Accelerator Physics Homework #2
P470 (Problems: 1-5)

1. Let \((\hat{x}, \hat{s}, \hat{z})\) be local polar coordinates inside a dipole. The particle coordinate is

\[
\vec{r} = (\rho + x)\hat{x} + z\hat{z},
\]

where \(\rho\) is the bending radius. The momentum of the particle is \(\vec{p} = \gamma m \vec{v}\), where \(\gamma\) is constant in the static magnetic field, and the overdot corresponds to the derivative with respect to time \(t\). Similarly, \(d\vec{p}/dt = \gamma m \vec{v}\).

(a) Using the geometry of polar coordinate system listed in Eq. (2.8), show that

\[
\vec{r} = \hat{x} \hat{x} + (\rho + x)\hat{s} + \hat{z} \hat{z} = \left[\hat{x} - (\rho + x)\hat{\theta}\right]\hat{\theta} + \left[2\hat{x}\hat{\theta} + (\rho + x)\hat{\theta}\right]\hat{s} + \hat{z} \hat{z},
\]

where \(\theta = s/\rho\) is the angle associated with the reference orbit, i.e. \(ds = \rho d\theta\).

(b) Using \(d\vec{p}/dt = \pm e\vec{v} \times \vec{B}\), with \(\vec{B} = B_x \hat{x} + B_z \hat{z}\), show that

\[
\ddot{x} - (\rho + x)\dot{\theta}^2 = \pm \frac{v_s^2 B_z}{B\rho}, \quad \ddot{z} = \mp \frac{v_s^2 B_z}{B\rho},
\]

where \(B\rho = \gamma mv_s/e\) is the momentum rigidity, \(v_s\) is the longitudinal velocity, and the signs at the right hand side of these equations are for positive and negative charged particles respectively.

(c) Transforming the time coordinate \(t\) to the longitudinal distance \(s\), show that

\[
x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_z}{B\rho} (1 + \frac{x}{\rho})^2, \quad z'' = \mp \frac{B_z}{B\rho} (1 + \frac{x}{\rho})^2,
\]

where the prime is the derivative with respect to \(s\).

(d) Expand the magnetic field up to first order in \(x\) and \(z\), i.e.

\[
B_z = -B_0 + \frac{\partial B_z}{\partial x} x = \mp B_0 + B_1 x, \quad B_x = \frac{\partial B_z}{\partial x} z = B_1 z,
\]

where \(B_0/B\rho = 1/\rho\) signifies the dipole field in defining a closed orbit, and the quadrupole gradient function \(B_1 = \partial B_z / \partial x\) is evaluated at the closed orbit. Show that the betatron equation of motion becomes

\[
x'' + K_x(s)x = 0, \quad K_x = 1/\rho^2 \mp K_1(s), \quad (2)
\]
\[
z'' + K_z(s)z = 0, \quad K_z = \pm K_1(s), \quad (3)
\]

where \(K_1(s) = B_1(s)/B\rho\) is the effective focusing function, and the upper and lower signs correspond respectively to the positive and negative charged particles.
2. Consider a linear Hill’s equation: \( y'' + K(s)y = 0 \), where \( y \) stands for either \( x \) or \( z \) betatron coordinates, the prime corresponds to derivative with respect to the independent coordinate \( s \), and the focusing function \( K(s) \) for most accelerator magnets can be assumed to be piecewise constant.

(a) Show that the solution of the Hill’s equation can be expressed as

\[
\begin{pmatrix}
  y(s_2) \\
  y'(s_2)
\end{pmatrix}
= M(s_2|s_1)
\begin{pmatrix}
  y(s_1) \\
  y'(s_1)
\end{pmatrix}
\]  

(4)

where the transfer matrix \( M \) is given by

\[
K(s) = 0,
M(s_2|s_1) = \begin{pmatrix}
  1 & s \\
  0 & 1
\end{pmatrix},
\]

\[
K(s) = K \geq 0,
M(s_2|s_1) = \begin{pmatrix}
  \cos \sqrt{K}s & \frac{1}{\sqrt{K}} \sin \sqrt{K}s \\
  -\sqrt{K} \sin \sqrt{K}s & \cos \sqrt{K}s
\end{pmatrix},
\]

\[
K(s) = K < 0,
M(s_2|s_1) = \begin{pmatrix}
  \cosh |K|s & \frac{1}{|K|} \sinh |K|s \\
  \sqrt{|K|} \sinh |K|s & \cosh |K|s
\end{pmatrix},
\]

with \( s = s_2 - s_1 \).

(b) Show that the mapping matrix \( M \) for a short quadrupole of length \( \ell \), in the thin-lens approximation, is

\[
M = \begin{pmatrix}
  1 & 0 \\
  -\frac{1}{f} & 1
\end{pmatrix}
\]

where \( f = \lim_{\ell \to 0}(K\ell)^{-1} \), is the focal length of a quadrupole. For a focusing quad, \( f > 0 \); and for a defocusing quad, \( f < 0 \).

