VERTICAL BEAM EMITTANCE
CORRECTION WITH INDEPENDENT
COMPONENT ANALYSIS MEASUREMENT
METHOD

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Submitted to the faculty of the University Graduate School
in partial fulfillment of the requirement
for the degree
Doctor of Philosophy
in the Department of Physics,
Indiana University

May 2008
Accepted by the Graduate Faculty, Indiana University, in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

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WISSINK, SCOTT, Ph.D.
To my parents.
Acknowledgments

I would like to thank, first and foremost, my advisor, Dr. S. Y. Lee, for his guidance and support throughout my graduate career and during the completion of this thesis. Dr. Lee is the one professor who truly made a difference in my life. His expertise, understanding, and patience added considerably to my graduate experience. I would also like to thank the members of my committee who attend my defense: Dr. Mark Hess, Dr. Chen-Yu Liu and Dr. Scott Wissink. Their advice and patience are appreciated.

I am particularly grateful to Dr. Xiaobiao Huang who first introduced the Independent component analysis method to the beam diagnosis and was always helpful with advice. I would like to thank Dr. K. Y. Ng for the assistance he provided with the research project. I must also acknowledge Xiaoying Pang for her advice and assistance on the optical function measurements using the independent component analysis method. Appreciation also goes out to Shing-Shong Shei for all of his computer and technical assistance. I have profited from many discussions with Dr. M. H. Wang and Dr. Alexander W. Chao. I would like to express my thanks to Yoichi Sato, S. Wang, Dazhang Huang, Fanglei Lin, Lingyun Yang, Yue Hao, Qiong Wu, Wai Ming Tam, Zheng-Zheng Liu, and Jeffery Kolski for being not only excellent colleagues but also good friends, for making the group cheerful and bringing a lot of fun, both inside and outside the lab. Many thanks to all my friends who have been around these five years, who have been helping me at different occasions and in one way or another have influenced this thesis.

Finally, I would like to thank my parents, Yonghou and Jinhua Wang, for their encouragement and the support they provided me through my entire life. A very special thank you to my husband, Jingyuan Wang, without whose love and under-
standing during this whole event, I would not have finished this thesis. In the end I want to mention my son, Frederick G. Wang, who brings me much fun during this study, hope he can keep good memory during this special time for us.

This work is supported in part by grants from the US Department of Energy DE-FG0292ER40747 and the National Science Foundation NSF PHY-0552389.
Abstract

Fei Wang

VERTICAL BEAM EMITTANCE CORRECTION WITH INDEPENDENT COMPONENT ANALYSIS MEASUREMENT METHOD

The storage ring performance is determined by the vertical beam size, that is by the vertical emittance, which is determined by two factors: the vertical dispersion generated in the bending magnets, and the coupling of the oscillations in the vertical and horizontal plane. In this dissertation, a detailed study of the main source of the vertical emittance and effective correction methods are presented.

Simulations show that the vertical emittance is dominated by the contribution due to photon emission with non-zero vertical dispersion in bending magnets. An effective method to make vertical dispersion correction is to analyze the harmonics of the vertical dispersion and to eliminate the largest components of the stopband integral with harmonics near the vertical betatron tune. A stopband correction scheme is being implemented in which the excitation of skew-quadrupole correctors is determined from measurements of the resonance strengths (stopband widths) of major resonances. This method can correct the vertical dispersion function and the coupling strength simultaneously without identifying the source of errors. Studies show the coupling strength and the vertical dispersion can be controlled individually in the quadruple-bend achromatic low emittance lattice. Resulting improvement in machine performance is that the equilibrium vertical emittance is reduced by the factor of 7.

