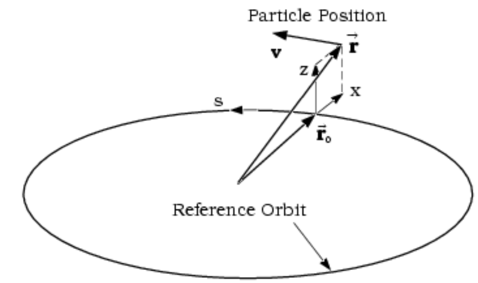
Transverse (Betatron) Motion

- Linear betatron motion
- Effects of imperfections of magnets
- Dispersion function of off momentum particle
- Simple Lattice design considerations

$$\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{\upsilon}\times\vec{B})$$

$$\vec{E} = -\nabla\Phi - \frac{\partial\vec{A}}{\partial t},\ \vec{B} = \nabla\times\vec{A}$$

$$H = e\Phi + c\sqrt{m^2c^2 + (\vec{p} - e\vec{A})^2}$$



Frenet-Serret coordinate system: We assume that there exists a closed orbit $r_0(s)$. The coordinates around the reference orbit is defined by

$$\hat{s} = \frac{d\vec{r}_0}{ds},\ \ \hat{x} = -\rho\frac{d\hat{s}}{ds},\ \ \hat{z} = \hat{x}\times\hat{s} \qquad \hat{x}' = \tfrac{1}{\rho}\hat{s} + \tau\,\hat{z},\ \ \hat{s}' = -\tau\,\hat{x}$$

$$\vec{r} = \vec{r}_0 + x\hat{x} + z\hat{z}$$

How to transform from the original coordinate system onto the Frenet-Serret coordinate system? **Generating function**!

$$F_3(P,x,s,z) = -\vec{P}\cdot(\vec{r}_0 + x\hat{x} + z\hat{z})$$

$$p_s = -\frac{\partial F_3}{\partial s} = (1+\frac{x}{\rho})\vec{P}\cdot\hat{s},\ \ p_x = -\frac{\partial F_3}{\partial x} = \vec{P}\cdot\hat{x},\ \ p_z = -\frac{\partial F_3}{\partial z} = \vec{P}\cdot\hat{z},$$

Canonical Transformations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q},\quad \frac{dq}{dt} = \frac{\partial H}{\partial p};\quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Canonical transformation: $(q,p) \longrightarrow (Q,P)$
Transformation that preserves Hamilton's equations of motion

$$G = F_1(q,Q,t):\ p = \frac{\partial F_1}{\partial q},\ P = -\frac{\partial F_1}{\partial Q};\quad \mathcal{H}(Q,P,t) = H(q,p,t) + \frac{\partial F_1}{\partial t}$$

$$G = F_2(q,P,t):\ p = \frac{\partial F_2}{\partial q},\ Q = \frac{\partial F_2}{\partial P};\quad \mathcal{H}(Q,P,t) = H(q,p,t) + \frac{\partial F_2}{\partial t};$$

$$G = F_3(p,Q,t):\ q = -\frac{\partial F_3}{\partial p},\ P = -\frac{\partial F_3}{\partial Q};\quad \mathcal{H}(Q,P,t) = H(q,p,t) + \frac{\partial F_3}{\partial t};$$

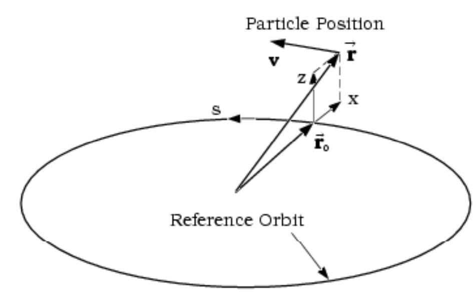
$$G = F_4(p,P,t):\ q = -\frac{\partial F_4}{\partial p},\ Q = \frac{\partial F_4}{\partial P};\quad \mathcal{H}(Q,P,t) = H(q,p,t) + \frac{\partial F_4}{\partial t}.$$

$$A_s = (1+\frac{x}{\rho})\vec{A}\cdot\hat{s},\ A_x = \vec{A}\cdot\hat{x},\ A_z = \vec{A}\cdot\hat{z},$$

$$H = e\Phi + c\left[m^2c^2 + \frac{(p_s - eA_s)^2}{(1+x/\rho)^2} + (p_x - eA_x)^2 + (p_z - eA_z)^2\right]^{1/2}$$

