In a well-known paper, Saffman \(^1\) derived the expression of the lift force experienced by a solid spherical particle moving in a low-Reynolds-number shear flow. The result he obtained is valid only for very high shear rates since he assumed that, over a distance of the order of the particle radius, the velocity difference induced by the shear is much larger than the slip velocity of the particle. This condition, however, is rarely attained in the experiments. Later McLaughlin \(^2\) revisited the same problem and extended Saffman’s analysis to the more realistic case of moderate shear rates. Starting from McLaughlin’s results, Mei and Klausner \(^3\) addressed the similar problem for the case of an inviscid bubble. However for a reason which will become clear later they only solved one part of the problem and their final result is not correct. Consequently it seems that the analytical expression giving the lift force experienced by a bubble, and more generally a drop of arbitrary viscosity, moving at low Reynolds number in a shear flow has not been obtained so far. The purpose of this note is to derive this result which may be useful in two-phase flows studies. As can be expected the present derivation follows closely the method developed by Saffman and in many places we refer to intermediate results which have been established in Ref. 1.

In the problem we consider, a spherical drop of radius \(R\) whose center of mass is located at the origin of a Cartesian frame of reference \((e_x, e_y, e_z)\) is embedded in a steady incompressible linear shear flow whose undisturbed velocity field is \(U=(U_0 + \alpha y^*)e_x, \ y^* = R y\) being the local distance along the \(e_x\)-axis. We denote by \(\rho\) the density of the outer fluid, and by \(\mu\) and \(\mu'\) the dynamic viscosities of the outer and inner fluids, respectively. The flow depends on three parameters, namely the Reynolds number \(Re=\mu U_0 R/\mu\), the nondimensional shear rate \(Sr=\alpha R U_0\), and the viscosity ratio \(R_{\mu} = \mu/\mu'\). Note that \(Sr\) will be positive or negative depending on the sign of the shear \(\alpha\). In what follows \(Re\) is supposed to be much smaller than unity. Under this condition it will be shown later that, at leading order, the lift force does not depend on the density ratio of the two fluids. The velocity outside the drop is written under the form \(U+V\), \(V\) being the velocity disturbance induced by the presence of the drop. The velocity inside the drop is denoted by \(V'\). The governing equations for the incompressible flow outside the drop can then be written in dimensionless form as

\[
-\nabla P + \nabla^2 V = Re[(1 + Sr y)(e_x \nabla)V + Sr (V \cdot e_x)e_x + V \cdot \nabla V].
\]

In Eq. (1) all the distances and velocities have been made dimensionless by \(R\) and \(U_0\), respectively, and the pressure \(P\) has been made dimensionless by \(\mu U_0 / R\). On the drop surface, the boundary conditions state that the normal velocities of both fluids are zero and that both the tangential velocities and the tangential stresses are continuous. Denoting by \(n\) the outward unit normal to the drop surface, these three conditions yield

\[
\begin{align*}
V \cdot n + (1 + Sr y)e_x \cdot n &= V' \cdot n = 0, \\
\hat{n} \times [V + (1 + Sr y)e_x] &= \hat{n} \times V', \\
\hat{n} \times [\nabla V + \nabla V' + Sr(e_x e_y + e_y e_x) \cdot n] &= R_{\mu} \hat{n} \times [\nabla V' + \nabla V] \cdot n
\end{align*}
\]

where \(r=(x^2+y^2+z^2)^{1/2}\). Finally the disturbance velocity \(V\) must vanish at infinity. Once the solution of Eqs. (1)–(2) is found, the hydrodynamic force acting upon the drop can be obtained by a standard integration of the stress on the surface. However finding the solution of Eqs. (1)–(2) for \(Re\neq 0\) is very tedious, especially for a drop because it requires matching the solutions of both media. Saffman \(^1\) established a very useful result which greatly simplifies the problem. This result [Eq. (2.9) of Ref. 1] shows that up to \(O(Re)\) the force \(F\) acting on a spherical particle can be obtained by evaluating three integrals on a sphere \(\Gamma\) of arbitrary radius concentric to the particle \((r>1)\), i.e.,

\[
F = -\int_{\Gamma} P r d\Gamma = \int_{\Gamma} \nabla d\Gamma + O(Re)
\]

where the dimensionless force \(F\) is equal to the actual force divided by \(\mu RU_0\).

