2.3 Off-momentum closed orbit and dispersion function

We have discussed the we discussed the closed orbit for a reference particle with momentum p_0, including dipole field errors and quadrupole misalignment. By using closed-orbit correctors, we can achieve an optimized closed orbit that essentially passes through the center of all accelerator components. This closed orbit is called the “golden orbit,” and a particle with momentum p_0 is called a **synchronous** particle. However, a beam is made of particles with momenta distributed around the synchronous momentum p_0. What happens to particles with momenta different from p_0? Here we study the effect of off-momentum on the closed orbit. For a particle with momentum p, the momentum deviation is $\Delta p=p-p_0$ and the fractional momentum deviation is $\delta=\Delta p/p_0$, which is typically small of the order of 10^{-6} to 10^{-3}. Since δ is small, we can study the motion of off-momentum particles perturbatively.

$$\begin{cases} x'' - \dfrac{\rho+x}{\rho^2} = \pm\dfrac{B_z}{B\rho}\dfrac{p_0}{p}\left(1+\dfrac{x}{\rho}\right)^2, \\ z'' = \mp\dfrac{B_x}{B\rho}\dfrac{p_0}{p}\left(1+\dfrac{x}{\rho}\right)^2, \end{cases} \qquad \begin{aligned} &B_z = \mp B_0 + B_1 x + \cdots, \\ &B_0/B\rho = 1/\rho \end{aligned}$$

$$x'' + K_x(s)x = 0, \qquad K_x = 1/\rho^2 \mp K_1(s),$$
$$z'' + K_z(s)z = 0, \qquad K_z = \pm K_1(s),$$

$$p = p_0 + \Delta p, \ \delta = \frac{\Delta p}{p_0}$$

$$x'' - \frac{\rho+x}{\rho^2} = \left(-\frac{1}{\rho} + Kx\right)\frac{1}{1+\delta}\left(1 + 2\frac{x}{\rho} + \frac{x^2}{\rho^2}\right)$$

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K}{(1+\delta)}\right)x = \frac{\delta}{\rho(1+\delta)}$$

$$x'' + \left(\frac{1}{\rho^2} - K(s) + \Delta K(s)\right)x = \frac{\delta}{\rho(1+\delta)}, \quad \Delta K(s) = \left(\frac{2}{\rho^2} - K(s)\right)\delta + O(\delta^2)$$

$$x'' - \frac{\rho+x}{\rho^2} = \pm\frac{B_z}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2, \qquad z'' = -\frac{B_x}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2.$$

Expanding the betatron equation of motion, we obtain

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}, \quad K(s) = \frac{B_1}{B\rho}, \quad B_1 = \frac{\partial B_z}{\partial x}$$

Because of the horizontal dipole for a complete revolution on the horizontal plane, the betatron equation of motion is in-homogeneous for particles with nonzero δ. The solution of the inhomogeneous equation is a linear combination of the particular solution and the solution of the homogeneous equation, i.e.

$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + \left(K_x(s) + \Delta K_x\right)x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + \left(K_x(s) + \Delta K_x\right)D = \frac{1}{\rho} + O(\delta)$$

The solution of the homogeneous equation is the betatron oscillation. The solution of the inhomogeneous equation is called the dispersion function, or the off-momentum closed orbit.

$$x'' + \left(\frac{1}{\rho^2} - K(s) + \Delta K(s)\right)x = \frac{\delta}{\rho(1+\delta)}, \quad \Delta K(s) = \left(\frac{2}{\rho^2} - K(s)\right)\delta + O(\delta^2)$$

$$x'' + \left(\frac{1}{\rho^2} - K(s)\right)x = \frac{\delta}{\rho(1+\delta)},$$

$$x = x_\beta + x_{\rm co} = x_\beta + D\delta$$

$$D'' + \left(\frac{1}{\rho^2} - K(s)\right)D = \frac{1}{\rho},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

$$\bar{d} = \begin{cases} \begin{pmatrix} \frac{1}{\rho K_x}(1-\cos\sqrt{K_x}s) \\ \frac{1}{\rho\sqrt{K_x}}\sin\sqrt{K_x}s \end{pmatrix} & \text{if } K_x > 0, \\ \begin{pmatrix} \frac{1}{\rho|K_x|}(-1+\cosh\sqrt{|K_x|}s) \\ \frac{1}{\rho\sqrt{|K_x|}}\sinh\sqrt{|K_x|}s \end{pmatrix} & \text{if } K_x < 0. \end{cases}$$

