Solution:

1. Lorentz force provides the central force of circular motion:

\[ \vec{F} = q\vec{v} \times \vec{B} = \frac{\gamma mv^2}{\rho} \]

\[ B\rho = \frac{\gamma mv}{Ze} = \frac{p}{Ze} \]

where \( Z \) is the charge number of the particle. Using \( 1 \text{ [GeV/c]} = 10^9 \text{ [eV]} / (2.9979 \times 10^8 \text{ [m/s]}) = 3.3357 \times e \) [kg m/s], we obtain

\[ B\rho \text{ [Tm]} = \frac{3.3357p \text{ [GeV/c]}}{Z} = 3.3357 \left( \frac{p}{A} \text{ [GeV/c/u]} \right) \frac{A}{Z} \] \hspace{1cm} (1)

\[ \left( \frac{p}{A} \text{ [GeV/c/u]} \right) = 0.29979 \frac{Z}{A} B\rho \text{ [Tm]}. \] \hspace{1cm} (2)

Note that the momentum for the heavy ion is measured in unit of momentum per nucleon: \( p/A \), where \( A \) is the nucleon number and the momentum in the second equality has a unit of \([\text{GeV/c/u}]\), i.e. \([\text{GeV/c}]\) per nucleon. The relations between kinetic energy, total energy and momentum are

\[ T + m_0c^2 = \gamma m_0c^2 \rightarrow \gamma = \frac{T}{m_0c^2} + 1 \rightarrow cp = m_0c^2\sqrt{\gamma^2 - 1} \]

<table>
<thead>
<tr>
<th></th>
<th>( T ) [GeV]</th>
<th>( p ) [GeV/c]</th>
<th>( B\rho \text{ [T \cdot m]} )</th>
<th>( B ) [T]</th>
<th>( 2\pi\rho \text{ [m]} )</th>
<th>( B ) [T]</th>
<th>( 2\pi\rho \text{ [m]} )</th>
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<tr>
<td>IUCF</td>
<td>0.5</td>
<td>1.0901</td>
<td>3.6362</td>
<td>1.7</td>
<td>13.439</td>
<td>-</td>
<td>-</td>
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<tr>
<td>RHIC</td>
<td>-</td>
<td>250</td>
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<td>1.7</td>
<td>3082.2</td>
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<td>12329</td>
<td>5</td>
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<tr>
<td>LHC</td>
<td>-</td>
<td>7000</td>
<td>23350</td>
<td>1.7</td>
<td>86301</td>
<td>8.0</td>
<td>18339</td>
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</table>

<table>
<thead>
<tr>
<th>( B\rho \text{ [Tm]} )</th>
<th>electron ( p ) [GeV/c]</th>
<th>proton ( p ) [GeV/c]</th>
<th>proton ( E ) (GeV)</th>
<th>( \text{Au}^{79+} ) ( E/A ) (GeV/u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.30</td>
<td>0.30</td>
<td>0.985</td>
<td>0.94</td>
</tr>
<tr>
<td>10</td>
<td>3.0</td>
<td>3.00</td>
<td>3.14</td>
<td>1.52</td>
</tr>
<tr>
<td>100</td>
<td>29.98</td>
<td>29.98</td>
<td>29.99</td>
<td>12.06</td>
</tr>
<tr>
<td>1000</td>
<td>299.79</td>
<td>299.79</td>
<td>299.79</td>
<td>120.22</td>
</tr>
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</table>

The momentum and energy is normally measured in units of \([\text{GeV/c/u}]\) and \([\text{GeV/u}]\) for heavy ion colliders. In the above exercise, \( \text{Au}^{79+} \) stands for gold ion having charge \( Z = 79 \) unit, and nucleon number \( A = 197 \). The energy and momentum are related by

\[ E^2 = M^2c^4 + p^2c^2, \quad \frac{E}{A} = \sqrt{\left( \frac{M}{A} \right)^2 c^4 + \left( \frac{p}{A} \right)^2 c^2} \approx \sqrt{u^2c^4 + \left( \frac{p}{A} \right)^2 c^2}. \]

We find \( M = 196.97u \) for the gold ion, where \( u \) is the atomic mass unit with \( uc^2 = 0.931494 \text{ GeV} \).
2. This exercise shows that at low energy electric field is more effective, while at high energy, magnetic field is preferred. Note that $1 \text{ N} = 6.24 \times 10^{18} \text{ eV/m}$!

<table>
<thead>
<tr>
<th>$\beta = v/c$</th>
<th>$F_{E=1} \text{ MV/m} [\text{N}]$</th>
<th>$F_{B=1} \text{ T} [\text{N}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>$1.602 \times 10^{-13}$</td>
<td>$4.803 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.01</td>
<td>$1.602 \times 10^{-13}$</td>
<td>$4.803 \times 10^{-14}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$1.602 \times 10^{-13}$</td>
<td>$4.803 \times 10^{-12}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$1.602 \times 10^{-13}$</td>
<td>$2.401 \times 10^{-11}$</td>
</tr>
<tr>
<td>1</td>
<td>$1.602 \times 10^{-13}$</td>
<td>$4.803 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

3. In a classical cyclotron, the rf cavity operates at a constant cyclotron frequency $\omega_0 = eB/mc$. On the other hand, the synchrotron frequency $\omega = eB/\gamma mc = \omega_0/\gamma$ depends on the relativistic energy factor $\gamma$. Particles gain energy at the Dees, where rf voltage is applied to accelerate particles.