The solution \((V^0, P^0)\) of Eqs. (1)–(2) corresponding to creeping flow conditions \((Re=0)\) can be obtained by various methods. Using spherical harmonics, it can be put under the general form used by Saffman [see Eqs. (2.2) and (2.3) of Ref. 1] where \(P^0\) is expressed in terms of coefficients \(p_n\) while \(V^0\) involves coefficients \(p_n, \varphi_n\) and \(\chi_n\).
It is found that only four of these coefficients are nonzero in the case of a spherical drop moving in a linear shear flow. They are

\[ p_1 = -\frac{1}{2} \left( 1 + \frac{2R_\mu}{R} \right) \alpha, \quad p_2 = -\frac{5 + 2R_\mu}{1 + R_\mu} \alpha y, \]

\[ \varphi_1 = -\frac{1}{2} \left( 1 + \frac{2}{1 + R_\mu} \alpha \right), \quad \varphi_2 = -\frac{1}{2} \frac{Sr}{1 + \frac{2}{1 + R_\mu} \alpha y}. \]  

The coefficients (4) agree with those found by Saffman in the limit \( R_\mu = 0 \). However, Saffman found a fifth nonzero term because he considered a solid sphere that could rotate with an arbitrary rotation rate \( \Omega \). This led him to a coefficient \( \chi_1 \) proportional to \( \Omega \cdot Sr/2 \). This term does not appear here because the conditions (2) imply that the drop must rotate at a rate \( Sr/(1 + R_\mu) \). The corresponding solution inside the drop, say \( (V^0, P^0) \), involves five nonzero coefficients which are not given explicitly here because they are not used in the remainder of the paper. Using the coefficients (4) and Eqs. (2.2) and (2.3) of Ref. 1, the two integrals needed to obtain the force through Eq. (3) can be readily evaluated. They are

\[ \int_V p^0 dV = -\frac{2\pi}{3} \alpha \epsilon_x, \quad \int_V \nabla^2 dV = -\frac{4\pi}{3} \alpha \epsilon_x, \]

where \( \alpha \) denotes the ratio \((3 + 2R_\mu)/(1 + R_\mu) \). Note that for the complete velocity field \((1 + Sr y)\epsilon_x + V^0 \) the solution corresponding to the Eqs. (5) would be

\[ \int_V \left[ (1 + Sr y)\epsilon_x + V^0 \right] dV = 4\pi \left( r - \frac{\alpha}{3} \right) \epsilon_x. \]

Using the results (5) the force \( F^0 \) corresponding to the creeping flow solution is immediately obtained as

\[ F^0 = 2 \pi \frac{\alpha}{3} \epsilon_x. \]

This is of course the well-known Hadamard–Ribczynski expression for the drag force acting on a fluid sphere in a uniform flow (see Ref. 4, p. 30). The result (7) shows that the presence of the shear does not modify the hydrodynamic force when \( Re = 0 \). Consequently, in the same manner as for a solid particle, the lift force on a drop moving at low but finite Reynolds number originates in small inertia effects.

Having established or recalled these preliminary results we turn now to the solution of the problem for the case of low but nonzero Reynolds numbers. This solution follows essentially the matching asymptotic expansion procedure carried out by Saffman.\footnote{Saffman\textsuperscript{1}} For that reason only the key points and the differences with the original work of Saffman will be mentioned here. The small parameter of the problem can be readily identified by inserting the creeping flow solution into the momentum Eq. (1). Since this solution indicates that far from the drop the leading term of \( V^0 \) behaves as \( r^{-1} \), it is immediate to see that the two advective terms of Eq. (1) proportional to \( Sr \) become comparable to the viscous term at a typical distance \( r_1 = O(ReSr)^{-1/2} \). In contrast the advective contribution in Eq. (1) which comes from the uniform flow \( U_0 \epsilon_x \) becomes of the same order as the viscous term at a distance \( r_2 = O(Re)^{-1} \). At this point it is useful to define the ratio \( \varepsilon = r_2/r_1 = (Sr/Re)^{1/2} \). Provided \( \varepsilon \) is much larger than unity (as assumed by Saffman\textsuperscript{1} or of order unity (as assumed by McLaughlin\textsuperscript{2}), the creeping flow solution first becomes invalid because of the inertia contribution linked with the existence of the shear. Under such conditions the perturbation parameter is necessarily \( \eta = (ReSr)^{1/2} \) and we assume in the following that \( \eta \) satisfies the condition \( \eta \ll 1 \).