For a pure dipole:

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -(1/\rho)\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix},$$

$$x = x_\beta + x_{\rm co} = x_\beta + D\delta$$

$$D'' + \left(\frac{1}{\rho^2} - K(s)\right) D = \frac{1}{\rho},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For a pure dipole:

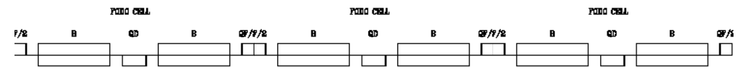
$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \ell & \frac{1}{2}\ell\theta^2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

For pure quadrupoles:

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell & 0 \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: FODO cell



$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) & 2L\theta(1 + \frac{L}{4f}) \\ -\frac{L}{2f^2} + \frac{L^2}{4f^3} & 1 - \frac{L^2}{2f^2} & 2\theta(1 - \frac{L}{4f} - \frac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}.$$

Using the Courant-Snyder parameterization for the transfer matrix, we obtain

$$\sin\frac{\Phi}{2} = \frac{L}{2f}, \quad \beta_{\rm F} = \frac{2L(1 + \sin(\Phi/2))}{\sin\Phi}, \quad \alpha_{\rm F} = 0,$$

$$D_{\rm F} = \frac{L\theta(1 + \frac{1}{2}\sin(\Phi/2))}{\sin^2(\Phi/2)}, \quad D'_{\rm F} = 0.$$

1) The dispersion is proportional to the cell length L times the bending angle θ, and inversely proportional to the square of the phase advance.
2) The dispersion at other locations can be obtained by using the transfer matrix $M(s_2,s_1)$.

The AGS (33 GeV proton synchrotron built in 1960) is simply made of 60 (5×12) FODO cells. The CPS (28 GeV) is simply made of 50 FODO cells.

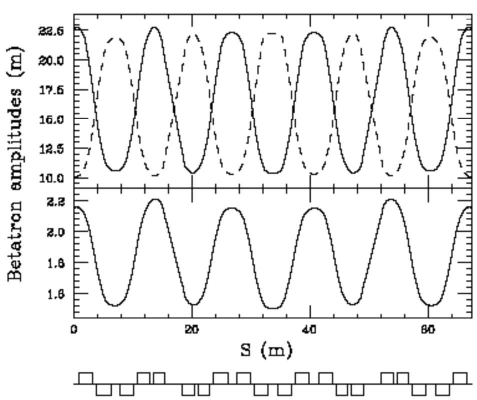


Figure 2.5: The betatron amplitude functions for one superperiod of the AGS lattice, which made of 20 combined-function magnets. The upper plot shows β_x (solid line) and β_z (dashed line). The middle plot shows the dispersion function D_x. The lower plot shows schematically the placement of combined-function magnets. Note that the superperiod can be well approximated by five regular FODO cells. The phase advance of each FODO cell is about 52.8°.

The transfer matrix of a periodic cell:

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \frac{M_{13}(1 - M_{22}) + M_{12}M_{23}}{2 - M_{11} - M_{22}} = \frac{M_{13}(1 - \cos\Phi + \alpha\sin\Phi) + M_{23}\beta\sin\Phi}{2(1 - \cos\Phi)},$$

$$D' = \frac{M_{13}M_{21} + (1 - M_{11})M_{23}}{2 - M_{11} - M_{22}} = \frac{-M_{13}\gamma\sin\Phi + M_{23}(1 - \cos\Phi - \alpha\sin\Phi)}{2(1 - \cos\Phi)},$$

Solving M_{13} and M_{23} as functions of D and D', the 3×3 transfer matrix is

$$\mathbf{M} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi & (1 - \cos\Phi - \alpha\sin\Phi)\ D - \beta\sin\Phi\ D' \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi & \gamma\sin\Phi\ D + (1 - \cos\Phi + \alpha\sin\Phi)\ D' \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{H}$-Function, Action, and Integral Representation