(a) Let $V_{rf} = V \cos \omega_0 t$ be the sinusoidal voltage applied to the dees, and let $\psi$ be the rf phase of the particle. The energy gain rate is

$mc^2 \frac{d\gamma}{dt} = \frac{\Delta E}{\Delta t} = \frac{2eV_{rf}}{2\pi/\omega} = \frac{\omega eV \cos \psi}{\pi}$

The equations of motion in a uniform acceleration approximation are

$\frac{d\gamma^2}{dt} = 2\gamma \frac{d\gamma}{dt} = 2\gamma \omega eV \cos \psi \frac{mc^2\pi}{\pi mc^2} = \frac{2\omega_0 eV}{\gamma mc^2} \cos \psi \equiv a \cos \psi$

$d\psi/dt = \omega - \omega_0 = (\gamma^{-1} - 1)\omega_0$,

where $a = 2\omega_0 eV/\gamma mc^2$ and $e$ and $m$ are the charge and mass of the particle.

Note that the synchrotron frequency slips with respect to the rf frequency.

(b) Defining a variable $q = a \cos \psi$, then

$\frac{dq}{dt} = -a \sin \psi \frac{d\psi}{dt} = -\sqrt{a^2 - q^2} \left( \frac{1}{\gamma} - 1 \right) = \sqrt{a^2 - q^2} \left( 1 - \frac{1}{\gamma} \right)$

$\frac{dq}{dt} = \frac{dq}{d\gamma} \frac{d\gamma}{dt} = \frac{dq}{d\gamma} 2\gamma = \sqrt{a^2 - q^2} \left( 1 - \frac{1}{\gamma} \right)$

where we use the equation of motion: $\frac{d\gamma^2}{dt} = q$. Thus we obtain:

$q \sqrt{a^2 - q^2} dq = 2(\gamma - 1)\omega_0 d\gamma$.

Integrating this equation from initial state where $\psi = \pi/2$ and $\gamma = 1$ to maximum energy state where $\psi = 0$:

$\int_0^{\alpha} \frac{q}{\sqrt{a^2 - q^2}} dq = -\sqrt{a^2 - q^2}^{\alpha}_0 = a$

$\int_1^{\gamma_m} (2\gamma - 2)\omega_0 d\gamma = (\gamma_m - 1)^2 \omega_0$
Then the maximum \( \gamma_m \) is

\[
\gamma_m = 1 + \sqrt{a/\omega_0}, \quad T_m = (\gamma_m - 1)mc^2 = \sqrt{a/\omega_0}mc^2 = \sqrt{2eVmc^2/\pi}
\]

We find that the maximum attainable kinetic energy is \( T_m = \sqrt{2eVmc^2/\pi} \). Substituting \( V \approx 250 \text{kV/turn} \), we find that the maximum attainable kinetic energy is about 12 MeV. This limit was alleviated by the invention of isochronous principle in cyclotron by L.H. Thomas.\(^5\)

4. The energy lost due to synchrotron radiation is propotional to \( \gamma^4 \). At high energy electron has a much larger \( \gamma \) than proton. For high energy electron rings, we need to compensate this energy lost.

<table>
<thead>
<tr>
<th>Particle</th>
<th>E</th>
<th>( U_0 ) [GeV/turn]</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEP ( e )</td>
<td>50GeV</td>
<td>0.1785</td>
</tr>
<tr>
<td>LEP ( e )</td>
<td>100GeV</td>
<td>2.856</td>
</tr>
<tr>
<td>SSC ( p )</td>
<td>20TeV</td>
<td>0.0001222</td>
</tr>
</tbody>
</table>

5. The \( H \) field inside the iron plates of infinity permeability is 0. Using Ampere’s law, we find \( H = NI/g \) where \( H \) is the magnetic field in the gap. Thus, \( B = \mu_0H = \mu_0NI/g \). For a given current \( I \) in the coil the total magnetic flux is \( \Phi = NBlw \), where \( w \) is the width and \( \ell \) is the length of the magnet. The inductance is then

\[
L = \frac{\Phi}{I} = \frac{\mu_0N^2lw/g}{g} = \mu_0n^2V,
\]

where \( V = lwg \) is the volume of the dipole and \( n = N/g \) is the number of turns per unit length.

