USPAS UNM HW08 06/24/2009

1.1: The force experienced by a charged particle with charge e in a magnetic field is $\gamma m v^2/\rho = evB$, or $p = \gamma m v = eB\rho$. Thus

$$B\rho\ [\mathrm{Tm}] = \frac{10^9}{c} p\,[\mathrm{GeV/c}] = 3.3357\,p\,[\mathrm{GeV/c}]\,.$$

The relation between p and kinetic energy, T, is $pc = \sqrt{E^2 - E_0^2} = \sqrt{T^2 + 2Tmc^2}$. The total length of dipoles is $L = 2\pi\rho$. The solution is shown in the table below.

	IUCF Cooler	RHIC	Tevatron	SSC
p [GeV/c]	1.0901	250	1000	20000
$B\rho$ [Tm]	3.6362	833.9	3335.7	66713.
B [T]	1.7	1.7	1.7	1.7
$L_1 = 2\pi\rho$ [m]	13.434	3082.2	12329.	246572.
B [T]	1.7	3.5	5	6.6
$L_2 = 2\pi\rho$ [m]	13.434	1497.	4191.7	63512.
circum (m)	86.82	3833.8	6283.2	87000

2.1.1: The Hamiltonian for the phase-space coordinates $(x, p_x, z, p_z, t, -H)$ is

$$\tilde{H} \approx -p\left(1 + \frac{x}{\rho}\right) + \frac{1 + x/\rho}{2p}\left[(p_x - eA_x)^2 + (p_z - eA_z)^2\right] - eA_s,$$

Thus the Hamilton's equations of motion are

$$x' = \frac{\partial \tilde{H}}{\partial p_x} = \left(1 + \frac{x}{\rho}\right)\frac{p_x}{p}, \qquad p_x' = -\frac{\partial \tilde{H}}{\partial x} = \frac{p}{\rho} - \frac{1}{2p\rho}[p_x^2 + p_z^2] + e\frac{\partial A_s}{\partial x}.$$

Neglecting higher order terms, we find

$$x'' - \frac{\rho + x}{\rho^2} = \pm\frac{B_z}{B\rho}\frac{p_0}{p}\left(1 + \frac{x}{\rho}\right)^2 + \cdots,$$

where ± signs are associated with the positive/negative charged particles respectively, and we have used $p_0 = |e|B\rho$ for the reference momentum and $B_z = \frac{1}{1+x/\rho}\frac{\partial A_s}{\partial x}$ for the magnetic induction. Let $B_z = \mp B_0 + B_1 x + \cdots$, where $B_0/B\rho = 1/\rho$ is the magnetic induction used to define the reference orbit, and $B_1 = \frac{\partial B_z}{\partial x}$ is the gradient. The resulting linearized Hill's equation for the on-momentum particle with $p = p_0$ becomes

$$x'' + \left(\frac{1}{\rho^2} \mp K\right)x = 0, \qquad K = \frac{B_1}{B\rho}.$$

Similarly, we find

$$z'' = \mp \frac{B_x}{B\rho}\frac{p_0}{p}\left(1+\frac{x}{\rho}\right)^2 .$$

The linearized Hill's equation becomes $z'' \pm Kz = 0$.

2.1.2: Without loss of generality, we use the Frenet-Serret coordinate system of Fig. 2.1 and derive equation of motion for positively charged ions in the accelerator. It is easy to modify the equations of motion for electrons.