We recall that we define the normalized betatron phase-space coordinates:

$$y^2+P_y^{\,2}=y^2+(\alpha y+\beta y')^2=2\beta J.$$

We define the normalized dispersion function coordinates:

$$\begin{cases} X_{\rm d} = \dfrac{1}{\sqrt{\beta_x}}D = \sqrt{2J_{\rm d}}\cos\Phi_{\rm d}, \\ P_{\rm d} = \sqrt{\beta_x}D' + \dfrac{\alpha_x}{\sqrt{\beta_x}}D = -\sqrt{2J_{\rm d}}\sin\Phi_{\rm d}, \end{cases}$$

The H-function of the dispersion invariant is defined as:

$$\mathcal{H}(D,D') = \gamma_x D^2 + 2\alpha_x DD' + \beta_x D'^2 = \frac{1}{\beta_x}[D^2 + (\beta_x D' + \alpha_x D)^2].$$

$$J_{\rm d} = \frac{1}{2}\mathcal{H}(D,D')$$

In a straight section, $J_{\rm d}$ is invariant and d, aside from a constant, is identical to the betatron phase advance. In a region with dipoles, $J_{\rm d}$ is not constant. The change of the dispersion function across a thin dipole is D = 0 and D′ = θ, i.e.

$$\Delta X_{\rm d} = 0, \quad \Delta P_{\rm d} = \sqrt{\beta_x}\,\Delta D' = \sqrt{\beta_x}\,\theta$$

For a FODO cell, the dispersion H-function at the defocussing quadrupole is larger than that at the focusing quadrupole, i.e. $H_F \le H_D$, where

$$\mathcal{H}_{\rm F} = \frac{L\theta^2\sin\Phi(1+\frac{1}{2}\sin\frac{\Phi}{2})^2}{2(1+\sin\frac{\Phi}{2})\sin^4\frac{\Phi}{2}}, \quad \mathcal{H}_{\rm D} = \frac{L\theta^2\sin\Phi(1-\frac{1}{2}\sin\frac{\Phi}{2})^2}{2(1-\sin\frac{\Phi}{2})\sin^4\frac{\Phi}{2}}$$

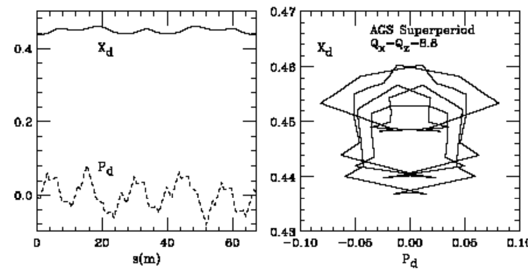


Figure 2.27: Left: Normalized dispersion phase-space coordinates $X_{\rm d}$ and $P_{\rm d}$ are plotted in a superperiod of the AGS lattice. Right: the coordinates are shown in $X_{\rm d}$ vs $P_{\rm d}$. The scales for both $X_{\rm d}$ and $P_{\rm d}$ are $\mathrm{m}^{1/2}$.

Integral representation of the dispersion function

$$\theta = \frac{\Delta p}{p_0}\frac{ds}{\rho}, \qquad \frac{\Delta p}{p_0}\frac{1}{\rho} \to \frac{\Delta B}{B\rho} \qquad y_{\rm co} = D(s)\frac{\Delta p}{p_0}, \qquad D(s) == \int_s^{s+C}\frac{G_x(s,t)}{\rho(t)}dt,$$

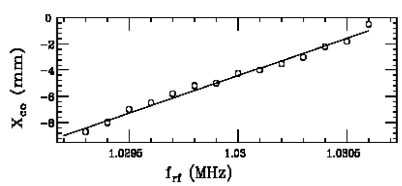
$$\begin{cases} X_{\rm d}(s) = \dfrac{1}{2\sin\pi\nu_x}\displaystyle\int_s^{s+C}\frac{\sqrt{\beta_x(t)}}{\rho}\cos(\psi_x(t)-\psi_x(s)-\pi\nu_x)dt \\ P_{\rm d}(s) = \dfrac{-1}{2\sin\pi\nu_x}\displaystyle\int_s^{s+C}\frac{\sqrt{\beta_x(t)}}{\rho}\sin(\psi_x(t)-\psi_x(s)-\pi\nu_x)dt. \end{cases}$$

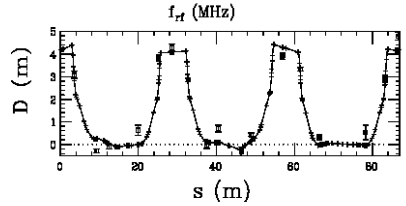
How to measure D(s)? $x_{co}(s)=D(s)\delta$.