Effective correction depends on precise beam measurements. Independent compo-
nent analysis for BPM turn-by-turn data has shown the potential to be a useful tool for diagnostics and optics verification. The effectiveness of employing the independent component analysis (ICA) method to measure the vertical dispersion function is studied. This method for extracting the beta function and phase advance for the beam position monitors is presented. The accuracy of optical functions thus calculated is affected by different factors in a different manner. The most influential factors on the accuracy are discussed through simulation studies of the quadrupole-bend achromatic lattice.
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Chapter 1

Motivation of Study

Verification and diagnosis of an accelerator optics is desirable because the accuracy of the constructed accelerator beamlines (compared to the designed lattice) directly determines the accelerator performance. Beta function, phase advance and dispersion function are basic optical functions characterizing the properties of an accelerator lattice. Accurate and efficient measurements of these quantities are important for commissioning and operating a machine. In most accelerators, beam position monitors (BPMs) are used to record the transverse position of the centroid motion of particle beam. Statistical analysis of beam position monitor data has shown remarkable potential to significantly enhance measurement capability for the accelerator optical functions. Recently the independent component analysis (ICA) was introduced to analyze multiple BPM turn-by-turn data simultaneously around a ring. The physical origin of ICA modes can by identified by their spatial and temporal functions. Thus the beta function, phase advance and dispersion function can be derived according to the ICA modes [4]. However sometimes the two largest ICA modes principally form betatron motion can still be mixed with additional sources signals such as synchrotron motion, random noises and etc., which affect the accuracy of optical functions. In
order to suppress the errors due to additional source signals and achieve high accuracy measurements of optical functions, the influence factors on the accuracy deserve further study.

The photon brilliance of synchrotron light source is inversely proportional to the product of the horizontal and vertical emittances. In an ideal storage ring, the horizontal and vertical motions are independent, resulting in an extremely small beam height. In practice, ring imperfections result in a coupling of horizontal and vertical transverse oscillations (betatron coupling) as well as vertical motion associated with energy oscillations (vertical dispersion). Thus the vertical beam emittance in a storage ring arises essentially from the linear betatron coupling and the photon emission in bending magnets due to the vertical dispersion function. So far, there is no systematic analysis and correction methods on the vertical emittance. So we intend to study the main sources of the vertical emittance and explore effective correction methods. Effective correction depends on precise beam measurements. An accurate measurement of vertical dispersion carried out using the independent component analysis for BPM turn-by-turn data triggered the motivation to measure the complex strength of the stopband integrals, which determine the excitation of skew-quadrupole correctors. It is important to study the effectiveness of employing the ICA method to measure the vertical dispersion function and the effectiveness of the stopband correction. The correction of vertical dispersion is of great importance. An example is given to the low emittance storage ring designs [9]. The vertical emittance of the Quadruple-bend achromatic (QBA) lattice is dominated by the contribution due to the vertical dispersion, which is roughly four times as much as the contribution due to the betatron coupling in the QBA lattice.

This thesis is organized into six chapters. This first chapter describes the motivation of study. Chapter two introduces the basics of accelerator physics. Chapter three presents the method of independent component analysis (ICA) for BPM turn-
Motivation of Study

by-turn data and its error analysis. Chapter four studies the main sources of the vertical emittance. Chapter five describes the stopband correction scheme we have developed and used to successfully reduce the vertical emittance of the QBA lattice. Conclusions are summarized in Chapter six.
Chapter 2

Introduction to Accelerator Physics

This chapter focuses on the fundamental principles of particle accelerators and beam dynamics. The coordinate system is introduced in section 2.1. Section 2.2 gives a brief description of the basic physical elements of accelerators. The transverse and longitudinal motions of particles in accelerators are provided in section 2.3 and section 2.4 respectively. Topic on the emittance of synchrotron radiation lattices is mentioned in section 2.5 only briefly.

2.1 Coordinate System

In an accelerator, the coordinate system for particle motion is a curvilinear system where the origin of the system moves at the reference energy along the reference orbit of the machine. This moving reference system is shown in Fig. 2.1. The longitudinal coordinate $s$ is the distance along the reference orbit and the transverse coordinates $x$ and $z$. For path length measurements the longitudinal coordinate $s$ is used as the
independent coordinate to describe trajectories of particles, and derivatives are taken with respect to the instantaneous tangent line to the reference orbit, i.e.

\[ x' = \frac{dx}{ds}, \quad z' = \frac{dz}{ds}. \]  

(2.1)

Figure 2.1: Frenet-Serret coordinate system [6].

