Answer to the Midterm exam

1. In \((\phi; \frac{\Delta E}{\omega_0})\) space, the phase-space area \(A\) is proportional to a factor \(F = \gamma/|\eta|\). When \(\gamma < \gamma_T\), \(F\) increases monotonously with \(\gamma\), and when \(\gamma > \gamma_T\), the factor \(F\) has a minimum at \(\gamma = \sqrt{3}\gamma_T\).

\[
\frac{dF}{d\gamma} = \frac{1}{\eta^2} \left[ \left( \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} \right) - \gamma \cdot \frac{2}{\gamma^3} \right]
\]

When \(\gamma = \sqrt{3}\gamma_T\), \(dF/d\gamma = 0\) and \(d^2F/d\gamma^2 > 0\), thus \(A\) has a minimum.

2. Using the results of Sec. II.3, we find

\[
\sigma_E = (\beta^2 E)\delta = A^{1/2} \left( \frac{heV\omega_0^2\beta^2 E}{2\pi^3|\eta|} \right)^{1/4} = \frac{A}{\pi\sigma_E}
\]

\[
\sigma_t = \frac{1}{\omega_0} \hat{\theta} = A^{1/2} \left( \frac{2|\eta|}{heV\pi\omega_0^2\beta^2 E} \right)^{1/4}
\]

<table>
<thead>
<tr>
<th></th>
<th>AGS</th>
<th>RHIC</th>
<th>FNAL-MI</th>
<th>FNAL-BST</th>
<th>Cooler</th>
</tr>
</thead>
<tbody>
<tr>
<td>K.E. [GeV]</td>
<td>25</td>
<td>250</td>
<td>120</td>
<td>8</td>
<td>0.045</td>
</tr>
<tr>
<td>(V_t) [MV]</td>
<td>0.3</td>
<td>0.3</td>
<td>2</td>
<td>0.95</td>
<td>0.0001</td>
</tr>
<tr>
<td>(h)</td>
<td>12</td>
<td>342</td>
<td>588</td>
<td>84</td>
<td>1</td>
</tr>
<tr>
<td>(\gamma_T)</td>
<td>8.5</td>
<td>24.5</td>
<td>21.8</td>
<td>5.446</td>
<td>4.6</td>
</tr>
<tr>
<td>(C) [m]</td>
<td>807.12</td>
<td>3833.84</td>
<td>3319.4</td>
<td>474.2</td>
<td>86.8</td>
</tr>
<tr>
<td>(A) [eV-s]</td>
<td>1.5</td>
<td>0.5</td>
<td>0.15</td>
<td>0.15</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

| \(\sigma_E\) (MeV) | 34.0 | 62.3 | 53.0 | 17.4 | 2.88 \times 10^{-3} |
| \(\sigma_t\) (ns)   | 14.0 | 2.6  | 0.90 | 2.75 | 11.0    |

3. We consider the TEM wave in the transmission line, e.g. a coaxial cable. Faraday’s and Ampere’s laws are

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{B} = \mu \varepsilon \frac{\partial \vec{E}}{\partial t},
\]

where \(\mu\) and \(\varepsilon\) are permeability and permittivity of the medium that the TEM wave propagates, and \(t\) be the time coordinate. Let \(\hat{s}\) be the direction of the propagation. The electromagnetic fields in the TEM wave is perpendicular to the direction of propagation. For the coaxial geometry, we find

\[
\frac{\partial E_r}{\partial s} = -\frac{\partial B_\phi}{\partial t}, \quad \frac{\partial B_\phi}{\partial s} = \mu \varepsilon \frac{\partial E_r}{\partial t},
\]
or

\[
\frac{\partial V}{\partial s} = \frac{\partial}{\partial s} \int_{r_1}^{r_2} E_r dr = -\frac{\partial}{\partial s} \int_{r_1}^{r_2} B_\phi dr = -\left( \frac{\mu}{2\pi} \int_{r_1}^{r_2} \frac{1}{r} dr \right) \frac{\partial I}{\partial t} = -L \frac{\partial I}{\partial t},
\]
\[
\frac{\partial I}{\partial s} = -C \frac{\partial V}{\partial t},
\]
where we use \(L = (\mu/2\pi) \ln(r_2/r_1)\), and \(C = 2\pi\epsilon/\ln(r_2/r_1)\). Combine these two
equations, we find the wave equation:

\[
\frac{\partial^2 V}{\partial s^2} - \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2} = 0 \quad \frac{\partial^2 I}{\partial s^2} - \frac{1}{v^2} \frac{\partial^2 I}{\partial t^2} = 0
\]

can be derived from Maxwell’s equation, where \(v = 1/\sqrt{LC}\) is the wave velocity, \(L\) and
\(C\) are the inductance and capacitance per unit length.