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta,$$

$$\Delta f/f_0 = -\eta\delta,$$

$$D = \frac{dx_{\rm co}}{d(\Delta p/p_0)} = -\eta\, f_0\frac{dx_{\rm co}}{df_0},$$





Dispersion Suppression and Dispersion Matching

The first-order achromat theorem states that a lattice of n repetitive cells is achromatic to first order if and only if $M^n = I$ or each cell is achromatic. Here M is the 2 × 2 transfer matrix of each cell, and I is a 2 × 2 unit matrix.

$$R = \begin{pmatrix} M & \bar{d} \\ 0 & 1 \end{pmatrix}, \quad \bar{d} = \begin{pmatrix} d \\ d' \end{pmatrix}$$

$$R^n = \begin{pmatrix} M^n & (M^{n-1}+M^{n-2}+\cdot+1)\bar{d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M^n & \bar{w} \\ 0 & 1 \end{pmatrix}$$

$$\bar{w} = (M^n - I)(M-I)^{-1}\bar{d}$$

$$\text{if } M^n = I \text{ or } \bar{d} = 0 \qquad \bar{w} = 0$$

Achromat Transport Systems

The double-bend achromat [OO] B {O QF O} B [OO]

$$\begin{pmatrix} D_c \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/(2f) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$f = \frac{1}{2}\left(L_1 + \frac{1}{2}L\right), \quad D_c = \left(L_1 + \frac{1}{2}L\right)\theta.$$

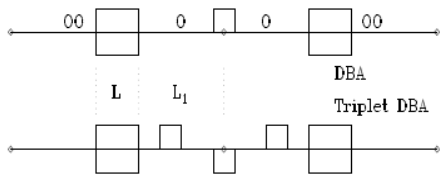
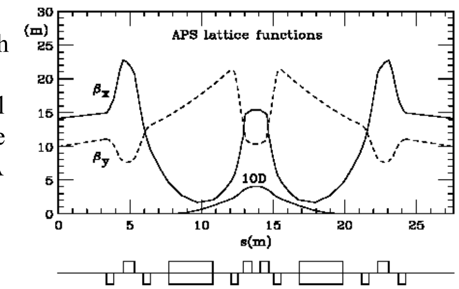
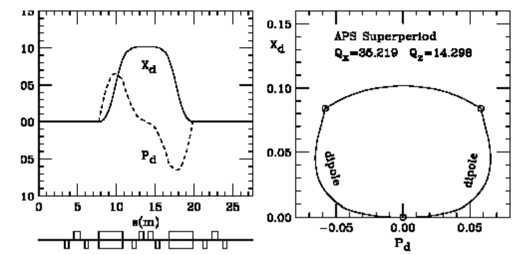


Figure 2.31: Schematic plots of DBA cells. Upper plot: standard DBA cell, where O and OO can contain doublets or triplets for optical match. Lower plot: triplet DBA, where the quadrupole triplet is arranged to attain betatron and dispersion function match of the entire module.

Example: APS lattice is made of 40 Double-bend Achromats (DBA) with a total length of 1104m. The momentum compaction factor for all DBA lattice is **$\alpha_c = \rho\theta^2/(6R)$**. Because of its simplicity and flexibility, DBA lattice is commonly used as basic cells of synchrotron light source design.