2.2 Basic Physical Elements

The main accelerator magnets are dipole magnets for particle bending, quadrupole magnets for focusing particles, and sextupole magnets for controlling the beam's chromaticity. These magnets fall into the categories of conventional iron dominated magnets and superconducting magnets for high energy accelerators.

2.2.1 Dipole

Dipole magnets are used to realize bends in the design trajectory of the particles. Fig. 2.2 shows a schematic drawing of a dipole magnet. The bending radius \( \rho \) is
determined by both the magnetic field and the momentum of the beam

\[ \rho = \frac{p_0}{ZeB}. \]  \hspace{1cm} (2.2)

Where \( p_0 \) is the momentum of the beam, and \( B\rho = p_0/Ze \) is the momentum rigidity of the beam. \( Ze \) is the total charge of the particle.

**Figure 2.2:** Schematic of a dipole magnet.

### 2.2.2 Quadrupole

The most suitable device that provides a material free aperture and the desired focusing field is a quadrupole magnet. The reference orbit usually runs through the center of the quadrupole. So the magnetic field inside a quadrupole magnet is

\[ B_x = B_1z, \]  \hspace{1cm} (2.3)

\[ B_z = B_1x. \]  \hspace{1cm} (2.4)

Where \( B_1 = \frac{\partial B_z}{\partial x} \bigg|_{x=0,z=0} \). Following Maxwell’s equation, \( \nabla \times \vec{B} = 0 \) in the current-
Figure 2.3: Left, Schematic of a normal quadrupole magnet. Right, Schematic of a skew quadrupole magnet.

free region, and the magnetic field can be derived from a magnetic potential, $\Phi_m$, with $\vec{B} = -\nabla \Phi_m$. The magnetic potential of a quadrupole field is $\Phi_m = -B_1xz$. The equipotential curve is $xz=$constant. The radius of a cylinder around the symmetry axis touching the magnet poles is called the pole inscribed radius or magnet aperture radius $a$. Thus the pole shapes of quadrupoles are hyperbolic curves with $xz = \pm a^2/2$. The pole-tip field is $B_{\text{pole-tip}} = B_1a$. To avoid the magnetic field saturation in iron, the pole-tip field in a quadrupole is normally designed to be less than 0.9 Tesla [6]. The achievable gradient is $B_1 = B_{\text{pole-tip}}/a$. The skew quadrupole is a normal quadrupole rotated by 45° about the $s$ axis.
2.2.3 Sextupole

Because the focusing function of a sextupole increases linearly with momentum, sextupoles are used to correct chromatic effects in accelerators. The magnetic field of sextupoles depends quadratically on transverse position such as

\[ B_x = B_2xz, \]
\[ B_z = B_2\frac{x^2 - z^2}{2}. \]

(2.5)

Where \( B_2 \) is the sextupole coefficient which is defined as \( B_2 = \frac{\partial^2 B_z}{\partial x^2} \bigg|_{x=0,z=0} \) [6]. It is normalized by the momentum rigidity \( B\beta \) to give the effective sextupole strength \( S(s) = -B_2/B\beta \).

2.2.4 RF cavity

RF cavities are used to accelerate, decelerate, or compensate energy loss of the beam. They also provide longitudinal focusing, and include references of phase focusing. The longitudinal electric field at an rf gap is

\[ \varepsilon = \varepsilon_0 \sin(\phi_{rf}(t) + \phi_s), \quad \phi_{rf} = \hbar \omega_0 t \] [6].

(2.7)

where \( \omega_0 = \beta_0 c / R_0 \) is the angular revolution frequency of the synchronous particle, \( \beta c \) and \( R_0 \) are respectively the speed and the average radius of the synchronous particle, \( \hbar \) is the integer harmonic number, and \( \phi_s \) is the phase angle for a synchronous particle. When the synchronous particle passes through the cavity gap, its energy gain per passage is

\[ \Delta E = e\varepsilon \beta c \int_{-g/2\beta c}^{g/2\beta c} \sin(h\omega_0 t + \phi_s) dt = e\varepsilon gT \sin \phi_s, \]