4. See p.258 of the textbook.

5. (a) Using \(y = x + j z\), the equation of particle motion in solenoid becomes

\[
y'' - 2j gy' - jg'y = 0.
\]

(b) Define the particle coordinate in the rotating frame \(\bar{y} = ye^{-j\theta(s)}\), where \(\theta = \int_0^s g ds\),
we find \(\bar{y}'' + g^2 \bar{y} = 0\).

(c) The solutions of the above equation in the rotating frame are

\[
(x + j z) = (x_0 + j z_0) \cos gs + (x'_0 + j z'_0)(1/g) \sin gs
\]
\[
(x' + j z') = -(x_0 + j z_0) g \sin gs + (x'_0 + j z'_0) \cos gs.
\]

Then one can get \(\bar{M}\) as shown in the problem where

\[
(\bar{x} \quad x' \quad z \quad z')^\dagger = \bar{M} (x_0 \quad x'_0 \quad z_0 \quad z'_0)^\dagger
\]

(d) Transforming back to the original frame with \(y = \bar{y}e^{j\theta}, y_0 = \bar{y}_0\), we define

\[
\bar{r} = (x \quad x' \quad z \quad z')^\dagger \quad \bar{r} = (\bar{x} \quad \bar{x}' \quad \bar{z} \quad \bar{z}')^\dagger
\]

and obtain

\[
\bar{r} = \bar{M}\bar{r}_0 \quad \bar{r} = M\bar{r}_0 \quad \text{with} \quad \bar{r}_0 = \bar{r}_0,
\]
i.e.

\[
(x + j z) = (x + j \bar{z})(\cos \theta + j \sin \theta) = (x \cos \theta - \bar{z} \sin \theta) + j(x \sin \theta + \bar{z} \cos \theta)
\]

\[
\bar{r} = \begin{pmatrix}
\cos \theta & 0 & -\sin \theta & 0
\vspace{1em}
0 & \cos \theta & 0 & -\sin \theta
\vspace{1em}
\sin \theta & 0 & \cos \theta & 0
\vspace{1em}
0 & \sin \theta & 0 & \cos \theta
\end{pmatrix} \bar{r} = \bar{R}(\theta)\bar{r}.
\]

So we get the intended result: \(\bar{r} = \bar{R}(\theta)\bar{r} = \bar{R}(\theta)\bar{M}\bar{r}_0 = \bar{R}(\theta)\bar{M}\bar{r}_0\).
6. For small amplitude synchrotron motion, the phase-space coordinates are (see Sec. II.3, Chap. 3)
\[ \varphi = \hat{\phi} \cos(\omega_s t + \chi), \quad \delta = -\frac{\nu_s}{\hbar|\eta|} \hat{\phi} \sin(\omega_s t + \chi). \]
The phase-space ellipse of a particle becomes
\[ \left( \frac{\delta^2}{\hat{\delta}} \right)^2 + \left( \frac{\varphi}{\hat{\phi}} \right)^2 = 1, \quad \frac{\hat{\delta}}{\hat{\phi}} = \left( \frac{eV \cos \phi_s}{2\pi \beta^2 \hbar|\eta|} \right)^{1/2} = \frac{\nu_s}{\hbar|\eta|}, \]
where \( \hat{\delta} \) and \( \hat{\phi} \) are maximum amplitudes of the phase-space ellipse. The phase-space area of the ellipse is \( \pi \hat{\delta} \hat{\phi} \). Similarly, for a single bunch, we have
\[ \left( \frac{\delta^2}{\hat{\delta}} \right)^2 + \left( \frac{\varphi}{\hat{\phi}} \right)^2 = 1, \quad \frac{\hat{\delta}}{\hat{\phi}} = \frac{\nu_s}{|\eta|}, \]
where \( \hat{\delta} \) and \( \hat{\phi} \) are maximum amplitudes of the phase-space ellipse. The phase-space area of the ellipse of a single bunch is \( \pi \hat{\delta} \hat{\phi} \).

