Fundamentals of rf systems:

Synchrotrons require rf cavities for particle acceleration and manipulation. The rf cavity is a device that can hold electromagnetic field and energy. Characteristics of RF cavities are listed as follows.

1. **Frequency** $\omega_{\text{rf}}$, harmonic number $h$: $\omega_{\text{rf}} = h \omega_0$

2. $V_{\text{rf}}$, shunt impedance, Q-factor, and transit time factor.

Let the cavity gap be $g$, electric field amplitude be $E$, and speed of the particle be $v$, the energy gain in a cavity with sinusoidal varying electric field is reduced by a transit time factor.

$$\Delta E = e \int_{-g/2}^{g/2} E \cos \frac{\omega s}{v} ds = eEgT_u, \quad T_u = \frac{\sin(\pi g / \beta \lambda)}{(\pi g / \beta \lambda)}$$

We normally include the transit time factor in the voltage amplitude of the accelerator rf cavity.

An rf cavity is a device to store electromagnetic energy at a particular frequency with minimum energy loss. The cavity can be designed with many different shapes and geometry.

**Pill-box cavity**

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \mu \frac{\partial \vec{E}}{\partial t}, \quad \nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

where $\varepsilon$ and $\mu$ are dielectric permittivity and permeability of the medium. The EM waves in the cavity can conveniently be classified into transverse magnetic (TM) mode, for which the longitudinal magnetic field is zero, and transverse electric (TE) mode, for which the longitudinal electric field is zero. The TM modes are of interest for beam acceleration in the rf cavity.

In an ideal acceleration cavity, the electromagnetic fields satisfy the boundary condition: $\hat{n} \times \vec{E} = 0, \quad \vec{n} \cdot \vec{H} = 0$, where $\hat{n}$ is the vector normal to the conducting surface. There is no tangential component of electric field, and no normal component of magnetic field.

**Pillbox cavity:** Assuming a time dependence factor $e^{i\omega t}$ for electric and magnetic fields, the TM standing wave modes in cylindrical coordinates $(r, \phi, s)$ are

$$E_s = A k_s^2 J_m(k_s r) \cos m \phi \cos k s$$
$$E_r = -A k_r^2 J_m(k_r r) \cos m \phi \sin k s$$
$$E_\phi = A (m \omega / c^2 r) J_m(k_r r) \sin m \phi \sin k s$$
$$B_s = 0$$
$$B_r = -i A (m \omega / c^2 r) J_m(k_r r) \sin m \phi \cos k s$$
$$B_\phi = -j A (m \omega / c^2 r) J_m(k_r r) \cos m \phi \cos k s$$

where $A$ is a constant, $s = 0$ and $t$ correspond to the beginning and end of the pillbox cavity, $m$ is the azimuthal mode number, $k_s, k_r$ are wave numbers in the longitudinal and radial modes, and

$$\frac{\omega}{c} = \sqrt{k_s^2 + k_r^2}.$$

The longitudinal wave number $k$ is determined by the boundary condition that $E_r = 0$ and $E_\phi = 0$ at $s = 0$ and $t$, i.e. $k_s(p) = \pi n / \ell, \quad p = 0, 1, 2, \cdots$

Similarly the radial wave number is determined by the boundary condition with $E_s = 0$ and $E_\phi = 0$ at $r = b$, i.e.

$$k_{s,mn} = j \frac{m \omega}{c}$$

$$k_{s,mn} = \sqrt{k_r^2 + k^{(p)}_r} = \sqrt{ \frac{2m^2}{b^2} + \frac{p^2 \pi^2}{\ell^2} } = \frac{\omega_{s,mn}}{c} = \frac{2\pi}{\lambda_{s,mn}}.$$

The resonance wave number $k$ for mode number $(m, n, p)$ is

$$k_{mn} = \sqrt{\frac{2m^2}{b^2} + \frac{p^2 \pi^2}{\ell^2}} = \frac{\omega_{mn}}{c} = \frac{2\pi}{\lambda_{mn}}.$$

The lowest frequency mode is usually called the fundamental mode. Other resonance frequencies are called high order modes (HOM). The art (science) of cavity design is to damp HOMs without affecting the fundamental mode. The EM field of the lowest mode TM$_{010}$ ($k_s = 0$) is

$$E_s = E_0 J_0(k r), \quad B_s = E_0 J_1(k r), \quad k_{010} = \frac{2.405}{b}, \quad \lambda = \frac{2\pi b}{2.405}.$$

The phase velocity, $\omega/k_{s,mn}$ for the traveling wave component of the TM$_{010}$ mode with $k_s = 0$ is infinite. Thus beam particles traveling at speed $v \leq c$ do not synchronize with the electromagnetic wave. To slow down the phase velocity, the cavity is loaded with one beam hole with an array of cavity geometries and shapes.
VI.2 Low Frequency Coaxial Cavities

Lower frequency rf systems usually resemble coaxial wave guides, where the length is much larger than the width.