(2.8)

where \( g \) is the rf cavity gap width, \( e \) is the charge of the particles, and the transit time factor \( T \) is defined as \( T = \frac{\sin(hg/2R_0)}{hg/2R_0} \). \( V = \varepsilon gT \) is the effective voltage of the orbiting particle.
2.2.5 Beam Position Monitor

Figure 2.4: Strip Beam Position Monitor [10]

A Beam Position Monitor (BPM) is a device to detect and record the location of the beam bunches moving through a particular segment of the accelerator, i.e. a strip beam position monitor shown in Fig. 2.4. This BPM consists of four metal strips on the inside of the accelerator structure, connected to wires that extend outside the structure and are grounded. As the beam passes by BPMs, the induced image electric charges on the metal strips are transmitted into a low impedance circuit. The entire apparatus is electrically isolated from the accelerator structure itself. The information from the four strips can be used to measure the number of electrons in the bunch and determine the position of the bunch as it passes between the strips.
2.3 Transverse Motion

The particle transverse motion consists of a small-amplitude betatron motion around a closed orbit which is defined as a closed particle trajectory. Under the alternating focusing or defocusing forces of gradient magnets, particles with small deviations from the closed orbit will perform betatron oscillations around the closed orbit. The deviation of the closed orbit is proportional to the fractional off-momentum deviation \( \delta = (p-p_0)/p_0 \), where \( p_0 \) is the momentum of a reference particle. Thus the dispersion function is defined as the derivative of the closed orbit with respect to \( \delta \) in the first-order approximation. The unavoidable magnet field errors, i.e. quadrupole roll error, will perturb the betatron motion.

The linearized betatron motion of a particle in the presence of external magnetic field is governed by Hill’s equation [6]

\[
x'' + K_x(s)x = 0, \quad (2.9)
\]
\[
z'' + K_z(s)z = 0, \quad (2.10)
\]

where \( K_x(s) = 1/\rho^2 - B_1(s)/B \rho \), \( K_z(s) = B_1(s)/B \rho \), and \( B \rho = p/e \) is the momentum rigidity of the particle, \( B_1 = \partial B_z/\partial x \) is the quadrupole field gradient. The focusing functions are periodic such as \( K_{x,z}(s+L) = K_{x,z} \). Where \( L \) is the length of a periodic structure in an accelerator. If \( y \) represent either \( x \) or \( z \), then Hill’s equation becomes

\[
y'' + K_y(s)x = 0. \quad (2.11)
\]

A bending dipole with perpendicular entrance and exit angles to the edge of the dipole field has a focusing function \( K_x = 1/\rho^2 \), and \( K_z = 0 \). A horizontally focusing quadrupole is also a vertical defocussing quadrupole because \( 1/\rho = 0 \) and \( K_x = -K_z \).

Hill’s equation describes particle motion through accelerator elements, i.e. dipole and quadrupole magnets. Higher order magnetic elements are usually treated as
perturbations to this equation. The general solution [6] of Hill’s equation are sine-like and cosine-like solutions, \( S(s) \) and \( C(s) \), with the boundary conditions such that

\[
\begin{pmatrix}
  y(s) \\
  y'(s)
\end{pmatrix} = M(s|s_0) \begin{pmatrix} y(s_0) \\
  y'(s_0)
\end{pmatrix},
\]

(2.12)

\[
M(s|s_0) = \begin{pmatrix}
  C(s, s_0) & S(s, s_0) \\
  C'(s, s_0) & S'(s, s_0)
\end{pmatrix}.
\]

(2.13)

Where \((y_0, y'_0)\) and \((y, y')\) are the particle phase-space coordinates at the entrance and exit of accelerator elements. \( C' \) and \( S' \) are the derivatives of \( C \) and \( S \) with respect to \( s \). The transport matrix is the \( 2 \times 2 \) matrix \( M \) which depends only on the focusing function \( K(s) \) of accelerator elements. For example, the transfer matrix of a quadrupole in thin-lens approximation is

\[
M_{\text{quad}} = \begin{pmatrix}
  1 & 0 \\
  -1/f & 1
\end{pmatrix},
\]

(2.14)

