VIII Introduction to Linear Accelerators

VIII.1 Historical Milestones

1. In 1924 G. Ising published a first theoretical paper on the acceleration of ions by applying a time varying electric field to an array of drift tubes via transmission lines;
2. 1928 R. Wideröe used a 1 MHz, 25 kV rf source to accelerate potassium ions up to 50 keV. The optimal choice of the distance between acceleration gaps is $d = \frac{\beta \lambda}{2} = \frac{\beta c}{2f}$, where $d$ is the distance between drift tube gaps, $\beta c$ is the velocity of the particle, and $\lambda$ and $f$ are the wavelength and frequency of the rf wave. In 1931–34 E.O. Lawrence, D. Sloan et al., at U.C. Berkeley, built a Wideröe type linac to accelerate Hg ions to 1.26 MeV using an rf frequency of about 7 MHz. At the same time (1931-1935) K. Kingdon at the General Electric Company and L. Snoddy at the University of Virginia, and others, accelerated electrons from 28 keV to 2.5 MeV.

To minimize the length of the drift region, which does not provide particle acceleration, a higher frequency rf source is desirable. For example, the velocity of a 1 MeV proton is $v = \beta c = 0.046 c$, and the length of drift space in a half cycle at rf frequency $f_r = 7$ MHz is $0.5v/f_r = 1$ m. As the energy increases, the drift length becomes too long. The solution is to use a higher frequency system, which became available from radar research during WWII. However, the accelerator is almost capacitive at high frequency, and it radiates a large amount of power $P = IV = \omega CV^2$, where $V$ is the accelerating voltage, $I = \omega CV$ is the displacement current, $C$ is the capacitance between drift tubes, and $\omega$ is the angular frequency. At high frequency, the Wiederoe linac will produce high power loss.

The solution is to enclose the gap between the drift tubes in a cavity that holds the electromagnetic energy in the form of a magnetic field by introducing an inductive load to the system. To attain a high gradient, the cavity must be designed such that the resonant frequency is equal to the frequency of the accelerating field. A cavity is a structure in which electromagnetic energy can be resonantly stored. A cavity or a series of cavities can be fed by an rf source.

In 1937 the Varian brothers invented the klystron at Stanford. Similarly, high power magnetrons were developed in Great Britain.

When two or more cavity gaps are adjacent to each other, the cavity can be operated at π-mode or 0-mode. In 0-mode, the resulting current is zero at the common wall so that the common wall is useless. Thus a group of drift tubes can be placed in a single resonant tank, where the field has the same phase in all gaps. Such a structure was invented by L. Alvarez in 1945.

In 1945–47 L. Alvarez, W.K.H. Panofsky, et al., built a 32 MeV, 200 MHz proton drift tube linac (DTL). Drift tubes in the Alvarez structure are in one large cylindrical tank and powered at the same phase. The distances between the drift tubes, $d = \beta \lambda$, are arranged so that the particles, when they are in the decelerating phase, are shielded from the fields.

In 1945 E.M. McMillan and V.I. Veksler discovered the phase focusing principle, and in 1952 J. Blewett invented electric quadrupoles for transverse focusing based on the alternating gradient focusing principle. These discoveries solved the 3D beam stability problem, at least for low intensity beams. Since then, Alvarez linacs has commonly been used to accelerate protons and ions up to 50–200 MeV kinetic energy.

In the ultra relativistic regime with $\beta \rightarrow 1$, cavities designed for high frequency operation are usually used to achieve a high accelerating field. At high frequencies, the klystron, invented in 1937, becomes a powerful rf power source. In 1947-48 W. Hansen et al., at Stanford, built the MARK-I disk loaded linac yielding 4.5 MeV electrons in a 9 ft structure powered by a 0.75 MW, 2.856 GHz magnetron. On September 9, 1967, the linac at Stanford Linear Accelerator Center (SLAC) accelerated electrons to energies of 20 GeV. In 1973 P. Wilson, D. Farkas, and H. Hogg, at SLAC, invented the rf energy compression scheme SLED (SLAC Energy Development) that provided the rf source for the SLAC linac to reach 30 GeV. In 1990’s, SLAC has achieved 50 GeV in the 3 km linac.
Another important idea in high energy particle acceleration is acceleration by traveling waves. The standing wave cavity in a resonant structure can be decomposed into two traveling waves: one that travels in synchronism with the particle, and the backward wave that has no net effect on the particle. Thus the shunt impedance of a traveling wave structure is twice that of a standing wave structure except at the phase advances 0 or π.

To regain the factor of two in the shunt impedance for standing wave operation, E. Knapp and D. Nagle invented the side coupled cavity in 1964. In 1972 E. Knapp et al. successfully operated the 800 MHz side coupled cavity linac (CCL) to produce 800 MeV energy at Los Alamos. In 1994 the last three tanks of the DTL linac at Fermilab were replaced by CCL to upgrade its proton energy to 400 MeV. Above β>0.3, CCL has been widely used for proton beam acceleration.

For the acceleration of ions, the Alvarez linac is efficient for β>0.04. The acceleration of low energy protons and ions relies on DC accelerators such as the Cockcroft-Walton or Van de Graaff. In 1980, R. Stokes et al. at Los Alamos succeeded in building an RFQ to accelerate protons to 3 MeV. Today RFQ is commonly used to accelerate protons and ions for injection into linacs or synchrotrons.

