eta}}{4 \sqrt{Q_s}} \tan \phi \right) I^{3/2}. \]
(c) After canceling $I^{3/2}$, the perturbing term in the resulting Hamiltonian is proportional to $I^2$.

(d) The average Hamiltonian

$$\langle H \rangle = \frac{1}{2\pi} \int H d\psi = \omega_0 Q_s I - \frac{\omega_0 h\eta}{16} \left( 1 + \frac{5}{3} \tan^2 \phi_s \right) I^2 + \cdots,$$

where the tune

$$Q = \frac{\partial \langle H \rangle}{\partial I} = Q_s \left[ 1 - \frac{h\eta}{8Q_s} \left( \frac{5}{3} \tan^2 \phi_s \right) \frac{\hat{\phi}^2}{I} \right] = Q_s \left[ 1 - \frac{1}{16} \left( \frac{5}{3} \tan^2 \phi_s \right) \frac{\hat{\phi}^2}{I} \right]$$

with $\hat{\phi}^2 = (2h\eta/Q_s) I$.

8. Since $\sigma_s^2 = C_q \gamma^2 / J_E \rho$, we can then calculate $\sigma_\theta$, $\sigma_t$, and $\cal{A}$. In the following table, we assume the damping partition $J_E = 2$.

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