Effects of Linear Magnetic field Error

\[ x'' + K_z(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_z}{B\rho} \]

\[ \Delta B_z + j\Delta B_x = B_0 \sum_n (b_n + j a_n)(x + jz)^n, \]

\[ b_n : \text{dipole, } \quad a_n : \text{skew (vertical) dipole; } \quad z = B_0 b_0, \quad x = B_0 a_0, \]

\[ b_1 : \text{quad, } \quad a_1 : \text{skew quad; } \quad z = B_0 b_1 x, \quad x = B_0 a_1 z, \quad z = -B_0 a_1 z, \quad x = B_0 a_1 x, \]

Effect of dipole field error:
We consider a single localized dipole error with the kick angle given by
\[ \theta = \frac{\Delta B}{B\rho}. \]
Because of the dipole field error, the reference orbit is perturbed! The idea is to find a new closed orbit that include the dipole field error.

\[ y'' + K_z(s)y = \theta \delta(s - s_0) \]

The closed orbit is given by the following condition:

\[ \begin{pmatrix} y_0 \\ y'_0 - \theta \end{pmatrix} = M \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} = \begin{pmatrix} \cos \Phi + \alpha_0 \sin \Phi & \beta_0 \sin \Phi \\ -\gamma_0 \sin \Phi & \cos \Phi - \alpha_0 \sin \Phi \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} \]

Where \( \Phi = 2\pi v, \) \( v \) is the betatron tune, the parameters \( \alpha_0, \beta_0, \) and \( \gamma_0 \) are values of the Courant-Snyder parameters at the kicker location. The solution is

\[ y_0 = \frac{\beta_0 \theta}{2\sin \pi v} \cos \pi v, \quad y'_0 = \frac{\theta}{2\sin \pi v} \left( \sin \pi v - \alpha_0 \cos \pi v \right) \]

For the distributed dipole field error, the closed orbit becomes

\[ y_{co}(s) = \sqrt{\beta(s)} \frac{y_C}{2\sin \pi v} \int ds_0 \sqrt{\beta(s_0)} \cos[\pi v - |\psi(s) - \psi(s_0)|] \frac{\Delta B(s_0)}{B\rho} \]

Making coordinate transformation:

\[ \phi(s) = \frac{1}{v} \int_{s_0}^{s} ds \beta(s), \quad \psi(s) = v \psi(s), \quad \psi(s_0) = v \psi(s_0) \]

The closed orbit becomes

\[ y_{co}(s) = \sqrt{\beta(s)} \frac{y_C}{2\sin \pi v} \int d\phi \left[ \beta^{3/2}(\phi) \frac{\Delta B(\phi)}{B\rho} \right] \cos \left[ \pi v - |\phi - \phi'| \right] \]

\[ f_k = \frac{1}{2\pi} \int \beta^{3/2}(\phi) \frac{\Delta B(\phi)}{B\rho} e^{-ik \phi} d\phi = \frac{1}{2\pi} \int \beta^{1/2}(\phi) \frac{\Delta B(\phi)}{B\rho} e^{-ik \phi} d\phi \]

\[ y_{co}(s) = \sqrt{\beta(s)} \sum_{k=\pm\infty} \frac{\beta^{1/2}(\phi) \frac{\Delta B(\phi)}{B\rho}}{V^2 - k^2} e^{ik \phi} \quad \text{Only harmonics near } v \text{ are important} \]
Effect of quadrupole field error:

The betatron tune of the accelerator is changed by the quadrupole field error. At the same time, the betatron amplitude function is also changed by the quadrupole field error. The betatron amplitude function can be obtained by a one-turn map, i.e.

\[ \mathbf{M}(s) = \mathbf{M}_0(s) \mathbf{m}(s) \mathbf{M}(s_1, s_2) \]

We assume that the transfer matrix of the unperturbed betatron can be described by

\[ \mathbf{M}_0(s) = I \cos \Phi_0 + J \sin \Phi_0, \quad \Phi_0 = 2 \pi \nu_0 \]

\[ J(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \]

The perturbation can be described by

\[ \mathbf{m}(s) = \begin{pmatrix} 1 & 0 \\ -k(s)ds_1 & 1 \end{pmatrix} \]

The transfer matrix of the one-turn map is

\[ \mathbf{M}(s) = \begin{pmatrix} \cos \Phi_0 + \alpha_i \sin \Phi_0 - \beta_i k(s_1)ds_1 & \beta_i \sin \Phi_0 \\ -\gamma_i \sin \Phi_0 - [\cos \Phi_0 + \alpha_i \sin \Phi_0]k(s_1)ds_1 & \cos \Phi_0 - \alpha_i \sin \Phi_0 \end{pmatrix} \]

\[ \cos \Phi = \cos \Phi_0 - \frac{1}{2} \beta_i k(s_1)ds_1 \sin \Phi_0 \]

\[ \Delta \Phi \approx \frac{1}{2} \beta_i k(s_1)ds_1, \quad \Delta \nu = \frac{1}{4\pi} \int \beta_i k(s_1)ds_1 \]

The perturbation to the betatron amplitude function becomes

\[ \frac{\Delta \beta(s)}{\beta(s)} = -\frac{1}{2 \sin \Phi_0} \int_s^{s+C} ds_1 \beta_i k(s_1) \cos 2 \nu_0 (\pi + \phi - \phi_i) \]