Since the first experiment on a superconducting linear accelerator at SLAC in 1965, the superconducting (SC) cavity has become a major branch of accelerator physics research. In the 1970’s, many SC post linacs were constructed for heavy ions. In 1980’s, more than 180 m of superconducting cavities have been installed in CEBAF for the 4 GeV continuous electron beams used in nuclear physics research. More than 400 m of SC cavities beyond 7 MV/m were installed in LEP energy upgrade, and reached 3.6 G V rf voltage for the operation of 104.5 GeV per beam in 2000. The TESLA project had also successfully achieved an acceleration gradient of 35 MV/m.

### LEP Accelerating Cavities

<table>
<thead>
<tr>
<th>Date</th>
<th>Copper Film</th>
<th>Niobium</th>
<th>Solid Niobium</th>
<th>( V_0 ) (MV)</th>
<th>E (GeV)</th>
<th>( U_0 ) (GeV)</th>
</tr>
</thead>
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<td>0</td>
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<td>45</td>
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<td>Nov 1995</td>
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<td>56</td>
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<td>750</td>
<td>70</td>
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<tr>
<td>June 1996</td>
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<td>140</td>
<td>4</td>
<td>1600</td>
<td>80.5</td>
<td>1.200</td>
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<td>160</td>
<td>16</td>
<td>1900</td>
<td>86</td>
<td>1.564</td>
</tr>
<tr>
<td>May 1997</td>
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<td>224</td>
<td>16</td>
<td>2500</td>
<td>91.5</td>
<td>2.004</td>
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<tr>
<td>May 1998</td>
<td>48</td>
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<td>16</td>
<td>2750</td>
<td>94.5</td>
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<td>16</td>
<td>2900</td>
<td>96</td>
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<td>16</td>
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<td>104.5</td>
<td>3.409</td>
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</tbody>
</table>

(Source: CERN courier)

### VIII.2 Fundamental Properties of Accelerating Structures

#### A. Transit time factor

We consider a standing wave accelerating gap, e.g. the Alvarez structure, and assume that the electric field in the gap is independent of the longitudinal coordinate s. If \( E \) is the maximum electric field at the acceleration gap, the accelerating field and the energy gain in traversing the accelerating gap are

\[
\Delta E = e \int_{-\frac{L}{2}}^{\frac{L}{2}}  E \cos \frac{\omega s}{v} ds = eEgT_{tr} = eV_0, \quad T_{tr} = \frac{\sin (\pi g/\beta \lambda)}{\pi g/\beta \lambda},
\]

where \( V_0=EgT_{tr} \) is the effective voltage of the gap, \( T_{tr} \) is the transit time factor, \( \lambda=2\pi c/\omega \) is the rf wavelength, and \( \pi g/\beta \lambda \) is the rf phase shift across the gap. The overall transit time factor for standing wave structures in DTL is about 0.8. The transit time factor is valid for the standing wave structure. The transit time factor for particle acceleration by a traveling wave is given in an example in Exercise 3.8.7.

#### B. Shunt impedance

Neglecting power loss to the transmission line and reflections between the source and the cavity, electromagnetic energy is consumed in the cavity wall and beam acceleration. The shunt impedance for an rf cavity is defined as

\[
R_{sh} = \frac{V_0}{P_d},
\]

where \( V_0 \) is the effective acceleration voltage, and \( P_d \) is the dissipated power. For a multi-cell cavity structure, it is also convenient to define the shunt impedance per unit length \( r_{sh} \) as

\[
r_{sh} = \frac{R_{sh}}{L_{cav}} = \frac{\pi g^2}{P_d/L_{cav}}, \quad \text{or} \quad \frac{dP_d}{ds} = \frac{\pi g^2}{r_{sh}}
\]

where \( E \) is the effective longitudinal electric field that includes the transit time factor, and \( dP_d/ds \) is the power dissipation per unit length. The power per unit length needed to maintain an accelerating field \( E \) is \( P_d/L=E^2/r_{sh} \) and the accelerating gradient for low beam intensity is \( E=(r_{sh}P_d/L_{cav})^{1/2} \). For a 200 MHz proton linac, we normally have \( r_{sh}=15-50 \) MΩ/m, depending on the transit time factors. At 10 GHz, \( r_{sh}>100 \) MΩ/m. The shunt impedance is generally proportional to \( \omega^{1.2} \). A high shunt impedance with low surface fields is an important guideline in rf cavity design. For example, using a 50 MW high peak power pulsed klystron, the accelerating gradient of a 10 GHz cavity may achieve as high as 70 MV/m. The working SLC S-band accelerating structure delivers about 20 MV/m.
C. The quality factor $Q$

The quality factor is defined by $Q = \frac{\omega}{W_{st}/P_d} \quad dW_{st}/dt = -P_d = -\omega W_{st}/Q,$

where $W_{st}$ is the maximum stored energy. In general, the $Q$-factor of an accelerating structure is independent of whether it operates in standing wave or traveling wave modes.

