2.2.11: The coordinate transformation of Eq. (2.90) is equivalent to the canonical transformation using the generating function:

$$F_1(y,\bar\psi) = -\frac{y^2}{2\beta}\left[\tan\left(\bar\psi + \chi(s) - \nu\theta\right) - \frac{\beta'}{2}\right],$$

where $\chi = \int^s ds/\beta$, and $\theta = s/R$. Performing the canonical transformation with the generating function, we find the new Hamiltonian as

$$\bar H = H + \partial F_1/\partial s = \frac{\nu\bar J}{R}.$$

Using θ as the independent variable, we obtain the new Hamiltonaian as $H = \nu\bar J$.

2.3.1:

1. Defining a new coordinate $\eta = y/\sqrt{\beta}$ and $\phi = (1/\nu)\int_0^s ds/\beta$, we find

$$\frac{d\eta}{d\phi} = \frac{ds}{d\phi}\frac{d\eta}{ds} = \nu\beta\left(\frac{1}{\sqrt{\beta}}y' - \frac{1}{2}\beta^{-3/2}\beta' y\right).$$

   Thus the equation of motion becomes

$$\frac{d^2\eta}{d\phi^2} + \nu^2\eta = \nu^2\beta^{3/2}\frac{\Delta B}{B\rho},$$

   where we have used the identity of Eq. (2.64).

2. The Green function can be expressed as

$$G(\phi-\phi_1) = \frac{1}{2\nu\sin\pi\nu}\cos\,\nu[\pi - 2(\phi-\phi_1)\theta(\phi-\phi_1)],$$

   where $\theta(t) = 1/2$ for $t > 0$ and $\theta(t) = -1/2$ for $t < 0$, with $\theta'(t) = \delta(t)$. Taking deivatives of the Green function with respect to ϕ, we verify the Green function solution. Using the Green function, we obtain easily the closed orbit solution given by Eq. (2.157).

3. The solution of part (a) can be attained by expanding the right-hand side in Fourier series as

$$\beta^{3/2}\frac{\Delta B}{B\rho} = \sum_{k=-\infty}^{\infty} f_k e^{jk\phi} \quad \text{with} \quad f_k = \frac{1}{2\pi\nu}\oint\sqrt{\beta}\frac{\Delta B}{B\rho}e^{-jk\phi}ds,$$

   where f_k are integer stopband integrals. Expressing the closed orbit as $\eta_{\rm co}(s) = \sum F_k e^{ik\phi}$, we find

$$y_{\rm co}(s) = \sqrt{\beta(s)}\sum_{k=-\infty}^{\infty}\frac{\nu^2 f_k}{\nu^2-k^2}e^{jk\phi}.$$

4. Using a single stopband approximation and limiting the closed orbit deviation to less than 20% of the beam size, we find

$$|y_{\rm co}| \approx 2|\frac{\sqrt{\beta}\nu^2 f_{[\nu]}}{\nu^2 - [\nu]^2}| \le 0.20 \times \sqrt{\beta\epsilon_{\rm rms}}.$$

   Approximating $\nu^2 - [\nu]^2 \approx 2\nu\Gamma_{[\nu]}$, where $\Gamma_{[\nu]}$ is the stopband width, we find $\Gamma_{[\nu]} \approx 5\nu|f_{[\nu]}|/\sqrt{\epsilon_{\rm rms}}$.

2.3.4: The closed-orbit solution of the inhomogeneous Hill equation

$$\frac{d^2y}{ds^2} + K(s)y = \frac{\Delta B}{B\rho},$$

where $\Delta B = \Delta B_z$ for horizontal motion and $\Delta B = -\Delta B_x$ for vertical motion, is

$$y_{\rm co}(s) = \int_s^{s+C} G(s,t)\,\frac{\Delta B(t)}{B\rho}\,dt,$$

where the Green function is

$$G(s,t) = \frac{\sqrt{\beta(s)\beta(t)}}{2\sin\pi\nu}\cos(\pi\nu - |\psi(s) - \psi(t)|)$$

with $\psi(s) = \int_0^s dt/\beta(t)$ as the betatron phase function.