The TEM wave in the coaxial wave guide section is converted to the TM mode at the cavity gap through the capacitive load. When the cavity is operating in 50 to 200 MHz range, it requires a very small amount of ferrite for tuning. When the cavity is operating at a few MHz range, the TEM wave guide is usually ferrite loaded with magnetic dipole or quadrupole fields for bias frequency tuning. At lower rf frequency, ferrite rings in the cavity are needed to slow down EM waves.

The inductance $\hat{L}$ and the capacitance $\hat{C}$ per unit length of the concentric coaxial wave guides are

$$\hat{L} = \frac{\mu}{2\pi} \ln \frac{r_2}{r_1} + \frac{\mu \delta_{\text{skin}}}{4\pi} \left( \frac{1}{r_1} + \frac{1}{r_2} \right), \quad \hat{C} = \frac{2\pi \epsilon}{\ln(r_2/r_1)}, \quad (3.240)$$

where $\mu$ is the permeability of the conductor, $\delta_{\text{skin}} = \sqrt{2/\omega \mu \sigma}$ is the skin depth of flux penetration. The inductance and capacitance of the coaxial cavity structure are respectively $L = \hat{L} \ell$ and $C = \hat{C} \ell$, where $\ell$ the length of the structure.

To understand the capacitive loading that converts the TEM wave into the TM wave at the cavity gap, we study the rf electromagnetic wave in the wave guide. The characteristic impedance of a wave guide is

$$Z_0 = R_c = \sqrt{L/C} \approx \frac{\mu}{2\pi} \sqrt{\ln \frac{r_2}{r_1}}.$$ 

For a standing wave with shorted end, i.e. $V(s=\ell) = 0$, we obtain

$$I(s, t) = I_0 \cos[k(s-\ell)] \quad \text{and} \quad V(s, t) = -jI_0 R_c \sin[k(s-\ell)].$$

The input impedance of the wave guide is

$$Z_{in} = \frac{V_0(t, 0)}{I(0, t)} = +jR_c \tan k\ell.$$

The length of the line is chosen at the resonance condition:

$$Z_{in} + Z_{gap} = 0, \quad \tan k\ell_{r} = \frac{1}{\omega R_c C_{gap}} = \frac{1}{g}.$$ 

If the loading capacitance is small, the resonance condition becomes $k\ell_r = \pi/2$, i.e. $\ell = \lambda/4$: the length of the coaxial cavity is $1/4$ of the wavelength of the TEM wave in the coaxial wave guide. Such a structure is also called a quarter-wave cavity.

The resulting voltage at the gap is

$$V_{rt} = +j I_0 R_c \sin k\ell_r = +j \frac{I_0 R_c}{\sqrt{1 + g^2}}.$$ 

A. Shunt impedance and Q-factor:

The surface resistivity and the resistance of the transmission line is

$$R_s = \frac{\mu_0 \sigma}{2\pi}, \quad R = \frac{R_s}{4} \left( 1 + \frac{1}{r_1} \right) \quad \frac{f [\text{MHz}]}{1000}$$

<table>
<thead>
<tr>
<th>$r_1$ [μm]</th>
<th>66</th>
<th>21</th>
<th>6.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{\text{skin}}$</td>
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<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>Q</td>
<td>1100</td>
<td>3500</td>
<td>11000</td>
</tr>
</tbody>
</table>

$$P_d = \frac{f^2 R_s}{2} \int \cos^2 x \, dx = \frac{f^2 R_s}{4(1 + g^2)} \left[ (1 + g^2) \cot^{-1} g + g \right]$$

$$R_{sh} = \frac{4}{\pi} \left[ \frac{\pi/2}{(1 + g^2) \cot^{-1} g + g} \right] R_s Q$$

$$\ln \left( \frac{r_2}{r_1} \right) \approx 1 \quad \text{and} \quad r_1 \approx 0.05 \, \text{m}, \quad \sigma_{eq} \approx 5.8 \times 10^7 \, [\text{F/m}]^{-1}$$
From the transmission-line point of view, the cavity gap presents a capacitance and resistive load shown in Fig. 3.26, where $Z_{in} = j\omega L_{eq}$ and $C_{eq} = C_{gap}$. The matching condition implies that the reactance of the cavity is zero on resonance, and the effective impedance is $R_{sh}$. The impedance of the rf system, represented by a parallel RLC circuit, becomes