For standing wave operation, the time for the field to decay to $1/e$ of its initial value is called the filling time of a standing wave cavity,

$$t_{F,sw} = 2Q\frac{L}{\omega}$$

For a traveling wave structure, we define the stored energy per unit length as $w_{st} = W_{st}/L_{cav}$, and the power loss per unit length becomes

$$\frac{dP_d}{ds} = -\frac{\omega w_{st}}{Q}, \quad \text{or} \quad Q = -\frac{\omega w_{st}}{dP_d/ds}.$$ 

The filling time for a traveling wave structure is $t_{F,tw} = L_{cav}/v_g$, where $L_{cav}$ is the length of the cavity structure and $v_g$ is the velocity of the energy flow. A useful quantity is the ratio $R_{sh}/Q$:

$$v_g = P/w_{st} \quad \frac{R_{sh}}{Q} = \frac{V_0^2}{\omega w_{st}}, \quad \text{or} \quad \frac{r_{sh}}{Q} = \frac{(V_0/L_{cav})^2}{\omega (W_{st}/L_{cav})} = \frac{\varepsilon^2}{\omega w_{sh}}.$$ 

VIII.3 Particle Acceleration by EM Waves

rf cavities for particle acceleration can be operated in **standing wave** or **traveling wave** modes. Standing wave cavities operating at steady state are usually used in synchrotrons and storage rings for beam acceleration or energy compensation of synchrotron radiation energy loss. The standing wave can also accelerate oppositely charged beams traveling in opposite directions. Its high duty factor can be used to accelerate long pulsed beams such as protons, and continuous wave (CW) electron beams in the Continuous Electron Beam Accelerator Facility (CEBAF).

Employing high power pulsed rf sources, a traveling wave structure can attain a very high gradient for the acceleration of an intense electron beam pulse.

A. EM waves in a cylindrical wave guide

$$E_s = E_0 j_0(k_r r) e^{-j[k_s - \omega t]},$$

$$E_r = \frac{k}{k_r} E_0 j_1(k_r r) e^{-j[k_s - \omega t]},$$

$$H_\phi = \frac{\omega}{cZ_0 k_r} E_0 j_1(k_r r) e^{-j[k_s - \omega t]},$$

$$E_\phi = 0, \quad H_s = 0, \quad H_r = 0,$$

where $(r, \phi)$ is the cylindrical coordinate, $s$ is the longitudinal coordinate, $k$ is the propagation wave number in the $s$ direction, $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the vacuum impedance, and $k^2 = (\omega/c)^2 - k_r^2$.

Left: Schematic of a cylindrical cavity. Right: Dispersion curve $(\omega/c)^2 = k^2 + (2.405/b)^2$ for the $TM_{01}$ wave. The phase velocity $\omega/k$ for a wave without cavity load is always greater than the velocity of light.

At high frequencies, where $k_r \to 0$, the phase velocity approaches the speed of light. However, the longitudinal component of the EM wave vanishes.

<table>
<thead>
<tr>
<th>$\omega/c$</th>
<th>$v_p = c$</th>
<th>$v_p = c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega/c = \sqrt{k^2 + (2.405/b)^2}$</td>
<td>$\omega_c = k_r c = 2.405c/b$</td>
<td></td>
</tr>
</tbody>
</table>

Unattenuated wave propagation at $\omega < \omega_c$ is not possible.

At high frequency, the phase velocity approaches $c$. However, the electromagnetic field is transverse; it becomes the transverse TEM wave, i.e.

$$\frac{E_s}{E_r} = \frac{j}{k} \to 0, \quad \frac{H_\phi}{H_r} = \frac{\omega}{c k Z_0} \to \frac{1}{Z_0}.$$ 

The propagation modes are determined by the boundary condition for $E_z = E_\phi = 0$ at the pipe radius $r = b$, i.e., $k_{m,n} = j_{m,n}/b$, where $j_{m,n}$ are zeros of the Bessel functions $J_m(j_{m,n}) = 0$ listed in the Table.

<table>
<thead>
<tr>
<th>$\omega/c$</th>
<th>$v_p = c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = \omega/c \left[ 1 - \left( \frac{\omega}{\omega_c} \right)^2 \right]^{1/2}$</td>
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</tbody>
</table>

Ordered zeros $j_{m,n}$ of $J_m(X)$

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
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<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.405</td>
<td>3.832</td>
<td>5.136</td>
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<td>2</td>
<td>5.520</td>
<td>7.016</td>
<td>8.417</td>
</tr>
<tr>
<td>3</td>
<td>8.654</td>
<td>10.173</td>
<td>11.620</td>
</tr>
<tr>
<td>4</td>
<td>11.792</td>
<td>13.324</td>
<td>14.796</td>
</tr>
</tbody>
</table>
B. Phase velocity and group velocity

For a quasi-monochromatic pulse at frequency $\omega_0$ in free space, the electric field can be represented by

$$ E(t, s) = A(t)e^{i(\omega_0 t - k_0 s + \delta)} $$

where $A(t)$ is the amplitude with a short time duration and we include the dispersion of the wave number in this equation. For a quasi-monochromatic wave at the angular frequency $\omega_0$, we expand the dispersion wave number around $\omega_0$:

$$ k(\omega) = k(\omega_0) + \frac{dk}{d\omega_0}(\omega - \omega_0) = k_0 + k' (\omega - \omega_0). $$

Thus the electric field in a dispersive medium is $E(t, s) = A(t - k' s)e^{i(\omega_0 t - k_0 s)}$, where the phase of the pulse propagates at a “phase velocity” of $v_p = \omega_0/k_0$, and the amplitude function of the EM pulse propagates at the “group velocity” $v_g = \frac{d\omega}{dk} = \frac{1}{k'}$.

For a single-mode wave propagation with $k^2 = (\omega/c)^2 - k_r^2$, we find $v_g = c^2/\omega$, or $v_g = c^2$. In fact, the group velocity is equal to the velocity of energy flow in the wave guide.