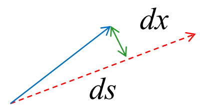
$$\dot{s} = \frac{\partial H}{\partial p_s}, \dot{p}_s = -\frac{\partial H}{\partial s};\quad \dot{x} = \frac{\partial H}{\partial p_x}, \dot{p}_x = -\frac{\partial H}{\partial x};\quad \dot{z} = \frac{\partial H}{\partial p_z}, \dot{p}_z = -\frac{\partial H}{\partial z}.$$

The phase space coordinates are ***(x,s,z)*** with independent coordinate ***t***. In one revolution, the time advances T_0, called the orbital period. In one orbital period, the particle orbit is equal to the circumference *C*. All accelerator components repeat in each orbital period. It would be nice to use ***s*** as the independent coordinate. How to make this coordinate transfer?



(x,px,z,pz,s,ps;t)→(x,px,z,pz,t,E;s)

$$x' = \frac{dx}{ds} = \frac{\dot{x}}{\dot{s}} = \left(\frac{\partial H}{\partial p_x}\right)\left(\frac{\partial H}{\partial p_s}\right)^{-1} = \frac{\partial(-p_s)}{\partial p_x},$$



$$dH = (\partial H/\partial p_x)dp_x + (\partial H/\partial p_s)dp_s = 0$$

$$t' = \frac{\partial p_s}{\partial H}, H' = -\frac{\partial p_s}{\partial t}; \quad x' = -\frac{\partial p_s}{\partial p_x}, p_x' = \frac{\partial p_s}{\partial x}; \quad z' = -\frac{\partial p_s}{\partial p_z}, p_z' = \frac{\partial p_s}{\partial z}.$$

These equations indicate that $-p_s$ becomes the new Hamiltonian with the $(x,p_x,z,p_z,t,-H)$ and s as the independent coordinate.

$$\tilde{H} = -\left(1+\frac{x}{\rho}\right)\left[\frac{(H-e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2\right]^{1/2} - eA_s,$$

$$\tilde{H} \approx -p\left(1+\frac{x}{\rho}\right) + \frac{1+x/\rho}{2p}[(p_x - eA_x)^2 + (p_z - eA_z)^2] - eA_s \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

$$x' = \frac{\partial \tilde{H}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{H}}{\partial x}, \quad z' = \frac{\partial \tilde{H}}{\partial p_z}, \quad p_z' = -\frac{\partial \tilde{H}}{\partial z}, \quad t' = \frac{\partial \tilde{H}}{\partial H}, \quad -H' = -\frac{\partial \tilde{H}}{\partial t}.$$

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s), \quad \phi_{n+1} = \phi_n + \frac{2\pi\eta}{\beta^2 E}\Delta E_{n+1}$$ Synchrotron motion

$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho}$$ Hill's equation

Transverse magnetic field: $\nabla\times A=B$, $\nabla\bullet B=0$. For 2D magnetic field, B can be represented by either one component of the vector potential A_s, or by a scaler potential Φ, i.e. $B_x=-\partial A_s/\partial z$, $B_z=\partial A_s/\partial x$, or $B_x=\nabla_x\Phi$, $B_z=\nabla_z\Phi$. Although the field can be represented two ways, only the vector potential serves as the "potential" in the betatron Hamiltonian. For two dimensional magnetic field, one can expand the magnetic field using **Beth representation**:

$$H = -p\left(1+\frac{x}{\rho}\right) + \frac{1+x/\rho}{2p}[p_x^2 + p_z^2] - eA_s$$

$$x'' - \frac{\rho + x}{\rho^2} = \pm\frac{B_z}{B\rho}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2, \qquad z'' = \mp\frac{B_x}{B\rho}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2.$$