Where \( f > 0 \) for a focusing quadrupole, \( f < 0 \) for a defocusing quadrupole. The focal length \( f \) is given by

\[
f = \lim_{x \to 0} \frac{1}{Kl}
\]

(2.15)

The transfer matrix of a dipole with a small bending angle and large bending radius \( \rho \) becomes

\[
M_{\text{dipole}} = \begin{pmatrix}
  1 & l \\
  0 & 1
\end{pmatrix},
\]

(2.16)

Where \( l \) is the length of the dipole. Particle motion can be tracked through accelerator elements by using these transfer matrices. For one repetitive period of length \( L \), the transfer matrix \( M \) is

\[
M(s + L|s) = M_n M_{n-1} \cdots M_2 M_1,
\]

(2.17)

\[
y(s + L) = M(s + L|s)y(s).
\]

(2.18)
2.3 Transverse Motion

Although it is convenient to track the particle phase space coordinates by transporting the particle through consecutive elements, it is also useful to obtain the general solution of Hill’s equation by assuming a Floquet transformation [6], with the form of the solution given by

\[ y(s) = aw(s)e^{i\psi(s)}, \quad y^*(s) = aw(s)e^{-i\psi(s)}, \quad (2.19) \]

where \( a \) is a constant, and \( w(s) \) and \( \psi(s) \) are the amplitude and phase functions. After substituting into Hill’s equation, the resulting solutions become

\[ y(s) = \sqrt{\epsilon \beta(s)} \cos (\psi(s) - \psi_0), \quad (2.20) \]
\[ \psi = \int_0^s \frac{1}{\beta(s)} ds. \quad (2.21) \]
\[ (2.22) \]

Where \( a, w(s) \) and \( \psi(s) \) can be related to the Courant-Snyder parameters [6] such as

\[ w^2 = \beta, \quad (2.23) \]
\[ \alpha = -ww' = -\beta'/2. \quad (2.24) \]

The Courant-Snyder parameter \( \beta(s) \) is called the beta function of the accelerator and depends exclusively on the magnetic function \( K(s) \) such as

\[ \frac{1}{2} \beta'' + K(s) \beta - \frac{1}{\beta} \left[ 1 + \left( \frac{\beta'^2}{2} \right) \right] = 0. \quad (2.25) \]

Then the betatron tune, the number of betatron oscillation in one oscillation, is defined by

\[ \nu_y = \frac{1}{2\pi} \int_s^{s+C} \frac{1}{\beta(s)} ds, \quad (2.26) \]

where \( C \) is the circumference of the machine. The phase advance of betatron motion increases by \( 2\pi \nu_y \) in one revolution.
Consider a particle with fractional momentum deviation $\delta = (p - p_0)/p_0$, the linearized inhomogeneous equation of motion is

$$x'' + \left( \frac{1 - \delta}{\rho^2 (1 + \delta)} - \frac{K(s)}{(1 + \delta)} \right) x = \frac{\delta}{\rho (1 + \delta)}. \quad (2.27)$$

Since $\delta$ is typically small, e.g. $\delta \leq 0.005$ for RHIC [6], $\delta$ is considered as a perturbation term in the equation of particle motion. The solution is a linear superposition of the particular solutions. The horizontal displacement of an off-momentum particle is given by [6]

$$x(s) = x_\beta(s) + \delta D(s), \quad (2.28)$$

where the betatron motion $x_\beta(s)$ around the off-momentum closed orbit $\delta D(s)$ is given by the solution of Hill's equation shown in Eq. (2.20). The off-momentum closed orbit is proportional to $\delta$ in the first-order approximation, and the dispersion function $D(s)$ is defined as the derivative of the off-momentum closed orbit with respect to $\delta$. The dispersion function $D(s)$ is a periodic function and satisfies the equations

$$D''_x + (K_x(s) + \Delta K_x) D_x = 1/\rho, \quad (2.29)$$
$$D''_z + (K_x(s) + \Delta K_z) D_z = 1/\rho, \quad (2.30)$$

Where $K_x = \frac{1}{\rho^2} - K(s), \Delta K_x = [-\frac{2}{\rho^2} + K(s)]\delta + o(\delta^2), K_z = \frac{1}{\rho^2} - K(s), \Delta K_z = -K(s)\delta + o(\delta^2), K = B_1/B\rho$ and $B_1 = \partial B_z/\partial x$. Neglecting the chromatic perturbation terms $\Delta K_x$ and $\Delta K_z$, the dispersion function is the solution of the inhomogeneous equation. In a planar synchrotron, the dispersion function is usually finite in the horizontal plane, and zero in the vertical plane.