C. TM modes in a cylindrical pillbox cavity

The standing wave solution of a closed pillbox cavity without beam holes with a time dependent factor $e^{i\omega t}$, the TM is

$$ E_\pi = Ck_r^2 J_m(k_r r) \sin m\phi \cos ks, $$

$$ E_\theta = -Ck_r J_m'(k_r r) \sin m\phi \sin ks, $$

$$ H_\varphi = Cnr^2 J_m(k_r r) \sin m\phi \cos ks, $$

$$ H_\theta = -Cn J_m'(k_r r) \sin m\phi \cos ks, $$

where $d$ is the length of the pillbox, $k_d$ is the phase advance of the EM wave in the cavity cell. We also use the boundary conditions $E_\pi = 0$ and $E_\theta = 0$ at $s = 0$ and $d$, we obtain $kd = p\pi$, where $b$ is the inner radius of the cylinder, and $j_m$ are zeros of the Bessel functions $J_m(j_m) = 0$. The resonance frequency $\omega$ for the TM$_{mnp}$ mode is

$$ \frac{\omega_{mnp}}{c} = \sqrt{\frac{j_m^2}{b^2} + \frac{p^2\pi^2}{d^2}}. $$

The solid lines are the dispersion curves of frequency $f$ vs phase shift $kd$ for TM$_{mnp}$ modes of a SLAC-like pillbox cavity with $a=18$ mm, $b=43$ mm, and $d=34.99$ mm. Because of the coupling between adjacent pillbox-cavities, the discrete mode frequencies become a continuous function of the phase advance $kd$, and the phase-velocity is effectively lowered. The dashed lines show the world line $v_p = c$.

<table>
<thead>
<tr>
<th>$b$ (mm)</th>
<th>$d$ (mm)</th>
<th>$kd$ (deg)</th>
<th>$f$ (GHz)</th>
<th>$R_{\text{eh}}$ (M$\Omega$)</th>
<th>Q</th>
<th>$e_{\text{eh}}$ (M$\Omega$/m)</th>
</tr>
</thead>
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<tr>
<td>42.475</td>
<td>17.495</td>
<td>60</td>
<td>2.8579</td>
<td>5.1067</td>
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<tr>
<td>41.805</td>
<td>30.616</td>
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<td>1.550</td>
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<td>50.92</td>
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<tr>
<td>41.685</td>
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<td>1.874</td>
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<td>41.415</td>
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<td>2.416</td>
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<td>41.290</td>
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<td>2.857</td>
<td>2.460</td>
<td>17464</td>
<td>46.98</td>
</tr>
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</table>
D. Alvarez structure

The Alvarez linac cavity resembles the TM_{010} standing wave mode. The tank radius and other coupling structures, such as rods and slugs inside the cavity, are designed to obtain a proper resonance frequency for the TM_{010} mode, and thus we have \( b \approx \frac{2.405c}{\omega} \). The resulting electric field is independent of \( s \). The total length is designed to have a distance \( \beta \lambda \) between two adjacent drift tubes (cells), where \( \beta \) is the speed of the accelerating particles. Since \( \beta \) increases along the line, the distance between drift tubes increases as well. The Table below shows some properties of an Alvarez linac, the SLAC cavity, and the CEBAF cavity.

<table>
<thead>
<tr>
<th>Machine</th>
<th>( f ) (MHz)</th>
<th>( b ) (cm)</th>
<th>( d ) (cm)</th>
<th>( N_{cell} )</th>
<th>( E ) (MV/m)</th>
</tr>
</thead>
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<tr>
<td>Alvarez linac</td>
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<td>57.0</td>
<td>( \sum \beta \lambda )</td>
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<td>1.60</td>
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<td>744</td>
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<td>55</td>
<td>2.0</td>
</tr>
<tr>
<td>Fermilab (cavity2)</td>
<td>45</td>
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<td>59</td>
<td>2.0</td>
</tr>
<tr>
<td>CEBAF SC cavity</td>
<td>1497</td>
<td>7.66</td>
<td>10.0</td>
<td>55</td>
<td>5 – 10</td>
</tr>
<tr>
<td>SLAC linac</td>
<td>2856</td>
<td>4.2</td>
<td>3.5</td>
<td>( \approx 100 )</td>
<td>20</td>
</tr>
</tbody>
</table>

\[ \frac{\omega}{k} = c, \quad k = \frac{\omega}{c} \]

For particle beam acceleration, we consider the TM guided wave, where E-field is in the direction of beam momentum. We observed in Sec. D that the propagating wave in an unloaded cylindrical wave guide has phase velocity \( v_p > c \). The phase velocity must be brought to the level of the particle velocity, i.e. \( v_p = c \). A simple method of reducing the phase velocity is to load the structure with disks, or washers. The size of the beam hole determines the degree of coupling and the phase shift from one cavity to the next. When the \( a, b \) parameters of the disk radii are tailored correctly, the phase change from cavity to cavity along the accelerator gives an overall phase velocity that is equal to the particle velocity. With the Floquet theorem for the periodic wave guide, the EM wave of an infinitely long disk loaded wave guide is

\[ E_\phi(r, \phi, s, t) = e^{-j[k_0s - \omega t]} E_\phi(r, \phi, s), \quad H_\phi(r, \phi, s, t) = e^{-j[k_0s - \omega t]} H_\phi(r, \phi, s), \]
\[ E_\phi(r, \phi, s + d) = E_\phi(r, \phi, s), \quad H_\phi(r, \phi, s + d) = H_\phi(r, \phi, s), \]
\[ E_\phi(r, \phi, s, t) = e^{-j[k_0s - \omega t]} \sum_{q=-\infty}^{\infty} E_{q\phi}(r, \phi)e^{-j2\pi q \phi/d} = e^{j\omega t} \sum_{q=-\infty}^{\infty} E_{q\phi}(r, \phi)e^{-j\omega q}, \]

where \( d \) is the period of the wave guide, the wave number is \( k_q = k_0 + 2\pi q/d \) (\( q = \text{integer} \)) for the \( q \)th “space harmonic”, and \( k_0 \) is the propagation wave number of the “fundamental space harmonic.” We note further that as \( k_0d \rightarrow 0 \) or \( \pi \), forward and backward traveling branches coincide and they will contribute to enhance the electric field.