Thus we now look for the inner expansion of the solution at first order in \( \eta \) under the form

\[ V = V^0 + \eta V^1, \quad P = P^0 + \eta P^1, \]

\[ F = F^0 + \eta F^1. \]

Inserting expansion (8) into Eq. (1) gives the governing equation for \( V^1 \) and \( P^1 \). Saffman [see Eqs. (2.13) and (2.14) of Ref. 1] obtained the solution of that problem integrated over the sphere \( \Gamma \) under the form,

\[ \int_V \frac{P^1 \cdot dV}{r} = \eta Q^1(r) + A + B r^3. \quad (9a) \]

\[ \int_V \frac{V^1 \cdot dV}{r} = \eta Q^1(r) + C r + D + \frac{1}{2} B r^3. \quad (9b) \]

The vectors \( Q^1 \) and \( Q^1 \) contain the particular solution corresponding to the advective terms resulting from the creeping flow solution \( V^0 \). Saffman showed that \( Q^1 \) and \( Q^1 \) are polynomial expressions and that their highest order terms are in \( r \) (this behavior can also be deduced directly from Eq. (1) since the highest order term of the right-hand side is in \( r^{-1} \), implying that \( V^1 \) must grow linearly with \( r \) at large distances). This means that, similarly to what happens in the classical Oseen’s problem, the particular solution is unable to vanish at infinity. Therefore the complementary solution which appears in Eqs. (9a)–(9b) is needed to obtain a uniformly valid expansion. This second part of the solution comes from the integration of the momentum Eq. (1) and involves four unknown constant vectors \( A, B, C, D \). At that point it is particularly relevant to evaluate the solution (9b) at distances from the sphere where the creeping flow solution becomes invalid, i.e., for \( r = O(\eta^{-1}) \). At such distances the leading terms of Eq. (1) produced by the creeping flow solution are \( O(\eta^5) \). Thus, in order that \( \nabla V^1 - \nabla P^1 \) balances the difference of these terms, \( V^1 \) must be \( O(1) \) there, implying that the largest terms in the right-hand side of Eq. (9b) must be \( O(\eta^{-1}) \). This constraint is obviously satisfied by the particular solution \( \eta Q^1(r) \) and it requires that \( C = O(1) \) and that \( B = O(\eta^2) \). Consequently close to the drop, i.e., for \( r = O(1) \), the solutions (9a)–(9b) becomes up to \( O(\eta) \)

\[ \int_V \frac{P^1 \cdot dV}{r} = A, \quad \int_V \frac{V^1 \cdot dV}{r} = C r + D. \quad (10) \]

Saffman\textsuperscript{1} obtained the relation between \( C \) and \( D \) by writing the second of Eqs. (10) right on the sphere surface, i.e., for \( r = 1 \). Since he was considering the case of a solid sphere, the no-slip condition led him immediately to \( \mathbf{D} = -C \). However this condition does not hold for a drop, as shown by Eq. (2). The fact that in that case the boundary conditions at \( r = 1 \) are not the same for the normal and the tangential components of
is the main difficulty of the extension of Saffman’s result to a drop of arbitrary viscosity. This difficulty can however be very easily overcome. For that purpose it is enough to remark that the complementary solution which appears in Eqs. (9a)–(9b) is nothing else than the solution of Eq. (1) with a zero right-hand side, i.e., the solution of a creeping flow problem around the drop. As pointed out by Saffman, B results from a flow which is parabolic when \( r \to \infty \), whereas \( C \) corresponds to a uniform flow at infinity. Consequently the part of this complementary solution that subsists at leading order in Eqs. (10) is nothing else than the creeping flow solution corresponding to a uniform flow whose strength and orientation are unknown at the present stage. Since Eqs. (5) and (6) give us the integrals similar to Eqs. (10) for the particular case of a unit flow along the \( e_\xi \)-axis [the \( \psi y \) term giving no contribution], for an arbitrary uniform flow we obtain by identification

\[
D = -\frac{R}{3} C + O(\eta) \quad \text{and} \quad A = \frac{1}{2} D + O(\eta).
\]

The fact that the flow defined by Eq. (11) is a solution of the creeping flow equations around the drop ensures that \( V^1 \) satisfies the condition of zero normal velocity for \( r = 1 \). Furthermore it ensures that a correction of the same order of magnitude \((V^1, P^1)\) can be added inside the drop in order to satisfy the two last conditions in Eq. (2). The solution \((V^1, P^1)\) can now be used to obtain the first-order correction to the hydrodynamic force. Inserting Eqs. (10) into Eq. (3) and using the result (11) yields

\[
F^1 = \frac{R}{2} C + O(\eta).
\]

Note that, as mentioned previously, up to this order the force does not depend on the densities of both fluids since \( F^0 \) and \( F^1 \) only involve creeping flow solutions in which no inertia effect appears.