A high energy particle with $\delta > 0$ has a weaker effective focusing strength because of its larger momentum rigidity. A low energy particle with $\delta < 0$ has a stronger effective focusing strength because of its smaller momentum rigidity. Thus the
betatron tunes depend on the particle momentum. That is the chromatic perturbation term $\Delta K_x$ generates the betatron tune shift given by

$$\Delta \nu_x = \frac{1}{4\pi} \int \beta_x \Delta K_x ds \approx \left( -\frac{1}{4\pi} \int \beta_x K_x ds \right) \delta = C_{x,nat} \delta, \quad (2.32)$$

$$\Delta \nu_z = \frac{1}{4\pi} \int \beta_z \Delta K_z ds \approx \left( -\frac{1}{4\pi} \int \beta_z K_z ds \right) \delta = C_{z,nat} \delta. \quad (2.33)$$

The natural Chromaticity, $C_{x,nat}$ or $C_{z,nat}$, is the variation of betatron tune with momentum due to chromatic aberrations of quadrupoles. Strong quadrupoles at the location with large beta functions contribute most to chromaticity. The natural chromaticity is usually very large. For example low emittance synchrotron light sources and high luminosity collider are about 3-4 times their betatron tunes with a beam momentum spread of $\delta = \pm 5 \times 10^{-3}$ [6]. Since large tune spread decreases the beam storage lifetime and dynamical aperture, chromaticity has to be corrected to zero by means of sextupoles. The resulting chromaticity becomes

$$C_x = -\frac{1}{4\pi} \int \beta_x [K_x - S(s) D(s)] ds, \quad (2.34)$$

$$C_z = -\frac{1}{4\pi} \int \beta_z [K_z + S(s) D(s)] ds. \quad (2.35)$$

Where $S(s) = -B_2/B\rho$. In order to compensate the chromatic aberrations of quadrupoles, chromatic sextupoles need to be located at nonzero dispersion place.

The trajectory of particle motion in phase space follows an ellipse described by the Courant-Snyder invariant [6]

$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = \epsilon, \quad (2.37)$$

where $\beta, \gamma, \alpha$, and $\epsilon$ are the ellipse parameters shown in Fig. 2.5. The phase space area $\pi \epsilon$ enclosed by the ellipse determines the beam emittance $\epsilon$, and it obeys Liouville’s Theorem. The beam emittance is constant when the beam is driven by the linear
forces, such as quadrupole and dipole forces. Under the influence of nonlinear forces, such as impedances and space charge forces, the emittance is not conserved. A realistic beam is composed of particle distributed in the phase space, each particle has its own invariant ellipse. The rms emittance is then defined as

$$\epsilon_{\text{rms}} = \sqrt{\sigma_y^2 \sigma_y'^{2} - \sigma_{yy}'.} \quad (2.38)$$

The rms emittance of a beam is equal to the phase-space area enclosed by the ellipse of the rms particle. The betatron oscillations of the rms particle is

$$y(s) = \sqrt{\epsilon \beta(s)} \cos (\psi(s) - \psi_o). \quad (2.39)$$

$$y'(s) = \sqrt{\epsilon \beta'(s)} \cos (\psi'(s) - \psi_o). \quad (2.40)$$
2.4 Synchrotron Motion

If the ring has no RF cavity, the beam will occupy the entire length of the ring, it is called the coasting beam. But we usually desire either beam bunching or bunched beam acceleration, which can be accomplished by applying RF voltage to the beam given by

\[ \varepsilon = \varepsilon_0 \sin(\phi_{rf}(t) + \phi_s), \quad \phi_{rf} = \hbar \omega_0 t. \]  \hspace{1cm} (2.41)