(c) Explain the equivalence of the particle trajectory to the geometric optics.

3. The angle that the particle enters a dipole makes difference to the particle motion when the bending radius is small.

(a) The particle orbit enters and exits a sector dipole magnet perpendicular to the dipole edges. Assuming that the gradient function of the dipole is zero, i.e. \( \partial B_z / \partial x = 0 \), show that the transfer matrix is

\[
M_x = \begin{pmatrix}
  \cos \theta & \rho \sin \theta \\
  -\frac{\sin \theta}{\rho} & \cos \theta
\end{pmatrix},
M_z = \begin{pmatrix}
  1 & \ell \\
  0 & 1
\end{pmatrix}
\]

where \( \theta \) is the bending angle, \( \rho \) is the bending radius, and \( \ell \) is the length of the dipole. Note that a sector magnet gives rise to horizontal focusing.

(b) When a particle enters a dipole at an angle \( \delta \) with respect to the normal edge of a dipole (see drawing below), there is a quadrupole effect. This phenomenon is usually referred to as edge focusing.\(^1\) We use the convention that \( \delta > 0 \) if the

\[^1\text{Using edge focusing, the zero-gradient synchrotron (ZGS) was designed and constructed in the 1960's at Argonne National Laboratory. The ZGS was made of 8 dipoles with a circumference of 172 m attaining the energy of 12.5 GeV. Its first proton beam was commissioned on Sept. 18, 1963. See L. Greenbaum, A Special Interest (Univ. of Michigan Press, Ann Arbor, 1971).}\]
particle trajectory is closer to the center of the bending radius. Show that the transfer matrices for the horizontal and vertical betatron motion due to the edge focusing are

\[ M_x = \begin{pmatrix} 1 & \frac{\tan \delta}{\rho} \\ \frac{\tan \delta}{\rho} & 1 \end{pmatrix} \quad M_z = \begin{pmatrix} 1 & \frac{\tan \delta}{\rho} \\ \frac{\tan \delta}{\rho} & 1 \end{pmatrix} \]

where \( \delta \) is the entrance or the exit angle of the particle with respect to the normal direction of the dipole edge. Thus the edge effect with \( \delta > 0 \) gives rise to horizontal defocussing and vertical focusing.

4. FODO cell in thin-lens approximation: A FODO cell (See the Figure below) is made of a pair of focusing and defocussing quadrupoles with or without dipoles in between, i.e.

\[ \begin{pmatrix} \frac{1}{2} QF & O & QD & O & \frac{1}{2} QF \end{pmatrix}, \]

where \( O \) represents either a dipole or a drift space \( L \). FODO cells are usually repetitively used for beam transport in arcs and transport lines.

A schematic plot of a FODO cell, where the transfer matrix for the dipoles (B) can be approximated by drift spaces, and QF and QD indicate the focusing and defocussing quadrupoles.

(a) Find the transfer matrix for the betatron motion in thin-lens approximation, assuming the focal lengths of the focusing and defocussing quadrupoles are \( f \) and \(-f\) with \( f = 3 \) m, and the length of the drift space between quadrupoles is \( L = 5 \) m.

(b) Find the evolution of particle phase space coordinates repeatedly passing through the FODO cell \( n = 1, 2, 3, 4, \ldots \) times.

(c) Plot the phase space trajectory turn-by-turn (Poincaré map), and determine the phase space area of the particle. What is the Poincaré map of this particle at a different location of the FODO cell?

(d) Assuming that an accelerator is made of 24-FODO cells, find the betatron tune. Find the betatron tune as a function of the focal length \( f \), and the half-cell length \( L \).

(e) Can you apply the Floquet theorem in simple FODO cell lattice? Explain.

5. Consider the Hill’s equation \( y'' + K(s)y = 0 \). Let the solution of the Hill’s equation be \( y = aw(s)e^{i\psi(s)} \), where \( w(s) \) and \( \psi(s) \) are respectively the amplitude and phase functions.
(a) Show that the equations for the amplitude and the phase functions are

\[
\begin{align*}
   w'' + K w - \frac{1}{w^3} &= 0, \\
   \psi' &= \frac{1}{w^2}.
\end{align*}
\]

(b) Show that the transfer matrix \(M(s_2|s_1)\) is given by Eq. (2.62) in Sec. II.3.

(c) Applying the Floquet theorem, and show that the transfer matrix is given by Eq. (2.67).

(d) Show that the betatron coordinate \(y\) can be expressed as Eq. (2.70). Use this result to prove the Courant-Snyder invariant shown in Eq. (2.97).

6. Show that \(\beta'' + 4\beta' K + 2\beta K' = 0\). Solve this equation for a drift space and a quadrupole respectively, and show that the solution of this equation must be one of the following forms:

\[
\begin{align*}
   \beta &= a + bs + cs^2, & \text{drift space} \\
   \beta &= a \cos 2\sqrt{K}s + b \sin 2\sqrt{K}s + c, & \text{focusing quadrupole} \\
   \beta &= a \cosh 2\sqrt{|K|}s + b \sinh 2\sqrt{|K|}s + c, & \text{defocussing quadrupole}.
\end{align*}
\]

(a) Express \(a, b, \) and \(c\) in terms of parameters \(\alpha_0, \beta_0\) and \(\gamma_0\) at the beginning of the element.