$$Z = \left( \frac{1}{R_{sh} + j\omega C_{eq} + \frac{1}{j\omega L_{eq}}} \right)^{-1} = \frac{R_{sh}}{1 + jQ(\omega - \omega_r/\omega)} \approx R_{sh} \cos \psi e^{-j\psi},$$

$$\omega_r = (L_{eq}C_{eq})^{-1/2},$$

$$Q = \frac{R_{sh}\sqrt{C_{eq}/L_{eq}}}{\omega_r},$$

$$\psi = \tan^{-1} \frac{2Q(\omega - \omega_r)}{\omega_r}$$

\(\psi\) is the cavity detuning angle

---

**D. Example: The rf cavity of the IUCF cooler injector synchrotron**

**Ferrite loaded cavity**

The IUCF cooler injector synchrotron (CIS) is a low energy booster for the IUCF cooler ring. It accelerates protons (or light ions) from 7 MeV to 225 MeV. The cavity is a quarter-wave coaxial cavity with heavy capacitance loading. To make the cavity length reasonably short and to achieve rapid tuning, required for synchrotron acceleration, ten Phillips 4C12 type ferrite rings are used. The \(\mu\) of the ferrite material is changed by a superimposed DC magnetic field provided by an external quadrupole magnet. The ferrite rings return the magnet flux between the two adjacent quadrupole tips.

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**VI.3 Beam Loading**

A passing beam charge can induce wakefields in rf cavity. The effective voltage at the rf gap is a superposition of voltage due to generator current and induced voltage due to induced rf current. Without proper compensation, the resulting rf voltage acting on the passing beam may cause beam deceleration in an uncontrollable manner. Thus beam loading needs to be considered in the operation of rf cavities.

**Phaser:** The rf voltage, oscillating at frequency \(\omega_{rf}\), can be considered as a vector rotating in the complex plane at an angular frequency \(\omega_{rf}\). The magnitude of the vector is equal to the amplitude of the rf voltage, and the rf voltage seen by the beam is the projection of the rotating vector on the real axis. We choose a coordinate system that rotates with the rf frequency, and thus the rf voltage is stationary in this rotating coordinate system. Let \(V_0\) and \(\theta\) be respectively the amplitude and the angle with respect to the real axis of the rf voltage vector, \(\phi_s\) is the synchronous phase angle, i.e.

$$V_0 \cos \theta = V_0 \sin \phi_s,$$

$$\hat{V} = V e^{j(\omega t + \theta)} \rightarrow \hat{V} = V e^{j\theta}$$

**B. Filling time:**

$$Q = \frac{R}{R} = \frac{P_{in}}{P_{out}} = \omega W_{in}, \quad \frac{dW}{dt} = -P_d = -\frac{\omega}{Q} W_i, \quad \therefore W_e = W_{in} \exp(-\frac{\omega}{Q} t)$$

The filling time is defined as the time for the electric field or potential to decrease 1/e of its initial value.

$$T_f = \frac{2Q}{\omega}$$
B. Fundamental theorem of beam loading

**Theorem:** A charged particle sees exactly 1/2 of its own induced voltage.

We assume that the stored energy in a cavity in any given mode is \( W = \alpha V^2 \). We assume that a fraction \( f \) of the induced voltage is seen by the inducing particle, and the effective voltage is \( V_e = fV_b \), where \( V_b \) is the induced voltage in each passage.

We assume further that the induced voltage lies at phase angle \( \chi \) with respect to the inducing current or charge.

Now, we consider two identical charged particles of charge \( q \), separated by phase angle \( \theta \), passing through the cavity. The total energy deposited in the cavity is

\[
W_e = \alpha(\frac{V_b}{2})^2 = \alpha \frac{V_b^2}{2}(1 + \cos \theta).
\]

The energy loss by these two particles is

\[
\Delta U = [qV_e] + [qV_e + qV_b \cos(\chi + \theta)],
\]

where the first and second brackets are the energy losses due to the first and second particles respectively. From the conservation of energy, \( \Delta U = W_e \), we obtain

\[
\chi = 0, \quad V_b = \frac{q}{2\alpha}, \quad V_e = \frac{1}{2}V_b, \quad f = \frac{1}{2} \quad (3.313)
\]