6. We define \( x = r - R \), where \( |x| \ll R \), and expand \( B_z \) as

\[
B_z = B_0 \left( \frac{r}{R} \right)^{-n} = B_0 \left( 1 + \frac{x}{R} \right)^{-n} \approx B_0 \left( 1 - n \frac{x}{R} \right)\,.
\]

From the relation \( \nabla \times \vec{B} = 0 \), the radial component is given by

\[
\frac{\partial B_r}{\partial z} = \frac{\partial B_z}{\partial r} \bigg|_{r=R} \approx -\frac{nB_0}{R} + \cdots, \quad B_r = -\frac{nB_0}{R}z + \cdots,
\]

where we assume \( B_r = 0 \) at \( z = 0 \). We define \( \xi = (r - R)/R \), and \( \zeta = z/R \) and obtain \( r = (1 + \xi)R \), and \( \dot{\theta} = v/r = \omega_0R/r = \omega_0/(1 + \xi) \approx \omega_0(1 - \xi) \). \( \gamma mv^2/R = evB_0 \).

(a) The radial equation of motion is given by $dp_r/dt - \gamma m r \dot{\theta}^2 = -e r \dot{\theta} B_z$, where $v = r \dot{\theta}$ and $p_r = \gamma m r = \gamma m R \dot{\xi}$. Thus we obtain $\gamma m R \ddot{\xi} - \frac{\gamma m v^2}{R(1 + \xi)} = -e v B_0 (1 - n \xi)$. Or

$$\ddot{\xi} + \omega_0^2 (1 - n) \xi = 0,$$

where $\omega_0 = v/R$.

(b) Similarly, the vertical equation motion: $dp_z/dt = e r \dot{\theta} B_r = -e v B_0 n \zeta$ gives

$$\ddot{\zeta} + \omega_0^2 n \zeta = 0.$$

The stablility condition for the classical cyclotron and the weak-focusing synchrotron is $1 > n > 0$.

7. If we use normalized potential $V > 0$, $V$ is just opposite to the actual voltage on the anode for the ion extraction. In the non-relativistic case, the energy of the ion is

$$(1/2) mv^2 = eV, \quad \text{or} \quad v = \sqrt{2eV/m}.$$

The charge density is $\rho = J/v = J (m/2e)^{1/2} V^{-1/2}$, where the current density $J$ is constant. Thus the 1-D Poisson equation becomes

$$\frac{d^2 V}{ds^2} = \frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0} \left( \frac{m}{2e} \right)^{1/2} V^{-1/2}.$$

For a space charge dominated beams, the electric potential is fully compensated by the space-charge field, i.e. $V = 0$ and $dV/ds = 0$ at $s = 0$; and $V = V_0$ at $s = a$. Using the ansatz $V = ks^n$ to solve the Poisson equation, we obtain $n = 4/3$ and $k = (9A/4)^{2/3}$, where $A = (m/2e)^{1/2} J/\epsilon_0$. Thus, we obtain Child’s law:

$$J = \frac{4\epsilon_0}{9} \left( \frac{2e}{m} \right)^{1/2} \frac{V_0^{3/2}}{a^2}.$$

8. In paraxial geometry, we use the cylindrical coordinate system. The electri potential is a function of $r$ and $s$ only. The Poisson equation becomes

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V}{\partial r} \right) + \frac{\partial^2 V}{\partial s^2} = 0.$$

Let the ansatz of the electric potential be

$$V(r, s) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots.$$

Substituting the ansatz to the Poisson equation, we find

$$a_1 = a_3 = 0, \quad a_2 = -\frac{1}{4} a_0^{(2)}, \quad a_4 = \frac{1}{4^3} a_0^{(4)}, \cdots.$$
Thus the electric potential $V(r, s)$ and the electric field $\vec{E} = -\nabla V = E_r \hat{r} + E_s \hat{s}$ are

$$V(r, s) = V_0(s) - \frac{V_0^{(2)}}{4} r^2 + \frac{V_0^{(4)}}{64} r^4 + \ldots,$$

$$E_r = \frac{V_0^{(2)}}{2} r - \frac{V_0^{(4)}}{16} r^3 + \ldots,$$

$$E_s = -V_0^{(1)} + \frac{V_0^{(3)}}{4} r^2 - \frac{V_0^{(5)}}{64} r^4 + \ldots,$$

where $V_0^{(n)}$ correspond to $n$th-derivative of $V_0$ with respect to $s$. Now, we consider the equation of motion. $m(\ddot{r} \hat{r} + \ddot{s} \hat{s}) = e \vec{E}$, and find

$$m \ddot{s} = e \left( -V' + \frac{V''}{4} r^2 + \frac{V^{(5)}}{64} r^4 + \ldots \right),$$

$$m \ddot{r} = e \left( \frac{V''}{2} r + \frac{V^{(7)}}{16} r^3 + \ldots \right),$$

where $V = V_0$ (without the subscript) is used to simplify the notation. Now, we change the independent coordinate $t$ to $s$, and find

$$\ddot{r} = \frac{d}{dt} \frac{dr}{dt} = \frac{d}{dt} \left( \frac{ds}{dt} \frac{r'}{dt} \right) = \frac{d^2 s}{dt^2} r' + \left( \frac{ds}{dt} \right)^2 r''.$$

Using the equation of motion, and using the fact that the kinetic energy is $m \left( \frac{ds}{dt} \right)^2 = -eV$, we find

$$V r'' + \frac{1}{2} V' r' + \frac{1}{4} V'' r = 0,$$

where higher order terms are neglected. This is known as the paraxial ray equation.