3.6.2: Inside the cavity, Maxwell's equations for electromagnetic fields are

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}, \quad \nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

where ϵ and μ are dielectric permittivity and permeability of the medium. The EM waves in the cavity can conveniently be classified into transverse magnetic (TM) mode, for which the longitudinal magnetic field is zero, and transverse electric (TE) mode, for which the longitudinal electric field is zero. The TM modes are of interest for beam acceleration in the rf cavity. Assuming a time dependence factor $e^{j\omega t}$ for electric and magnetic fields, the TM modes in cylindrical coordinates (r, ϕ, s) are

$$\begin{cases} E_s = A\, k_r^2\ J_m(k_r r) \cos m\phi \cos k_s s \\ E_r = -A\, k_s k_r\ J_m'(k_r r) \cos m\phi \sin k_s s \\ E_\phi = A\, (m k_s / r) J_m(k_r r) \sin m\phi \sin k_s s \\ B_s = 0 \\ B_r = -jA\, (mk/cr)\ J_m(k_r r) \sin m\phi \cos k_s s \\ B_\phi = -jA\, (k k_r / c)\ J_m'(k_r r) \cos m\phi \cos k_s s \end{cases}$$

where A is a constant, $s = 0$ and ℓ correspond to the beginning and end of the pillbox cavity, m is the azimuthal mode number, k_s, k_r are wave numbers in the longitudinal and radial modes, and $k = \omega/c = \sqrt{k_s^2 + k_r^2}$.

The longitudinal wave number k_s is determined by the boundary condition that $E_r = 0$ and $E_\phi = 0$ at $s = 0$ and ℓ, i.e.

$$k_s = \frac{p\pi}{\ell}, \quad p = 0, 1, 2, \cdots.$$

Similarly the radial wave number is determined by the boundary condition with $E_s = 0$ and $E_\phi = 0$ at $r = b$, i.e.

$$k_{r,mn} = \frac{j_{mn}}{b},$$

where j_{mn}, listed in Table 3.6, are zeros of Bessel functions $J_m(j_{mn}) = 0$.

3.8.3: In an Alvarez linac, the longitudinal equations of motion can be expressed as mapping equations:

$$\Delta\psi_{n+1} = \Delta\psi_n - \frac{L_{\rm cell}\omega}{mc^3\beta^3\gamma^3}\ \Delta E_n,$$
$$\Delta E_{n+1} = \Delta E_n + eV \cos\psi_{\rm s}\ \Delta\psi_{n+1},$$

where $\psi_n, \Delta E_n$ are the synchrotron phase-space coordinates at the nth cell, $L_{\rm cell}$ is the length of the drift tube cell, and eV is the energy gain in this cell. The transfer matrix is

$$M = \begin{pmatrix} 1 & -A \\ eV\cos\phi_{\rm s} & 1 - eVA\cos\phi_{\rm s} \end{pmatrix},$$

where $A = \omega L_{\rm cell}/mc^3\beta^3\gamma^3$. Using the Courant-Snyder representation, we obtain

$$\cos\Phi_{\rm syn} = 1 - \frac{1}{2}eVA\cos\phi_{\rm s}, \quad \text{or} \quad \sin^2\frac{\Phi_{\rm syn}}{2} = \frac{\pi\lambda e\mathcal{E}_{\rm av}\cos\phi_{\rm s}}{2\beta\gamma^2 E},$$

where the cell length of Alvarez Linac is $L_{\rm cell} = \beta\lambda$, the frequency is $\omega = 2\pi f = 2\pi c/\lambda$, and the average acceleration field is $\mathcal{E}_{\rm av} = V/L_{\rm cell}$. In the small phase advance approximation, the synchrotron tune per period is

$$\nu_{\rm syn} = \frac{\Phi_{\rm syn}}{2\pi} = \sqrt{\frac{e(\beta\lambda\mathcal{E}_{\rm av})\cos\phi_{\rm s}}{2\pi\beta^2\gamma^2 E}}.$$

This is the synchrotron tune formula of synchrotrons with $h = 1$, $\eta = -1/\gamma^2$, and $V = \beta\lambda\mathcal{E}_{\rm av}$.