(b) In a drift space, where there are no quadrupoles, show that the betatron amplitude function is

\[
\beta(s) = \beta^* + \frac{(s - s^*)^2}{\beta^*},
\]

where \(\beta^*\) is the betatron function at the symmetry point \(s = s^*\) with \(\beta' = 0\). Show also that \(\gamma = (1 + \alpha^2)/\beta\) is equal to \(1/\beta^*\), i.e. \(\gamma\) is constant in a drift space.

(c) Show that the Courant–Snyder parameters \(\alpha_2, \beta_2, \gamma_2\) at \(s_2\) are related to \(\alpha_1, \beta_1, \gamma_1\) at \(s_1\) by

\[
\begin{pmatrix}
   \beta_2 \\
   \alpha_2 \\
   \gamma_2
\end{pmatrix} =
\begin{pmatrix}
   M_{11}^2 & -M_{11}M_{12} & M_{12}^2 \\
   -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\
   M_{21}^2 & -2M_{21}M_{22} & M_{22}^2
\end{pmatrix}
\begin{pmatrix}
   \beta_1 \\
   \alpha_1 \\
   \gamma_1
\end{pmatrix},
\]

where \(M_{ij}\) are the matrix elements of \(M(s_2|s_1)\). Use these equations to verify your solution to part (a).

7. The Courant–Snyder phase-space ellipse of a synchrotron is \(\gamma y^2 + 2\alpha y y' + \beta y'^2 = \epsilon\), where \(\alpha, \beta\) and \(\gamma\) are the Courant–Snyder parameters. If the injection optics is mismatched with \(\gamma_1 y^2 + 2\alpha_1 y y' + \beta_1 y'^2 = \epsilon\), find the emittance growth factor.\(^2\)

\(^2\)The easiest way to estimate the emittance growth is to transform the injection ellipse into the normalized coordinates of the ring optics. The deviation of the injection ellipse from a circle in the normalized phase space corresponds to the emittance growth.
(a) Transform the injection ellipse into the normalized coordinates of the ring lattice, and show that the injection ellipse becomes

\[
\left( \frac{\beta}{\beta_1} + \frac{(\alpha_1 \beta - \beta_1 \alpha)^2}{\beta \beta_1} \right) Y^2 + 2 \frac{\alpha_1 \beta - \alpha \beta_1}{\beta} Y P + \frac{\beta_1 P^2}{\beta} = \epsilon,
\]

where

\[ Y = \frac{1}{\sqrt{\beta}} y, \quad P = \frac{1}{\sqrt{\beta}} (\alpha y + \beta y'). \]

(b) Transform the ellipse to the upright orientation, and show that the major and minor axes of the ellipse are

\[ F_+ = \left( X_{mm} + \sqrt{X_{mm}^2 - 1} \right)^{1/2}, \quad F_- = \left( X_{mm} - \sqrt{X_{mm}^2 - 1} \right)^{1/2}, \]

where the mismatch factor \( X_{mm} \) is (see Exercise 2.2.14)

\[ X_{mm} = \frac{1}{2} (\gamma_1 \beta + \beta_1 \gamma - 2 \alpha_1 \alpha) = \frac{1}{2 \epsilon_{rms}} (\beta \sigma_{x'}^2 + \gamma \sigma_x^2 + 2 \alpha \sigma_{xx'}). \]

Note that the rms quantities \( \sigma_x, \sigma_{x'} \) and \( \sigma_{xx'} \) can be measured from the injected beam. What happens to the beam if the beam is injected into a perfect linear machine where there is no betatron tune spread? Show that the tune of the envelope oscillations is twice the betatron tune (see Exercise 2.2.15).

(c) In general, nonlinear betatron detuning arises from space-charge forces, nonlinear magnetic fields, chromaticities, etc. Because the betatron tune depends on the betatron amplitude, the phase-space area of the mis-injected beam will decohere and grow. Show that the emittance growth factor is

\[ F_+^2 = \left( X_{mm} + \sqrt{X_{mm}^2 - 1} \right)^{1/2}. \]

(d) Let the betatron amplitude function at the injection point be \( \beta_x = 17.0 \) m and \( \alpha_x = 2.02 \). The injection ellipse of a beam with emittance 5\( \pi \) mm-mrad is given by \( x^2/a^2 + x'^2/b^2 = 1 \), where \( a = 5.00 \) mm and \( b = 1.00 \) mrad. Find the final beam emittance after nonlinear decoherence.

8. Study the properties of the Cooler Injector Synchrotron lattice located at [http://physics.indiana.edu/ shylee/p570/cis.txt](http://physics.indiana.edu/ shylee/p570/cis.txt)