1. The closed orbit of a three-bump system is

$$y(s) = \frac{\sqrt{\beta}}{2\sin\pi\nu}\sum_{i=1}^{3}\sqrt{\beta_i}\theta_i\cos(\pi\nu - |\psi - \psi_i|).$$

   Using the condition $y(s_3) = y'(s_3) = 0$, Then

$$\begin{cases} \sqrt{\beta_1}\theta_1\cos(\pi\nu + \psi_{13}) + \sqrt{\beta_2}\theta_2\cos(\pi\nu + \psi_{23}) + \sqrt{\beta_3}\theta_3\cos\pi\nu = 0 \\ \sqrt{\beta_1}\theta_1\sin(\pi\nu + \psi_{13}) + \sqrt{\beta_2}\theta_2\sin(\pi\nu + \psi_{23}) + \sqrt{\beta_3}\theta_3\sin\pi\nu = 0 \end{cases}$$

   where $\psi_{13} = \psi_1 - \psi_3$ and $\psi_{23} = \psi_2 - \psi_3$, we find

$$\theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}, \qquad \theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\frac{\sin\psi_{12}}{\sin\psi_{23}}.$$

2. When $\psi_{31} = n\pi$, we find $\theta_2 = 0$, i.e. only two steering dipoles are needed for a local bump. Since $\psi_{32} = \psi_{31} - \psi_{21} = n\pi - \psi_{21}$, we have $\sin\psi_{32} = (-1)^{n-1}\sin\psi_{31}$, and $\theta_3 = (-1)^{n-1}\sqrt{\beta_1/\beta_2}\,\theta_1$.

2.4.1: Transfer matrix for the dispersion function

1. The solution of the homogeneous equation, $D'' + KD = 0$ with $K > 0$ is

$$D = a\cos\sqrt{K}s + b\sin\sqrt{K}s,$$

and the special solution for $D'' + KD = 1/\rho$ is $D = 1/K\rho$. Thus the general solution is

$$D = a\cos\sqrt{K}s + b\sin\sqrt{K}s + 1/K\rho.$$

Using the initial conditions: $D = D_0$ and $D' = D'_0$ at $s = 0$, we obtain

$$a = D_0 - \frac{1}{K\rho} \qquad b = \frac{D'_0}{\sqrt{K}}.$$

The transfer matrix the 3×3 matrix becomes

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{K}s & \frac{1}{\sqrt{K}}\sin\sqrt{K}s & \frac{1}{K\rho}(1-\cos\sqrt{K}s) \\ -\sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s & \frac{1}{\rho\sqrt{K}}\sin\sqrt{K}s. \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}.$$

2. The general solution for $K < 0$ is

$$D = a\cosh\sqrt{|K|}s + b\sinh\sqrt{|K|}s - \frac{1}{|K|\rho}.$$

With initial conditions, the solutions becomes

$$\begin{aligned} D &= D_0\cosh\sqrt{|K|}s + \frac{D'_0}{\sqrt{|K|}}\sinh\sqrt{|K|}s + \frac{1}{|K|\rho}(-1+\cosh\sqrt{|K|}s), \\ D' &= D_0\sqrt{|K|}\sinh\sqrt{|K|}s + D'_0\cosh\sqrt{|K|}s + \frac{1}{\rho\sqrt{|K|}}\sin\sqrt{|K|}s. \end{aligned}$$

3. For a sector magnet, we substitute $K = 1/\rho^2$ and $\sqrt{K}s = \theta$ into part (a) and obtain

$$M_x = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

4. In a rectangle magnet, edge angle $\delta = \theta/2$. The transfer matrix becomes

$$\begin{aligned} M_x &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho\sin\theta & \rho(1-\cos\theta) \\ 0 & 1 & 2\tan\frac{\theta}{2} \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

5. In thin lens approximation, we have $\ell \to 0$ and $K\ell \to 1/f$ for a quadrupole; and $\theta \to 0$, $\rho \sin\theta \to l$ and $\cos\theta \to 1$ for a dipole, the transfer matrix becomes

$$M_x = \begin{pmatrix} 1 & \ell & \ell\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$