E. Loaded wave guide chain and the space harmonics

If the wave guide is loaded with wave reflecting structures such as iris, nosecone, etc., the propagating EM waves can be reflected by obstruction disks. The reflected waves for a band of frequencies interfere destructively so that there is no radial field at the irises.

Since the irises play no role in wave propagation, this gives rise to a minor perturbation in the propagating wave. At some frequencies the reflected waves from successive irises are exactly in phase so that the irises force a standing wave pattern. At these frequencies, unattenuated propagation is impossible, so that the EM wave becomes a standing wave and the group velocity again becomes zero, i.e. the phase advance \( kd = \pi \). Such a chain of loaded wave guides can be used to slow the phase velocity of EM waves.

The field components of the lowest TM_{00} mode with cylindrical symmetry become

\[ E_\phi = \sum_{q} E_{0q} J_0(k_q r)e^{-j[k_q s - \omega t]}, \]
\[ E_\phi = \sum_{q} k_q E_{0q} J_1(k_q r)e^{-j[k_q s - \omega t]}, \]
\[ E_\phi = \sum_{q} \frac{k_0}{k_q} E_{0q} J_1(k_q r)e^{-j[k_q s - \omega t]}, \]

where the wave number and the phase velocity at a given frequency \( \omega \) are

\[ k_q^2 = \left( \frac{\omega}{c} \right)^2 - k_{0q}^2, \quad v_{p,q} = \frac{\omega}{k_q} = \frac{\omega}{k_0 + 2\pi q/d}. \]

Note that \( k_{0q} = 0 \) and \( J_0(k_q r) = 1 \) for \( v_{p,q} = c \). This indicates that the electric field of the \( q \)th space harmonic is independent of the transverse position. One may wonder how to reconcile the fact that the tangential electric field component \( E_\phi \) must be zero at \( r = b \). The statement that the electric field is independent of transverse position is valid only near the center axis of loaded wave-guide structures.
The dispersion curve of a periodic loaded wave-guide structure (or slow wave structure) is a typical Brillouin-like diagram. The solid lines are forward traveling wave, and the branches with dashed dots are backward traveling wave.

The dispersion curve is a simple translation of $2\pi/d$, and these curves must join, they must have zero slope at the lower frequency $\omega_0/c$, where $k_0d = 0$, and at the upper frequency $\omega_\pi/c$, where $k_0d = \pi$. The range of frequencies $[\omega_0, \omega_\pi]$ is called the pass band, or the propagation band.

The electric field at a snapshot is shown schematically in the Figure. At an instant of time, it represents a traveling wave or the maximum of a standing wave. The upper plot shows the snapshot of an electromagnetic wave. The lengths of $kd = \pi, 2\pi/3, \pi/2$ cavities are also shown. The arrows indicate the maximum electric field directions. The lower plot shows a similar snapshot for $kd = 0, \pi/2, 2\pi/3$ and $\pi$ cavities.

Example: capacitive coupling

3.8.6: Identical resonator LC circuits are coupled with disk or washer loading by parallel capacitors $2C_p$ shown in the figure.

\[
\frac{i_{n+1}}{2j\omega C_p} + \left( \frac{1}{j\omega C_p} + \frac{1}{j\omega C_s} + j\omega L \right) i_n - \frac{i_{n-1}}{2j\omega C_p} = 0.
\]

Bloch theorem: the eigenfunctions of the periodic wave equation are of the form $i_n = e^{jkd} i_0$, where $i_0$ is a periodic function of the circuit. Comparing the above equation with the condition of Bloch theorem: $i_{n+1} - 2 \cos(kd) i_n + i_{n-1} = 0$, we find

\[
\begin{align*}
\cos(kd) &= 1 + \frac{C_s}{C_p} - \omega^2 C_p L \\
\omega^2 &= \omega_0^2 [1 + \kappa(1 - \cos kd)] \\
\omega_0 &= 1/\sqrt{LC_s} \\
\kappa &= C_s/C_p
\end{align*}
\]
F. Standing wave, traveling wave, and coupled cavity linacs

The Alvarez linac operates at the standing wave TM_{010} mode, with drift tubes used to shield the electric field at the decelerating phase. The effective acceleration gradient is reduced by the transit time factor and the time the particle spends inside the drift tube. A wave guide accelerator, where the phase velocity is equal to the particle velocity, can effectively accelerate particles in its entire length. A wave guide accelerator is usually more effective if the particle velocity is high. There are two ways to operate high-\(\beta\) cavities: standing wave or traveling wave. In a storage ring, a standing wave can be used to accelerate beams of oppositely charged particles moving in opposite directions.

