3.1.1: The energy gain for a non-synchronous particle is

$$\begin{aligned}\Delta E &= \frac{e}{\Delta t}\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}} V(t)dt = \frac{eV_{\rm g}}{h\omega_0\Delta t}\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}} \sin(h\omega_0 t+\phi)d(h\omega_0 t)\\ &= \frac{eV_{\rm g}}{h\omega_0\Delta t}\cdot 2\sin\phi\sin\left(h\omega_0\frac{\Delta t}{2}\right) = eV_{\rm g}\frac{\sin(hg/2R)}{(hg/2R)}\sin\phi\cdot = e(V_{\rm g}T)\sin\phi,\end{aligned}$$

where $\omega_0\Delta t = g/R$, T is the transit time factor, $V_{\rm g}$ is the gap voltage, and $V = V_{\rm g}T$ is the effective voltage. Thus we find that the effective energy gain is multiplied by a transit time factor as that of a synchronous particle.

3.1.3: Particle momentum and energy

$$pc = ecB(t)\rho, \qquad E = \sqrt{(mc^2)^2+(pc)^2} = \sqrt{(mc^2)^2+[ecB(t)\rho]^2}.$$

Then

$$\omega_{rf} = 2\pi h\frac{\beta c}{L} = \frac{hc}{R}\sqrt{1-\frac{(mc^2)^2}{E^2}} = \frac{hc}{R}\sqrt{\frac{B^2(t)}{[B^2(t)+(mc^2/ec\rho)^2]}}$$

3.2.3: In $(\phi, \frac{\Delta E}{\omega_0})$ space, the phase-space area $\mathcal{A}$ is proportional to a factor $F = \gamma/|\eta|$. When $\gamma < \gamma_{\rm T}$, F increases monotonously with γ, and when $\gamma > \gamma_{\rm T}$, the factor F has a minimum at $\gamma = \sqrt{3}\gamma_{\rm T}$.

$$\frac{dF}{d\gamma} = \frac{1}{\eta^2}\left[\left(\frac{1}{\gamma_{\rm T}^2}-\frac{1}{\gamma^2}\right)-\gamma\cdot\frac{2}{\gamma^3}\right]$$

When $\gamma = \sqrt{3}\gamma_{\rm T}$, $dF/d\gamma = 0$, $\mathcal{A}$ will be a minimum.

3.2.6: The anti-protons produced from the Main Injector (Main Ring) pulses have the following characteristics: $p_0 = 8.9$ GeV/c, $\sigma_t = 0.15$ ns, $\sigma_E = 180$ MeV, or $\Delta p/p_0 = \pm 2\%$. The antiprotons are captured in the Debuncher into the 53.1 MHz ($h = 90$) rf bucket with $V = 5$ MV, $\gamma_T = 7.7$, and $R = 83$ m.

1. With $V = 5$ MV, $h = 90$, $\gamma_{\rm T} = 7.7$, and $R = 83$ m, we find

$$\begin{aligned}&E = 8949.3\ \text{MeV} \quad \gamma = 9.5381 \quad \beta = 0.9954, \quad \eta = 0.0059\\ &f_0 = 3.612\ \text{MHz} \qquad T_0 = 0.2769\ \mu\text{s}\\ &\nu_{\rm s} = 6.89\times 10^{-3} \quad T_{\rm s} = 40.38\ \mu\text{s} \qquad \delta_{\rm B} = 0.0259\end{aligned}$$

2. A matched bunch ellipse is given by and momentum height is given by

$$\frac{\delta^2}{\hat{\delta}^2} + \frac{\theta^2}{\hat{\theta}^2} = 1, \quad \text{with} \quad \frac{\hat{\delta}}{\hat{\theta}} = \frac{\nu_{s1}}{\eta}.$$

When a beam is injected into the bucket with an ellipse

$$\frac{\delta^2}{\delta_1^2} + \frac{\theta^2}{\theta_1^2} = 1,$$

where $\delta_1/\theta_1 \neq \nu_{s1}/\eta$, it will rotate like a cigar. After 1/4 synchrotron period, the beam height will become the width and the width becomes the height, i.e.

$$\delta_2 = \frac{\nu_{s1}}{\eta}\theta_1 \qquad \theta_2 = \frac{\eta}{\nu_{s1}}\delta_1$$

To match the new shape of the beam, we must have

$$\frac{\delta_2}{\theta_2} = \frac{\nu_{s2}}{\eta} \qquad \text{or} \qquad \frac{\nu_{s2}}{\eta} = \frac{\delta_2}{\theta_2} = \left(\frac{\nu_{s1}}{\eta}\right)^2 \cdot \frac{\theta_1}{\delta_1}$$

Using the relations:

$$\nu_{\rm s} \sim \sqrt{V} \qquad \theta = \omega_0 \sigma_t \qquad \delta = \frac{1}{\beta^2 E}\sigma_E$$

it is easy to get

$$\frac{V_2}{V_1} = \left(\frac{\nu_{s2}}{\nu_{s1}}\right)^2 = \left(\frac{\nu_{s1}}{\eta}\right)^2 \left(\frac{\theta_1}{\delta_1}\right)^2 = \left(\frac{\nu_{s1}}{\eta}\omega_0\sigma_{t1}\right)^2 \left(\frac{\sigma_E}{\beta^2 E}\right)^{-2}$$

Inserting all the known parameters, we get $V_2 = 4.99$ keV.

3. From (b),

$$\sigma_{E2} = \delta_2(\beta^2 E) = \frac{\nu_{s1}}{\eta}\theta_1(\beta^2 E) = \left[\frac{\nu_{s1}}{\eta}(\omega_0\sigma_{t1})\right]\beta^2 E$$

The final energy spread is $\sigma_{E2} = 5.6$ MeV.