Question: identify dispersion phase space coordinates with accelerator elements



Path length, momentum compaction and phase-slip factors:

We recall the Frenet-Serret coordinate system. The path length of the reference orbit in one complete revolution is

$$d\ell = \sqrt{(1+\tfrac{x}{\rho})^2(ds)^2 + (dx)^2 + (dz)^2} = ds[(1+\tfrac{x}{\rho})^2 + x'^2 + z'^2]^{1/2} \cong ds[1+\tfrac{x}{\rho}]$$

$$C = \oint d\ell \cong \oint ds + \oint \frac{x}{\rho} ds = C_0 + \oint \frac{x_\beta + D\delta}{\rho} ds$$

$$\Delta C = \oint \frac{D}{\rho} ds\,\delta, \qquad \alpha_c \equiv \frac{d\Delta C}{C d\delta} = \frac{1}{C}\oint \frac{D}{\rho} ds \cong \frac{1}{C}\sum_i \langle D_i \rangle \theta_i$$

Here α_c is called the **momentum compaction factor**, which is a measure of the compactness of the orbit length for particles with different momenta. The important of the orbit length is that the particles in synchrotron must synchronize with the rf accelerating voltage. Note that the orbiting time for particle is T=C/v. Thus

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta, \qquad \eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}.$$

Here η is called the phase slip-factor.

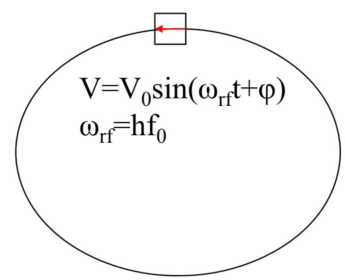
$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta, \qquad \eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}.$$

$$V_s = V_0 \sin\phi_s, \quad \dot{E}_0 = f_0 e V_0 \sin\phi_s, \quad \dot{E} = f e V_0 \sin\phi,$$

$\phi = -h\theta$, where θ is the actual angular position of the particle

$$\frac{d}{dt}(\phi - \phi_s) = -h\Delta\omega = h\omega_0 \frac{\Delta T}{T_0} = h\eta\omega_0 \frac{\Delta p}{p_0} = \frac{\eta h \omega_0^2}{\beta^2 E_0}\frac{\Delta E}{\omega_0}.$$

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi} e V_0 (\sin\phi - \sin\phi_s),$$



$$\frac{\Delta \upsilon}{\upsilon} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0} = \frac{1}{\beta^2\gamma^2}\frac{\Delta E}{E_0}$$

$$\frac{\dot{E}}{\omega} - \frac{\dot{E}_0}{\omega_0} = \frac{\dot{E} - \dot{E}_0}{\omega_0} - (\frac{1}{\omega_0} - \frac{1}{\omega})\dot{E} \approx \frac{\dot{E} - \dot{E}_0}{\omega_0} + \dot{E}\frac{\Delta(1/\omega_0)}{\Delta E}\Delta E = \frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right)$$

Phase stability and the synchrotron equation of motion:

$$\frac{d}{dt}(\phi-\phi_s) = -h\Delta\omega = h\omega_0\frac{\Delta T}{T_0} = h\eta\omega_0\frac{\Delta p}{p_0} = \frac{\eta h\omega_0^2}{\beta^2 E_0}\frac{\Delta E}{\omega_0}.$$

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi}eV_0(\sin\phi-\sin\phi_s),$$

$$\frac{d^2(\phi-\phi_s)}{dt^2} = \frac{\eta h\omega_0^2 eV_0}{2\pi\beta^2E_0}(\sin\phi-\sin\phi_s) \approx \frac{\eta\cos\phi_s h\omega_0^2 eV_0}{2\pi\beta^2E_0}(\phi-\phi_s).$$

$$\eta\cos\phi_s < 0,$$

$$\begin{cases} 0\le\phi_s\le\pi/2 & \text{if } \gamma<\gamma_T \text{ or } \eta<0, \\ \pi/2\le\phi_s\le\pi & \text{if } \gamma>\gamma_T \text{ or } \eta>0. \end{cases}$$