Standing wave operation of a module made of many cells may have a serious problem of many nearby resonances. For example, if a cavity has 50 cells, it can have standing waves at

\[kd = \pi, \frac{49\pi}{50}, \frac{48\pi}{50}, \ldots\]

Since \(d\omega/dk = 0\) for a standing wave at \(kd = 0\) or \(\pi\), these resonances are located in a very narrow range of frequency. A small shift of rf frequency will lead to a different standing wave mode. This problem can be minimized if the standing wave operates at the \(kd = \pi/2\) condition, where \(d\omega/dk\) has its highest value. However, the shunt impedance in \(kd = \pi/2\) mode operation is reduced by a factor of 2, because only half of the cavity cells are used for particle acceleration.

Since every other cavity cell has no electric fields in \(kd = \pi/2\) standing wave operation, these empty cells can be shortened or moved outside. This led to the invention of the coupled cavity linac (CCL) by E. Knapp and D. Nagle in 1964. The idea is schematically shown below. The CCL cavities operate at \(\pi/2\) mode, where field free cells are located outside the main cavity cells. These field free cells are coupled to the main accelerating cavity in the high magnetic field region. The electric field pattern of the main accelerating cavity cells looks like that of a \(\pi\)-mode cavity. Such a design regains the other half of the shunt impedance and provides very efficient proton beam acceleration for \(\beta > 0.3\).

A schematic drawing of the \(\pi/2\) phase shift cavity structure (top), where the field free regions are shortened (middle), and moved outside to become a coupled cavity structure (bottom).

VIII.4 Longitudinal Particle Dynamics in a Linac

Phase focusing of charged particles by a sinusoidal rf wave is the essential core of longitudinal stability in a linac. Let \(t_s, \psi_s, W_s\) be the time, rf phase, and energy of a synchronous particle, and let \(t, \psi, W\) be the corresponding physical quantities for a non-synchronous particle. We define the synchrotron phase space coordinates as \(\Delta t = t - t_s, \Delta \psi = \psi - \psi_s = \omega (t - t_s), \Delta W = W - W_s\). The accelerating electric field is \(E = E_0 \sin \omega t = E_0 \sin(\psi_s + \psi), \) where the coordinate \(s\) is chosen to coincide with the proper rf phase coordinate. The change of the phase coordinate is

\[
\frac{d\Delta \psi}{ds} = \omega \left( \frac{dt}{ds} - \frac{ds}{ds} \right) = \omega \left( \frac{1}{\nu_s} - 1 \right) \approx -\frac{\omega}{mc^2\beta_s^2}\Delta W,
\]

where \(v = ds/dt\) and \(v_s = ds/dt_s\) are the velocities of a particle and a synchronous particle, and the subscript \(s\) is used for physical quantities associated with a synchronous particle. This equation is in fact identical to Eq. (3.12), where \(\omega/\beta_s c\) is equivalent to the harmonic number per unit length, \(\Delta W/\beta_s^2 E\) is the fractional momentum spread, and \(-1/\gamma_s^2\) is the equivalent phase slip factor. Since the momentum compaction in a linac is zero, the beam in a linac is always below transition energy. The energy gain from rf accelerating electric fields is

\[
\frac{d\Delta W}{ds} = eE_0 \left[ \sin(\psi_s + \Delta \psi) - \sin \psi_s \right] \approx eE_0 \cos \psi_s \Delta \psi.
\]

The Hamiltonian for the synchrotron motion becomes

\[
H = -\frac{\omega}{2mc^2\beta_s^2}\Delta W^2 + eE_0 \left[ \cos(\psi_s + \Delta \psi) + \Delta \psi \sin \psi_s \right].
\]

Hereafter, \(\beta_s\) and \(\gamma_s\) are replaced by \(\beta\) and \(\gamma\) for simplicity. The linearized synchrotron equation of motion is simple harmonic:

\[
\frac{d^2\Delta W}{ds^2} = -k_{syn}^2 \Delta W, \quad k_{syn} = \sqrt{\frac{eE_0\omega \sin \psi_s}{mc^2\beta^3\gamma^3}},
\]

where \(k_{syn}\) is the wave number of the synchrotron motion. For medium energy proton linacs, \(k_{syn}\) is about 0.1 to 0.01 \(m^{-1}\). Synchro-beta coupling can be important if \(k_{syn}\) is near that of betatron motion. For high energy electrons, \(k_{syn} \approx 1/\gamma^{3/2}\) is small. The beam particles move rigidly in synchrotron phase space, and thus one choose \(\psi_s = \phi_s = \pi/2\), i.e. electron bunches are riding on top of the crest of the rf wave.

In contrast to synchrotrons, the linac usually do not have repetitive periodic structures, the concept of synchrotron tune is not necessary. However, if there is a quasiperiodic external focusing structures such as periodic solenoidal focusing systems, FODO focusing systems, or periodic doublet focusing systems, etc., the synchrotron tune can be defined as the \(\nu_{syn} = k_{syn}L/(2\pi)\), where \(L\) is the length of the periodic focusing system. Parametric synchrotron resonances can occur if \(m\nu_{syn} = l\) is satisfied, where \(m\) and \(l\) are integers. Near a parametric synchrotron resonance, the longitudinal phase space will form islands.
A. The capture condition in an electron linac with $v_p = c$

In an electron linac operating at a phase velocity equal to $c$, what happens to the injected electrons with velocities less than $c$? Let $\psi$ be the phase angle between the wave and the particle. Assuming constant gradient acceleration, the electric field seen by the electron is $E_0 \sin \psi$. Since the phase velocity and the particle velocity are different, the path length difference between the EM wave and the particle in time interval $dt$ is

$$dl = (c - v)dt = \frac{\lambda}{2\pi} d\psi,$$

$$\frac{d\psi}{dt} = \frac{2pc}{\lambda}(1 - \beta),$$

where $\lambda = 2\pi c/\omega$ is the rf wavelength, $d\psi = d\omega/2\pi$, and $\beta = v/c$. The particle gains energy through the electric field, i.e.