For distributed quadrupole field error, the perturbation to the betatron amplitude function becomes

\[ \frac{\Delta \beta(s)}{\beta(s)} = -\frac{V_0}{2 \sin \Phi_0} \int_0^{\pi/2} d\phi \beta^2(\phi) k(\phi) \cos 2 \nu_0 (\pi + \phi - \phi_i) \]

\[ \Delta \beta(s) = -\frac{V_0}{2 \sin \Phi_0} \int_0^{\pi/2} d\phi \beta^2(\phi) k(\phi) \cos 2 \nu_0 (\pi + \phi - \phi_i) \]

Note that the betatron amplitude function diverges when the betatron tune is integer or half-integer!
Beam measurements:

\[ I_+ = \frac{V_+}{Z_0}, \quad V_+ = (-I_+ + \alpha I_b) R_u \]

\[ V_0 = \alpha I_b \left( R_d \frac{Z_R}{R_u + Z_0} - R_u \frac{Z_R}{R_d + Z_0} \right) \rightarrow \frac{1}{2} \alpha I_b \]

\[ V = V_+ \]

\[ V_1 = V_+ \]

\[ V_{i1} = \alpha I_b \frac{R_s Z_0}{R_u + Z_0} \times (1 + \frac{R_d - Z_0}{R_d + Z_0}) = \alpha I_b \frac{R_s Z_0}{R_u + Z_0} \left( \frac{2R_d}{R_u + Z_0} \right) \]

\[ V_{i2} = -\alpha I_b \frac{R_s Z_0}{R_d + Z_0}, \quad V_2 = V_{i1} + V_{i2} = \alpha I_b \left( \frac{R_u - Z_0}{R_u + Z_0} \right) \frac{R_s Z_0}{R_d + Z_0} \rightarrow 0 \]

\[ \tilde{V}_1 = \alpha I_b \left( \frac{R_s Z_0}{R_u + Z_0} \right) \left( \frac{R_d - Z_0}{R_d + Z_0} \right) - \frac{R_s Z_0}{R_d + Z_0} = -\frac{1}{2} \alpha I_b Z_0 \]

The voltage at \( t=0 \) is

\[ V_1(\omega, 0) = \frac{I_b Z_0}{2\pi} \left( \sin \frac{\omega t}{c} \right) ^2 \]

Splitting a detector into half or 4 pieces

\[ y \approx \frac{w}{2} \left( \frac{U_+ - U_-}{U_+ + U_-} \right) = \frac{w}{2} \Delta \Sigma \]

\[ \Delta \Sigma \text{ is called } \Delta \text{-signal, and } \Sigma \text{ is called } \Sigma \text{ sum signal,} \]

\[ \Delta \Sigma \text{ is called the normalized signal. The beam position is generally given by} \]

\[ \text{the half width of the beam position monitor times the normalized signal.} \]

\[ x_2' = M(s_2 | s_1) \left( x_1 \right) \]

\[ M(s_2 | s_1) = \left( \frac{\beta_1}{\sqrt{\beta_1 \beta_2}} \right) \left[ \cos(\psi) + \alpha_1 \sin(\psi) \right] \sqrt{\beta_1 \beta_2 \sin(\psi)} \]

\[ x_1' = \csc \psi_2 \frac{\beta_2}{\sqrt{\beta_1 \beta_2}} x_2 - \left( \cot \psi_2 + \alpha_1 \right) x_1 \]

\[ p_1 = \alpha_1 x_1 + \beta_1 x_1' = \sqrt{\beta_1 \beta_2} \csc \psi_2 x_2 - \csc \psi_2 x_1 \]

\[ x_1^2 + p_1^2 = 2 \beta_1 J \]

\[ x_1^2 + \left( \frac{\beta_1}{\beta_2} \csc \psi_2 x_2 - \cot \psi_2 x_1 \right)^2 = 2 \beta_1 J \]
When the beam is not a delta-function, the beam spectrum cannot extend to infinite frequency! The frequency is limited by \( \Delta \omega \Delta t \approx 1 \).

\[
I(t) = e^{\sum_{n=-\infty}^{\infty} \delta(t - nT_0)}
\]

This yields:

\[
I(t) = \frac{e}{T_0} \sum_{n=-\infty}^{\infty} e^{j\omega_0 t} = \frac{e}{T_0} + 2 \frac{e}{T_0} \sum_{n=1}^{\infty} \cos n\omega_0 t
\]

Here, \( \omega_0 = 2\pi/T_0 \)

When the beam is not a delta-function, the beam spectrum cannot extend to infinite frequency! The frequency is limited by \( \Delta \omega \Delta t \approx 1 \).

\[
I(t) = N_\rho e^{\sum_{n=0}^{\infty} \rho(t - nT_0)}
\]

\[
\int_{-T_0/2}^{T_0/2} \rho(t - nT_0) dt = 1
\]

\[
\rho(t) = \frac{1}{2\Delta} \begin{cases} 
1/2 & \text{if } -\Delta < t - nT_0 < \Delta, \\
0 & \text{otherwise}, 
\end{cases}
\]

\[
I(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} I(t) e^{-j\omega t} dt
\]

\[
= \left[ N_\rho e^{\omega_0} \sin \frac{\omega \Delta}{2\pi} \right] \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0).
\]

\[
\rho(t) = \frac{1}{\sqrt{2\pi} \sigma_t} e^{-\frac{(t-nT_0)^2}{2\sigma_t^2}}
\]

\[
I(\omega) = \left[ N_\rho e^{\omega_0} e^{-\omega^2 \sigma_t^2/2} \right] \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0).
\]