In the bunch rotation manipulation, the evolution of the beam bunch is
\[ (V_1, \delta_0, \theta_0) \xrightarrow{\text{adiabatic}} (V_2, \delta_1, \theta_1) \xrightarrow{\text{non-adiabatic}} (V_1, \delta_1, \theta_1) \xrightarrow{\text{rotate}} (V_1, \delta_2, \theta_2) \]
The phase-space area enclosed by the ellipse is invariant in linear synchrotron motion, i.e.
\[ \pi \delta_0 \theta_0 = \pi \delta_1 \theta_1 \]
From the relations
\[ \delta_0 = \frac{\nu_s^2}{\eta} \theta_0, \quad \delta_1 = \frac{\nu_s^2}{\eta} \theta_1 \]
we get \( \theta_1^2 \nu_s^2 = \theta_0^2 \nu_s^1 \). From (3.57), the rotation in the last step has the relations
\[ \delta_2 = \frac{\nu_s^1}{\eta} \theta_1, \quad \theta_2 = \frac{\eta}{\nu_s^1} \delta_1 \]
The final bunch length is
\[ \theta_2 = \frac{\eta}{\nu_s^1} \theta_1 = \frac{\nu_s^2}{\nu_s^1} \theta_1 = \left( \frac{\nu_s^2}{\nu_s^1} \right)^{\frac{1}{2}} \theta_0 = \left( \frac{V_2}{V_1} \right)^{\frac{1}{2}} \theta_0 \]
From (3.57) and above, we find
\[ \omega_0 = 0.5679 \text{ MHz}, \quad \theta_2 = 8.518 \times 10^{-5}, \quad \theta_0 = 2.487 \times 10^{-4}, \quad V_2 = 469.2 \text{ (keV)} \]
The antiproton bunch has 0.15 ns bunch length. If the energy spread is \( \Delta E = E_t \cdot (\pm 3\%) = 0.534 \text{ (GeV)} \), the phase-space area of the antiprotons is 0.08 eV-s. Since the antiproton bunch has \( 6 \times 10^{10} \times 10^{-5} = 6 \times 10^5 \) particles, the phase-space density is \( 7.5 \times 10^6/(\text{eV-s}) \).
7. The anti-protons produced from the Main Injector (Main Ring) pulses have the following characteristics: \( p_0 = 8.9 \text{ GeV/c}, \sigma_t = 0.15 \text{ ns}, \sigma_E = 180 \text{ MeV}, \) or \( \Delta p/p_0 = \pm 2\% \). The antiprotons are captured in the Debuncher into the 53.1 MHz \((h = 90)\) rf bucket with \( V = 5 \text{ MV}, \gamma_T = 7.7, \) and \( R = 83 \text{ m} \).

(a) With \( V = 5 \text{ MV}, h = 90, \gamma_T = 7.7, \) and \( R = 83 \text{ m} \), we find
\[
E = 8949.3 \text{ MeV} \quad \gamma = 9.5381 \quad \beta = 0.9954, \quad \eta = 0.0059
\]
\[
f_0 = 3.612 \text{ MHz} \quad T_0 = 0.2769 \mu s
\]
\[

\nu_s = 6.89 \times 10^{-3} \quad T_s = 40.38 \mu s \quad \delta_B = 0.0259
\]

(b) A matched bunch ellipse is given by and momentum height is given by
\[
\frac{\delta^2}{\bar{\delta}^2} + \frac{\theta^2}{\bar{\theta}^2} = 1, \quad \text{with} \quad \frac{\delta}{\theta} = \frac{\nu_s \theta_1}{\eta}.
\]

When a beam is injected into the bucket with an ellipse
\[
\frac{\delta^2}{\bar{\delta}^2} + \frac{\theta^2}{\bar{\theta}^2} = 1,
\]
where \( \delta_1/\theta_1 \neq \nu_s \eta/\eta_1 \), it will rotate like a cigar. After 1/4 synchrotron period, the beam height will become the width and the width becomes the height, i.e.
\[
\delta_2 = \frac{\nu_s \theta_1}{\eta} \quad \theta_2 = \frac{\eta}{\frac{\nu_s}{\eta_1}} \delta_1.
\]

To match the new shape of the beam, we must have
\[
\frac{\delta_2}{\theta_2} = \frac{\nu_s \delta_2}{\eta} \quad \text{or} \quad \frac{\nu_s \delta_2}{\eta} = \frac{\theta_2}{\theta_2} = \left( \frac{\nu_s}{\eta} \right)^2 \cdot \frac{\theta_1}{\delta_1}.
\]

Using the relations:
\[
\nu_s \sim \sqrt{V} \quad \theta = \omega_0 \sigma_t \quad \delta = \frac{1}{\beta^2 E \sigma_E}
\]

it is easy to get
\[
\frac{V_2}{V_1} = \left( \frac{\nu_s}{\nu_s} \right)^2 = \left( \frac{\nu_s}{\eta} \right)^2 \left( \frac{\theta_1}{\delta_1} \right)^2 = \left( \frac{\nu_s}{\eta} \omega_0 \sigma_t \right)^2 \left( \frac{\sigma_E}{\beta^2 E} \right)^{-2}
\]

Inserting all the known parameters, we get \( V_2 = 4.99 \text{ keV} \).

(c) From (b),
\[
\sigma_{E2} = \delta_2 (\beta^2 E) = \frac{\nu_s \theta_1 (\beta^2 E)}{\eta} = \left[ \frac{\nu_s}{\eta} (\omega_0 \sigma_t) \right] \beta^2 E
\]

The final energy spread is \( \sigma_{E2} = 5.6 \text{ MeV} \).

8. See textbook