The filling time \( T_f = \frac{2Q_L}{\omega_I} \) is the cavity time constant or the cavity filling time. Here \( Q_L \) is the loaded cavity quality factor, taking into account the generator resistance \( R_g \) in parallel with the RLC circuit of the cavity, i.e.

\[
Q_L = \frac{R_g}{R_g + R_e} = \frac{Q_0}{1 + d}, \quad d = \frac{R_e}{R_g}
\]

The filling time of the loaded cavity is reduced by a factor \( 1/(1 + d) \). The cavity detuning angle \( \psi \) and the rf phase shift are

\[
\psi = \tan^{-1}\left[\frac{2Q_b(\omega - \omega_I)}{\omega_I}\right] = \tan^{-1}\left[(\omega - \omega_I)T_f\right],
\]

\[
\phi = (\omega - \omega_I)T_b = + (T_b/T_f) \tan \psi = + \lambda \tan \psi,
\]

where \( \omega \) is the cavity operation frequency. For rf cavities used in accelerators, we have \( \lambda = T_b/T_f = \omega_I/2Q_L \ll 1 \), and the induced voltage seen by the beam is

\[
V_b = I_b R_{sh} \left[1 + \frac{1 - e^{-\lambda \gamma_0}}{1 - e^{-\lambda \gamma}}\right] \approx I_b R_{sh} \frac{\cos \psi e^{-j\psi}}{\lambda} \quad (\lambda \to 0),
\]

where \( I_b \) is the rf image current, \( V_{b0} = I_b R_{sh} T_b / T_f \), and the term \(-1/2\) is neglected.

The beam induced voltage across the rf gap at the steady state is exactly the rf image current times the impedance of the rf cavity:

\[
V_b = \frac{V_{b0}}{1 - e^{-\lambda j \phi}} \frac{V_{b0}}{\lambda + j \phi} = I_b R_{sh} \cos \psi e^{-j\psi}
\]

The result can be summarized as follows:

1. The induced voltage of a beam must have a phase maximally opposite the motion of the charge, i.e. the phase angle \( \chi = 0 \).
2. \( V_e = V_b / 2 \). The particle sees exactly 1/2 of its own induced voltage.
3. \( V_e = \alpha V_b^2 = q^2/4\alpha = k q \), where \( k \) is the loss factor, \( k = V_b^2/(4W_e) \).
4. \( V_b = 2k q \) or \( V_e = k q \).

C. Steady state solution of multiple bunch passage

Consider an infinite train of bunches, separated by time \( T_b \), passing through an rf cavity gap. When the cavity is on resonance, the induced voltage seen by the particle is

\[
V_b = \frac{1}{2}V_{b0} + V_{b0} e^{-\lambda j \phi} + e^{-2\lambda j \phi} + \ldots = V_{b0} e^{-\lambda j \phi},
\]

where \( \phi = (\omega - \omega_I)T_b \) is the relative bunch arrival phase with respect to the cavity phase at the rf gap, \( \omega_I \) is the resonance frequency of the rf cavity, and \( \lambda = T_b/T_f \) is the decay factor of the induced voltage between successive bunch passages. A cavity oscillates with \( \exp(j\omega_I t) \). Subtracting \( \exp(-j\omega_I t) \) factor of the phaser-rotating frame, we find the relative phase becomes \( \phi \) shown in the above formula.

**Beam loading Compensation**

The cavity can be excited by external power supply. In fact, the image charge of the beam is also a natural power source at the right frequency, i.e.

\[
I(t) = e \sum_{i} \sum_{n=-\infty}^{\infty} \delta(t - t_i - nT_0) = N e f_0 + 2 N e f_0 \sum_{n=0}^{\infty} \cos(n \omega_0 t)
\]

The beam bunch has an image current equal to twice the average current at all revolution harmonics. In particular the image current at \( \omega_0 \) is twice the average current.

\[
\begin{align*}
\text{If the cavity is tuned on resonance, it requires large beam loading compensation. If it is not compensated, the effective acceleration voltage can be highly reduced, even polarity reversed. It is more economical to solve the problem by detuning cavity frequency.}
\end{align*}
\]
The equation for proper cavity detuning becomes
\[
\begin{align*}
I_0 &= I_0 e^{i\theta} \quad \text{generator current necessary for accelerating voltage in the absence of beam} \\
I_g &= I_e e^{i(\theta + \phi_s)} \quad \text{required generator current with beam} \\
V_g &= V_0 e^{i\theta} \quad \text{rf beam image current, } I_i \text{ is a positive quantity} \\
\psi &= \tan^{-1}\left(\frac{2(\omega - \omega_r)}{\omega_r}\right) \quad \text{detuning angle} \\
Y &= I/I_0 \quad \text{ratio of image current to unloaded generator current}
\end{align*}
\]