$$\eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}.$$

$$\omega_{syn} = \omega_0\sqrt{\frac{heV_0|\eta\cos\phi_s|}{2\pi\beta^2E_0}}.$$

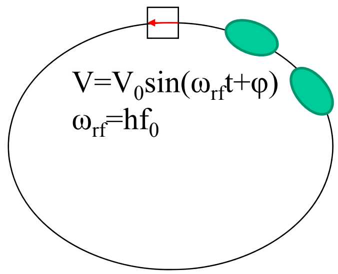
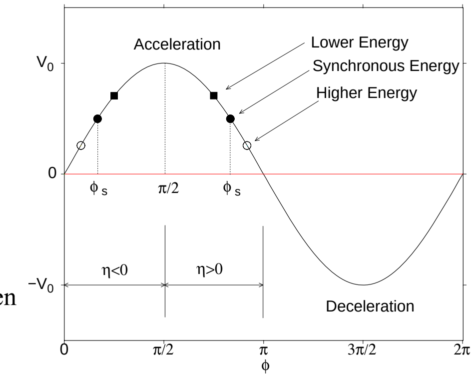


Illustration of the Phase stability: A beam bunch consists of particles with slightly different momenta. A particle with momentum p has its own off-momentum closed orbit Dδ. Since the energy gain depends sensitively on the synchronization of rf field and particle arrival time, what happens to a particle with a slightly different momentum when the synchronous particle is accelerated?



The key answer is the discovery of the phase stability of synchrotron motion by McMillan and Veksler. If the revolution frequency *f* is higher for a higher momentum particle, i.e. d*f*/dδ>0, the higher energy particle will arrive at the rf gap earlier, i.e. $\varphi<\varphi_s$. Therefore if the rf wave synchronous phase is chosen such that $0<\varphi_s<\pi/2$, higher energy particles will receive less energy gain from the rf gap.

Similarly, lower energy particles will arrive at the same rf gap later and gain more energy than the synchronous particle. This process provides the phase stability of synchrotron motion. In the case of d*f*/dδ<0, phase stability requires $\pi/2<\varphi_s<\pi$.

Synchrotron equation of motion:

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s) \qquad \phi_{n+1} = \phi_n + \frac{2\pi\eta}{\beta^2E}\Delta E_{n+1}$$

$$\omega_s = \omega_0\sqrt{\frac{heV|\eta_0\cos\phi_s|}{2\pi\beta^2E}} = \frac{c}{R}\sqrt{\frac{heV|\eta\cos\phi_s|}{2\pi E}},$$

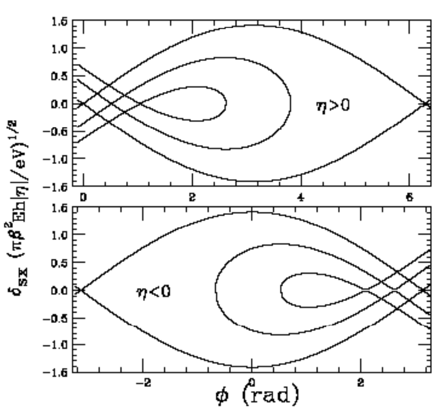


Figure 3.3: The separatrix orbits for $\eta>0$ with $\phi_s = 2\pi/3, 5\pi/6, \pi$, and for $\eta<0$ with $\phi_s = 0, \pi/6, \pi/3$. The phase space area enclosed by the separatrix is called the bucket area. The stationary buckets that have largest phase space areas correspond to $\phi_s =$ 0 and π respectively.

Summary:

Particle motion in an accelerator can be described by 3D simple harmonic motion. The transverse degree of freedom is called **betatron motion** and the longitudinal degree of freedom is called the **synchrotron motion**. The betatron tunes are number of betatron oscillations per revolution, and the synchrotron tune is the number of synchrotron oscillations per period. The betatron tunes increase with the size of the accelerator, while the synchrotron tune is about 10^{-4} to 10^{-2}. The momentum compaction factor plays an important role in the accelerator. Typically, the momentum compaction factor for FODO cell lattice is $\boldsymbol{\alpha_c \sim 1/\nu_x^2}$. Thus the transition energy is $\gamma_T \sim \nu_x$. However, the momentum compaction for accelerators can be changed by changing the dispersion function in dipoles.

$$\Delta C = \oint\frac{D}{\rho}ds\,\delta, \qquad \alpha_c \equiv \frac{d\Delta C}{Cd\delta} = \frac{1}{C}\oint\frac{D}{\rho}ds \cong \frac{1}{C}\sum_i\langle D_i\rangle\theta_i$$