Magnetic Field in Frenet-Serret Coordinate System

$$h_x = 1, \quad h_s = 1+\frac{x}{\rho}, \quad h_z = 1$$

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{x} + \frac{1}{h_s}\frac{\partial\Phi}{\partial s}\hat{s} + \frac{\partial\Phi}{\partial z}\hat{z},$$

$$\nabla\cdot\vec{A} = \frac{1}{h_s}\left[\frac{\partial(h_s A_1)}{\partial x} + \frac{\partial A_2}{\partial s} + \frac{\partial(h_s A_3)}{\partial z}\right],$$

$$\nabla\times\vec{A} = \frac{1}{h_s}\left[\frac{\partial A_3}{\partial s} - \frac{\partial(h_s A_2)}{\partial z}\right]\hat{x} + \left[\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right]\hat{s} + \frac{1}{h_s}\left[\frac{\partial(h_s A_2)}{\partial x} - \frac{\partial A_1}{\partial s}\right]\hat{z},$$

$$\nabla^2\Phi = \frac{1}{h_s}\left[\frac{\partial}{\partial x}h_s\frac{\partial\Phi}{\partial x} + \frac{\partial}{\partial s}\frac{1}{h_s}\frac{\partial\Phi}{\partial s} + \frac{\partial}{\partial z}h_s\frac{\partial\Phi}{\partial z}\right],$$

where $A_1 = \vec{A}\cdot\hat{x}$, $A_2 = \vec{A}\cdot\hat{s}$, and $A_3 = \vec{A}\cdot\hat{z}$.

$$\vec{B} = B_x(x,z)\hat{x} + B_z(x,z)\hat{z}$$

$$B_x = -\frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial z} = -\frac{1}{h_s}\frac{\partial A_s}{\partial z}, \quad B_z = \frac{1}{h_s}\frac{\partial(h_s A_2)}{\partial x} = \frac{1}{h_s}\frac{\partial A_s}{\partial x},$$

For h_s=1 or $\rho=\infty$, one obtains

$$B_z + jB_x = B_0\sum_n (b_n + ja_n)(x+jz)^n, \qquad A_s = \mathrm{Re}\left\{B_0\sum_n \frac{1}{n+1}(b_n + ja_n)(x+jz)^{n+1}\right\}$$

b_0 : dipole, a_0 : skew (vertical) dipole; $B_z = B_0 b_0$, $B_x = B_0 a_0$,
b_1 : quad, a_1 : skew quad; $B_z = B_0 b_1 x$, $B_x = B_0 b_1 z$, $B_z = -B_0 a_1 z$, $B_x = B_0 a_1 x$,
b_2 : sextupole, a_2 : skew sextupole;

$$\frac{1}{B\rho}(B_z + jB_x) = \mp\frac{1}{\rho}\sum_{n=0}^{\infty}(b_n + ja_n)(x+jz)^n,$$

$$\tilde{H} \approx -p\left(1+\frac{x}{\rho}\right) + \frac{1+x/\rho}{2p}\left[(p_x - eA_x)^2 + (p_z - eA_z)^2\right] - eA_s.$$

$$x' = \frac{\partial\tilde{H}}{\partial p_x}, \quad p_x' = -\frac{\partial\tilde{H}}{\partial x}, \qquad z' = \frac{\partial\tilde{H}}{\partial p_z}, \quad p_z' = -\frac{\partial\tilde{H}}{\partial z}.$$

$$\begin{cases} x'' - \dfrac{\rho+x}{\rho^2} = \pm\dfrac{B_z}{B\rho}\dfrac{p_0}{p}\left(1+\dfrac{x}{\rho}\right)^2, \\ z'' = \mp\dfrac{B_x}{B\rho}\dfrac{p_0}{p}\left(1+\dfrac{x}{\rho}\right)^2, \end{cases}$$

Exercise 2.1.2

$$\ddot{r} - r\dot{\theta}^2 = \frac{ev_s B_z}{\gamma m} = \pm\frac{v_s^2 B_z}{B\rho}, \qquad \ddot{z} = \mp\frac{v_s^2 B_x}{B\rho},$$

$$B_z = -B_0 + \frac{\partial B_z}{\partial x}x = \mp B_0 + B_1 x, \qquad B_x = \frac{\partial B_z}{\partial x}z = B_1 z,$$

$$n = -\frac{R}{B_0}\left(\frac{dB_z}{dr}\right)_{r=R}$$

$$n(s) = \rho^2 K_1(s)$$

$$x'' + K_x(s)x = \pm\frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \quad K_z(s) = \pm\frac{B_1}{B\rho}$$