To complete the determination of \( F^1 \) it is obviously necessary to specify \( C \). This requires us to obtain the outer expansion of \( V \) and to match it with the inner expansion determined above. The momentum equation governing the outer expansion is classically found by rewriting Eq. (1) in terms of the strained coordinates \( \tilde{x} = \eta x, \tilde{y} = \eta y, \tilde{z} = \eta z \), and \( \tilde{r} = \eta r \) and introducing the corresponding operator \( \overline{\nabla} \). The effect of the particle on the far flow is obtained by introducing a point-force of strength \(-F^0\) located at \( \tilde{r} = 0 \). Thus for \( \tilde{r} = O(1) \) the momentum equation governing the outer expansion of \( V \), say \( v \), is in the limit \( |v| \to 0 \),

\[
-\overline{\nabla} p + \overline{\nabla}^2 v = f^0 \delta(\tilde{r}) + [(-\epsilon^{-1} + \gamma)(e_\xi \cdot \overline{\nabla})v + (v \cdot e_\xi)e_\xi],
\]

where \( f^0 = \eta F^0 \) and \( p = \rho f/\eta \). Saffman \(^1\) solved Eq. (13) in Fourier space in the limit \( \epsilon \to \infty \). McLaughlin \(^2\) generalized his solution to finite values of \( \epsilon \). In both cases the value of \( |F^0| \) was set to 6 \( \pi \) since only the case of the solid sphere was considered. However since the problem is obviously linear in \( F^0 \) the outer solution obtained by the previous authors is valid whatever the nature of the particle, provided the result is expressed as a function of \( |F^0| \). Once \( v \) is determined, \( f_{a}vd\Gamma \) can be evaluated and matched with \( f_{a}(V^0 + \eta V^1)d\Gamma \) in the common limit \( \tilde{r} \to 0 \) and \( r \to \infty \). This matching procedure shows that the \( y \)-component of the vector \( C \) is given by

\[
C_{y} = \frac{1}{\pi^{2}} \frac{\text{Sr}}{|\text{Sr}|} J(\epsilon) + O(\eta),
\]

where \( J(\epsilon) \) is a three-dimensional integral which was evaluated numerically by McLaughlin \(^3\) and \( \text{Sr}/|\text{Sr}| \) accounts for the sign of the shear. In the limit \( \epsilon \to \infty \) considered by Saffman the corresponding value of \( J(\infty) = 2.255 \). The lift force is finally obtained by inserting Eq. (14) into Eq. (12) and using the result (7). Neglecting terms of order \( \text{Re} |\text{Sr}| \) this procedure yields

\[
F_L = \eta \text{Re} F^1 \cdot e_y = \frac{\text{Re} |\text{Sr}|^{1/2} \text{Sr}}{\pi} J(\epsilon).
\]

In the limit \( \text{Re} \to 0 \), one has \( \text{Re}^{2} = 9 \) and Eq. (15) coincides with the result established by McLaughlin for a solid sphere [see Eq. (3.19) of Ref. 2 where the force is normalized differently]. In contrast in the limit \( \text{Re} \to \infty \) corresponding to the case of an inviscid bubble, one has \( \text{Re}^{2} = 4 \) so that Eq. (15) shows that the ratio of the lift forces experienced by a spherical inviscid bubble and a solid sphere is \((2/3)^{2}, \text{i.e.}, 0.444 \). In a previous attempt to determine the lift force on a spherical bubble, Mei and Klauser \(^4\) found incorrectly this ratio to be \( 2/3 \). This is simply because these authors reconsidered only the outer expansion of the problem and used Saffman’s result for the inner expansion; they obtained the result (14) but instead of using (12) they maintained the relation \( F^1 = 3/2C \) which is only valid for a solid sphere. Finally it is interesting to discuss the result (15) in terms of vorticity. It is well known that when a body experiences a creeping motion, the pressure gradient in the flow and the drag force are both directly proportional to the strength of the vorticity generated at the surface of the body. Thus the expression of \( p_{1} \), given in Eq. (4) or the expression (7) of \( F^0 \) shows that the interfacial vorticity on the drop is proportional to \( \epsilon \). The reason why the lift force (15) is proportional to \( \epsilon^{2} \) can then be easily understood. For a given slip velocity, the disturbance produced in the far field by the drop is directly proportional to \( F^0 \) and \( C \) is thus proportional to \( \epsilon \). Then the uniform flow of strength \( C \) required to satisfy the condition at infinity induces a correction of vorticity proportional to \( \epsilon C \) at the surface of the drop. This vorticity results in a correction of similar intensity \( F^1 \) in the hydrodynamic force. The fact that \( F_L \) is proportional to \( \epsilon^{2} \) follows directly.

\(^{1}\)P. G. Saffman, “The lift force on a small sphere in a slow shear flow,” J. Fluid Mech. 22, 385 (1965); Corrigendum, ibid. 31, 624 (1968).