$$\frac{d(\gamma mv)}{dt} = mc \frac{d\psi}{dt} \left(\frac{\beta}{(1 - \beta^2)^{1/2}}\right) = eE_0 \sin \psi.$$

Substituting $\beta = \cos \zeta$, we obtain

$$\frac{d\psi}{dt} = -\frac{eE_0}{mc} \frac{\sin \psi}{\sin^2 \zeta} \sin \psi.$$

Using the chain rule $d\psi/dt = (d\psi/d\zeta)(d\zeta/dt)$, we can integrate the equation of motion to obtain

$$\cos \psi_2 - \cos \psi_1 = \frac{2\pi mc^2}{eE_0 \lambda} \tan \left(\frac{\zeta}{2}\right) = -\frac{2\pi mc^2}{eE_0 \lambda} \left(1 - \beta_1\right)^{1/2} = -Y_{inj},$$

The capture condition favors a linac with a higher acceleration gradient $E_0$. If $Y_{inj} = 1.5$, particles within an initial phase $-\pi/3 < \psi_1 < \pi/3$ will be captured inside the phase region $\pi > \psi_2 > 2\pi/3$. If the factor $Y_{inj} = 1$, all particles within $-\pi/2 < \psi_1 < \pi/2$ will be captured into the region $\pi > \psi_2 > \pi/2$. In particular, particles distributed within the range $\Delta > \psi_2 > -\Delta$ will be captured into the range $\pi/2 < \psi_2 < \pi/2 + \Delta/2$ ($\Delta \ll 1$). For example, all injected beam with phase length 20º will be compressed to a beam with a phase length 3.5º in the capture process.

The capture efficiency and energy spread of the electron beam can be optimized by a prebuncher. A prebuncher is usually used to prebunch the electrons from a source, which can be thermionic or rf gun. We assume a thermionic gun with a DC gun voltage $V_0$, which is usually about 80–150 kV. Consider a cavity with electric field $E_0 \sin(\omega t)$ at a prebuncher. Electrons that arrive earlier are slowed and that arrive late are sped up. At a drift distance away from the prebuncher, the faster electrons catch up the slower ones. Thus electrons are prebunched into a smaller phase extension to be captured by the buncher and the main linac. Then all captured high energy electrons can ride on top of the rf wave in order to gain maximum energy from the rf electric field.

B. Energy spread of the beam

In a multi-section linac, individual adjustment of each klystron phase can be used to make a bunch with phase length ride on top of the rf crest, i.e. $\psi_s = \pi/2$ . The final energy spread of the beam becomes

$$\Delta W = \frac{1 - \cos \Delta}{2} = \frac{\Delta^2}{8}.$$

This means that a beam with a phase spread of 0.1 rad will have an energy spread of about 0.13 % . Thus the injection match is important in minimizing the final energy spread of the beam. Other effects that can affect the beam energy are beam loading, wakefields, etc. A train of beam bunches extracts energy from the linac structure and, at the same time, the wakefield induced by the beam travels along at the group velocity. Until an equilibrium state is reached, the energies of individual beam bunches may vary.

C. Synchrotron motion in proton linacs

Since the speed of protons in linacs is not highly relativistic, the synchronous phase angle $\psi_s$ cannot be chosen as $\pi/2$ . The synchrotron motion in ion linac is adiabatic. The longitudinal particle motion follows a torus of the Hamiltonian flow.

$$H = -\frac{1}{2} \frac{\omega^3}{mc^3 \beta \gamma^3} \left(\frac{\Delta W}{\omega}\right)^2 - \frac{1}{2} eE_0 \cos \psi_s (\Delta \psi)^2.$$

The table below lists bucket area and bucket height for longitudinal motion in proton linacs, where $\alpha_0(\psi_s)$ and $Y(\psi_s)$ are running bucket factors. The rf phase region for stable particle motion can be obtained from $\psi_s$ and $\pi - \psi_s$ identical to those in synchrotrons.

<table>
<thead>
<tr>
<th>Bucket Area</th>
<th>$16 \left(\frac{m \beta \gamma^2 \omega^2 E_0}{\omega^3}\right)^{1/4} \alpha_0(\psi_s)$</th>
<th>$16 \left(\frac{\omega^3 \beta \gamma^3}{mc^3 \omega^2 E_0}\right)^{1/4} \alpha_0(\psi_s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bucket Height</td>
<td>$2 \left(\frac{m^2 \beta^2 \gamma \omega^2 E_0}{\omega^3}\right)^{1/2} Y(\psi_s)$</td>
<td>$2 \left(\frac{\omega^3 \beta \gamma^3}{mc^3 \omega^2 E_0}\right)^{1/2} Y(\psi_s)$</td>
</tr>
</tbody>
</table>

$$\sigma_{\Delta W} \sim (\omega E_0)^{1/4} (\beta \gamma)^{3/4}, \quad \text{and} \quad \sigma_\tau \sim (\omega E_0)^{-1/4} (\beta \gamma)^{-3/4}.$$

$$A_{\text{rms}} = \pi \sigma_{\Delta W/\omega} \sigma_\Delta \psi \quad \text{and} \quad \sigma_{\Delta p/p} = \sqrt{\frac{A_{\text{rms}}}{\pi}} \left(\frac{\omega E_0 \cos \psi_s}{m^2 c^3 \beta \gamma}\right)^{1/4}.$$
VIII.5 Transverse Beam Dynamics in a Linac

The electric field lines between electrodes in an acceleration gap, e.g., the drift tubes of an Alvarez linac or the irises of a high-\(\beta\) linac, is shown at the right plot.