The equation for proper cavity detuning becomes
\[
\begin{align*}
\tilde{V}_g &= I_0 R_{sh} e^{i\theta} = [I_g e^{i(\theta + \phi_s)} - I_i] R_{sh} \cos \psi e^{-i\psi} \\
\tan \theta_g &= \tan \psi - Y \sin \theta \\
I_g &= I_0 \frac{1 + Y \cos \theta}{\cos \theta_g},
\end{align*}
\]

To minimize the generator current, we choose \( \theta_g = 0 \).
\[
I_g = I_0 (1 + Y \sin \phi_s) \\
\tan \psi = Y \cos \phi_s
\]

The Robinson stability condition becomes
\[
1 - Y \frac{\sin \psi \cos \psi}{\cos \phi_s} \geq 0 \quad \text{or} \quad 1 - \frac{\sin^2 \psi}{\cos^2 \phi_s} \geq 0
\]

This means that Robinson stability requires \( \psi < |\frac{1}{2} \pi - \phi_s| \). In general, Eq. (3.328) is applicable to all higher order modes. For those modes, Robinson stability can be described as follows.

Below transition energy, with \( \cos \phi_s > 0 \), Robinson stability can be attained by choosing \( \sin \psi < 0 \), i.e. the cavity frequency is detuned with \( \omega < \omega_r \). Above transition energy, with \( \cos \phi_s < 0 \), the cavity should be detuned so that \( \sin \psi > 0 \) or \( \omega > \omega_r \) in order to gain Robinson stability. Since the stability condition is a function of bunch intensity, stability is a self-adjusting process. Beam loss will appear until the Robinson stability condition can be achieved. Active feedback systems are used to enhance the stability of bunched beam acceleration.\(^{30}\)

Robinson instability of dipole mode

We consider a small perturbation by shifting the arrival time of all bunches by a phase factor \( \xi \). The accelerating rf voltage will be perturbed by the same phase factor,
\[
V_{acc} = V_0 \cos(\theta - \xi) = V_0 \cos \theta + \xi V_0 \sin \theta = V_{0s} \sin \phi_s + \xi V_0 \cos \phi_s,
\]

where the first term is the intended accelerating voltage and the second term is the effect of phase perturbation due to an error in arrival time. The wrong arrival time shifts the image beam current by a phase angle \( \xi \). The perturbation to the image rf current and voltage are
\[
\Delta I_i = j \xi I_i = -j \xi I_i, \quad \Delta V_g = -j \xi I_i R_{sh} \cos \psi e^{-i\psi}
\]

The induced accelerating voltage is equal to the projection of the phasor voltage onto the real axis: \( V_0 = -\xi Y V_0 \cos \psi \sin \psi \).

The net change in accelerating voltage seen by the bunch becomes
\[
\Delta V_{acc} = \xi V_0 \cos \phi_s \left[ 1 - Y \frac{\sin \psi \cos \psi}{\cos \phi_s} \right]. \quad (3.326)
\]

Since \( \frac{\Delta \omega}{\omega_0} = -\frac{\Delta E}{\beta^2 E_0} \), a higher beam energy has a smaller revolution frequency above the transition energy. If the cavity is detuned so that \( h\omega_0 > \omega_r \), where \( \omega_r \) is the resonance frequency of the cavity, the beam bunch at higher energy sees a higher shunt impedance and loses more energy, and the beam bunch at lower beam energy sees a lower shunt impedance and loses less energy. Thus the centroid of the beam bunch will damp in the presence of beam loading, and the dipole mode of beam motion is Robinson damped. Similarly, if the cavity is detuned such that \( h\omega_0 < \omega_r \), Robinson stability will be attained below transition energy.

![Figure 3.39: A schematic drawing of the real part of impedance arising from a wakefield induced by the circulating beam. To avoid Robinson instability, the cavity should be detuned to \( h\omega_0 > \omega_r \) above transition energy and \( h\omega_0 < \omega_r \) below transition energy. Above transition energy, higher energy particles have a smaller revolution frequency and thus lose more energy if the cavity detuning is \( h\omega_0 > \omega_r \). A similar argument applies to rf cavities operating below transition energy.](image-url)