1. The coordinate of a particle is $\vec{\mathbf{r}} = (\rho + x)\hat{x} + z\hat{z}$. Using Eq. (2.6) and (2.8), we obtain

   $$\dot{\hat{x}} = \frac{d\hat{x}}{dt} = \dot{s}\frac{d\hat{x}}{ds} = \frac{\dot{s}}{\rho}\hat{s}, \quad \dot{\hat{s}} = \dot{s}\frac{d\hat{s}}{ds} = \dot{s}\left(-\frac{\hat{x}}{\rho}\right) = -\dot{\theta}\hat{x},$$

   i.e. $\dot{\vec{\mathbf{r}}} = \dot{x}\hat{x} + (\rho + x)\dot{\theta}\hat{s} + \dot{z}\hat{z}$, and

   $$\begin{aligned}\ddot{\vec{\mathbf{r}}} &= (\ddot{x}\hat{x} + \dot{x}\dot{\theta}\hat{s}) + \dot{x}\dot{\theta}\hat{s} + (\rho + x)\ddot{\theta}\hat{s} + (\rho + x)\dot{\theta}\dot{\hat{s}} + \ddot{z}\hat{z} \\ &= [\ddot{x} - (\rho + x)\dot{\theta}^2]\hat{x} + [2\dot{x}\dot{\theta} + (\rho + x)\ddot{\theta}]\hat{s} + \ddot{z}\hat{z}.\end{aligned}$$

2. From Newton's law:

   $$\frac{d\vec{\mathbf{p}}}{dt} = \gamma m\ddot{\vec{\mathbf{r}}} = e\vec{\mathbf{v}} \times \vec{\mathbf{B}} = ev_s\hat{s} \times (B_x\hat{x} + B_z\hat{z}) = ev_sB_z\hat{x} - ev_sB_x\hat{z},$$

   we obtain

   $$\ddot{x} - (\rho + x)\dot{\theta}^2 = \frac{ev_sB_z}{\gamma m} = \frac{v_s^2 B_z}{B\rho}, \qquad \ddot{z} = -\frac{ev_sB_x}{\gamma m} = -\frac{v_s^2B_x}{B\rho}.$$

3. Transformation from t to s for the independent coordinate with

   $$\dot{s}\frac{ds}{dt} = \frac{v_s}{1 + x/\rho}, \qquad \dot{\theta} = \frac{v_s}{\rho + x}, \qquad \frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \dot{s}\frac{d}{ds}\left(\dot{s}\frac{d}{ds}\right) = (\dot{s})^2\frac{d^2}{ds^2},$$

   we obtain

   $$x'' - \frac{\rho + x}{\rho^2} = \frac{B_z}{B\rho}\left(1 + \frac{x}{\rho}\right)^2, \qquad z'' = -\frac{B_x}{B\rho}\left(1 + \frac{x}{\rho}\right)^2$$

2.2.8: Differentiating $\beta'' + 2K\beta - 2\gamma = 0$, we obtain $\beta''' + 4\beta'K + 2\beta K' = 0$. The solutions for piecewise constant K are

- Drift space: $K = 0$, $\beta = a + bs + cs^2$.
- Focusing quadrapole: $K = \text{constant} > 0$, $\beta = a\cos 2\sqrt{K}s + b\sin 2\sqrt{K}s + c$.

- Defocusing quadrapole: $K = \text{constant} < 0$, $\beta = a\cosh 2\sqrt{|K|}s + b\sinh 2\sqrt{|K|}s + c$.

1. Using the initial conditions at $s = s_0$:

$$\beta = \beta_0, \quad \beta_0' = -2\alpha_0, \quad \beta_0'' = -2K\beta_0 + \frac{2}{\beta_0}(1+\alpha_0^2) = -2K\beta_0 + 2\gamma_0,$$

   we obtain

   - Drift space: $a = \beta_0, \quad b = \beta_0' = -2\alpha_0, \quad c = \frac{1}{2}\beta_0'' = \gamma_0$.

$$\beta = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 = \gamma_0\,(s - s^*)^2 + 1/\gamma_0,$$

     where $s^* = \alpha_0/\gamma_0$.