In an electrostatic accelerator, the constant field strength gives rise to a global focusing effect because the particle at the end of the gap has more energy so that the defocusing force is weaker. This has been exploited in the design of DC accelerators such as the Van de Graaff or Cockcroft-Walton accelerators.

For rf linear accelerators, phase stability requires \(0 < \psi_s < \pi/2\) (below transition energy), and field strength increases with time during the passage of a particle. Thus the defocusing force experienced by the particle at the exit end of the gap is stronger than the focusing force at the entrance of the gap.

The EM field of TM\(_{010}\) mode is

\[ E_y = E_0 \sin \psi, \quad E_x = -\frac{\omega_T}{2v_p} E_0 \cos \psi, \quad B_\phi = -\frac{\omega_T}{2v_p} E_0 \cos \psi, \]

where \(\psi = (\omega t - \omega \int ds/v_p)\). The transverse force on particle motion is

\[ \frac{d(\gamma m r)}{dt} = -eE_x - evB_\phi = -\frac{c\omega e_0}{2v_p} (1 - \frac{v}{c}) \cos \psi \]

For a synchronous particle with \(v = v_p\), we obtain

\[ \frac{d(\gamma m r)}{dt} = -\frac{c\omega e_0 \sin \psi_s}{2\beta^2 c} r. \]

For a relativistic particle with \(\gamma \gg 1\), the transverse defocusing force becomes negligible because the transverse electric force and the magnetic force cancel each other. Assuming a zero defocusing force, we find

\[ \frac{dp_x}{dt} = 0 \quad \text{or} \quad \frac{dx}{ds} = 0. \quad \frac{dx}{ds} = \text{constant} = \gamma_0 x_0. \]

Assuming \(\gamma = \gamma_0 + \gamma' s\), where \(\gamma' = d\gamma/ds\), we obtain

\[ x - x_0 = \left( \frac{\gamma_0}{\gamma'} \ln \frac{\gamma}{\gamma_0} \right) x_0' \]

If no other external force acts on the particle, the orbit displacement increases only logarithmically with distance along a linac (Lorentz contraction). In reality, quadrupoles are needed to focus the beam to achieve good transmission efficiency and emittance control in a linac.
In the presence of a wakefield, the equation of motion is
\[
\frac{d^2}{ds^2} x(t, s) + k^2(t, s) x(t, s) = \frac{e^2}{\gamma(t, s)} \int^{\infty}_t dt' \rho(t') W_\perp(t - t') x(t', s),
\]
where \(t\) describes the longitudinal position of a particle, \(s\) is the longitudinal coordinate along the accelerator, \(x(t, s)\) is the transverse coordinate of the particle, \(k(t, s)\) is the betatron wave number (also called the focusing function), \(\rho(t)\) is the density of particle distribution, and \(W_\perp(t' - t)\) is the transverse wake function.

We consider a simple macro-particle model by dividing an intense bunch into two macro-particles separated by a distance \(\ell = 2\sigma\). Each macro-particle represents half of the bunch charge. They travel at the speed of light \(c\). The equation of motion in the smoothed focusing approximation is
\[
x_1'' + k_1^2 x_1 = 0,
\]
\[
x_2'' + k_2^2 x_2 = \frac{e^2 N W_\perp(t)}{2E} x_1 = G x_1,
\]
where \(eN/2\) is the charge of the leading macro-particle, \(x_1\) and \(x_2\) are transverse displacements, \(W_\perp(t)\) is the wake function evaluated at the position of the trailing particle, and \(k_1\) and \(k_2\) are betatron wave numbers for these two macro-particles.

If, for some reason, the leading particle begins betatron oscillation, the trailing particle can be resonantly excited, i.e.
\[
x_1 = \hat{x}_1 \sin k_1 s,
\]
\[
x_2 = \frac{k_1}{k_2} \hat{x}_1 \sin k_2 s + \frac{G \hat{x}_1}{k_2^2 - k_1^2} \left( \sin k_1 s - \frac{k_2}{k_1} \sin k_2 s \right).
\]

An interesting and effective method to alleviate the beam break up instabilities is BNS (V.Balakin, A.Novokhatsky, and V.Smironov) damping. If the betatron wave number for the trailing particle is higher than that for the leading particle by
\[
\Delta k = k_2 - k_1,
\]
the linear growth term vanishes. This means that the dipole kick due to the wakefield is exactly canceled by the extra focusing force. The bunch will perform rigid coherent betatron oscillations without altering its shape. The BNS damping depends on the beam current. The BNS damping can be achieved either by applying rf quadrupole field across the bunch length or by lowering the energy of trailing particles. The SLC linac uses the latter method by accelerating the bunch behind the rf crest early in the linac, and then ahead of the rf crest downstream, to restore the energy spread at the end of the linac. Since the average focusing function is related to the energy spread by the chromaticity
\[
\frac{\Delta k}{k_1} = C_x \frac{\Delta E}{E},
\]
and the chromaticity \(C_x \approx -1\) for FODO cells, the energy spread is equivalent to a spread in focusing strength. This method can also be used to provide BNS damping. It is also worth pointing out that the smooth focusing approximation provides a good approximation for the description of particle motion in a linac.