   - Focusing quadrapole:

$$a = -\frac{\beta_0''}{4K} = \frac{\beta_0}{2} - \frac{\gamma_0}{2K}, \quad b = \frac{\beta_0'}{2\sqrt{K}} = -\frac{\alpha_0}{\sqrt{K}}, \quad c = \beta_0 - a = \frac{\beta_0}{2} + \frac{\gamma_0}{2K}.$$

   - Defocusing quadrapole:

$$a = \frac{\beta_0}{2} + \frac{\gamma_0}{2K} \quad b = -\frac{\alpha_0}{\sqrt{|K|}} \quad c = \beta_0 - a = \frac{\beta_0}{2} - \frac{\gamma_0}{2K}$$

2. In a drift space with a symmetry condition at $s = s^*$, we have

$$\beta = \beta^*, \qquad \beta' = 0, \qquad \beta'' = 2/\beta^*.$$

   Using previous result, $\beta = a + bs + cs^2$, we obtain

$$c = \frac{\beta''}{2} = \frac{1}{\beta^*}, \quad b = -2cs^* = -\frac{2s^*}{\beta^*} \quad a = \beta^* s - bs^* - cs^{*2} = \beta^* + \frac{s^{*2}}{\beta^*}.$$

   Thus $\beta = \beta^* + \frac{1}{\beta^*}(s - s^*)^2$, and

$$\alpha = (s - s^*)/\beta^*, \quad \gamma = (1+\alpha^2)/\beta = 1/\beta^* = \text{constant}.$$

   The phase advance from the IP to the high β quad becomes

$$\psi = \int_{s*}^{s} \frac{ds}{\beta^* + \frac{1}{\beta^*}(s-s^*)^2} = \beta^* \int_0^{s-s*} \frac{du}{\beta^{*2} + u^2} = \arctan\left(\frac{s - s*}{\beta^*}\right) \to \frac{\pi}{2},$$

   where we use the fact that $s - s^* \gg \beta^*$.

3. The transfer matrices are related through similarity transformation, Eq. (2.46), with

$$M(s_2) = M(s_2|s_1)M(s_1)[M(s_2|s_1)]^{-1}.$$

Now the transfer matrix can be expressed in Courant-Snyder parametrization, Eq. (2.50),

$$\begin{aligned} M(s_2) &= I\cos\Phi + \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \sin\Phi = I\cos\Phi + J_2\sin\Phi \\ M(s_1) &= I\cos\Phi + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \sin\Phi = I\cos\Phi + J_1\sin\Phi. \end{aligned}$$

Thus

$$J_2 = M(s_2|s_1)J_1[M(s_2|s_1)]^{-1},$$

i.e.

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \cdot \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}.$$

Then α_2, β_2 and γ_2 are related to α_1, β_1, and γ_1 by the transfer matrix. Substituting appropriate transfer matrices, we find

- Drift space:

$$\beta = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

- Focusing quadrapole:

$$\begin{aligned} \beta &= \beta_0 \cos^2\sqrt{K}s - 2\frac{\alpha_0}{\sqrt{K}}\cos\sqrt{K}s\sin\sqrt{K}s + \frac{\gamma_0}{K}\sin^2\sqrt{K}s \\ &= \left(\frac{\beta_0}{2} - \frac{\gamma_0}{2K}\right)\cos 2\sqrt{K}s - \frac{\alpha_0}{\sqrt{K}}\sin 2\sqrt{K}s + \left(\frac{\beta_0}{2} + \frac{\gamma_0}{2K}\right) \end{aligned}$$

- Focusing quadrapole:

$$\begin{aligned} \beta &= \beta_0 \cosh^2\sqrt{|K|}s - 2\frac{\alpha_0}{\sqrt{|K|}}\cosh\sqrt{|K|}s\sinh\sqrt{|K|}s - \frac{\gamma_0}{K}\sinh^2\sqrt{|K|}s \\ &= \left(\frac{\beta_0}{2} + \frac{\gamma_0}{2K}\right)\cosh 2\sqrt{|K|}s - \frac{\alpha_0}{\sqrt{|K|}}\sinh 2\sqrt{|K|}s + \left(\frac{\beta_0}{2} - \frac{\gamma_0}{2K}\right). \end{aligned}$$

Note that $\beta(s)$ oscillates twice the betatron frequency.