A synchronous particle is defined as a particle with the energy \( E_0 \) that synchronize with the rf wave with a frequency of \( \omega_{rf} = \hbar \omega_0 \), where \( \omega_0 \) is the revolution frequency and \( \hbar \) is an integer. Thus a synchronous particle encounters the rf voltage at the same phase angle \( \phi_s \) every revolution. It will gain or lose energy \( \Delta E = eV \sin \phi_s \) per passage. The acceleration rate for the synchronous particle is \( \dot{E} = \frac{\omega}{2\pi} eV \sin \phi_s \). Also the acceleration rate of a non-synchronous particle is determined by the rf phase angle \( \phi \) such as \( \dot{E}_0 = \frac{\omega}{2\pi} eV \sin \phi \). Particles with energy higher than the synchronous particle arrive at the RF cavity early with respect to the synchronous particle and receive a negative voltage kick, while particles with energy lower than the synchronous particle arrive at the RF cavity late and receive a positive voltage kick. Thus, beam bunching is established.

The non-synchronous particle will oscillate around the synchronous one under the RF focusing in the longitudinal direction, its phase angle \( \phi \) and energy difference \( \Delta E = E - E_0 \) obey the longitudinal equation of motion given by [6]

\[ \dot{\phi} = \hbar \omega_0 \eta \delta = \frac{\hbar \omega_0^2 \eta}{\beta^2 E} \left( \frac{\Delta E}{\omega_0} \right), \]  \hspace{1cm} (2.42)

\[ \frac{d^2}{dt^2} (\phi - \phi_s) = \frac{\hbar \omega_0^2 eV \eta_0 \cos \phi_s}{2\pi \beta^2 E} (\phi - \phi_s), \]  \hspace{1cm} (2.43)

The stability condition for synchrotron oscillation is \( \eta_0 \cos \phi_s = \left( \frac{1}{\gamma^2} - \frac{1}{\gamma} \right) \cos \phi_s < 0 \). The phase angle \( \phi \) of the synchronous particle with the energy \( \gamma \) below the transition
energy $\gamma_T$ should be $0 < \phi_s < \pi/2$. Above the transition energy, the synchronous phase angle should be shifted to $\pi/2 < \phi_s < \pi$. The synchrotron frequency of the beam is $\omega_s = \omega_0 \sqrt{\frac{\hbar V_0 \gamma_0 \cos \phi_s}{2\pi J^2 E}}$. The synchrotron tune, the number of synchrotron oscillations per revolution is defined as $\nu_s = \frac{\omega_s}{\omega_0}$. The typical synchrotron tune is of the order of $\leq 10^{-3}$, e.g. 0.008 for all typical 3 GeV light sources. The synchrotron tune can be so high as 0.1 for rapid cycling accelerators such as the Fermilab Booster and high energy electron-positron colliders such as B factories and LEP [6].

### 2.5 Emittance of Synchrotron Radiation Lattices

Lattices for synchrotron radiation sources are usually arranged such that many insertion devices can be installed to enhance coherent radiation while attaining minimum emittance for the beam. Popular arrangements includes the double-bend achromat, the quadrupole-bend achromat, etc.

#### 2.5.1 The Double-Bend Lattice

Double-bend(DB) cells were often used to accommodate many straight sections for insertion devices(IDs), such as the wigglers and undulators, that enhance the brilliance and wavelength of the radiation. There are the achromatic mode to achieve the minimum emittance double-bend achromat(MEDBA) and the non achromatic mode aiming for the theoretical minimum emittance(TME) in the DB lattice. Because the non-achromatic mode can reach an emittance three times smaller than that at the achromatic condition [6], most light sources operate in the non-achromatic mode in order to reach a smaller emittance. A double-bend cell is made of two dipoles located symmetrically with respect to the center of the cell and the dispersion matching section made of a single quadrupole, a doublet, or a triplet. If the achromatic condition is
Figure 2.6: Schematic of a triplet DBA, where the quadrupole triplet is arranged to attain betatron and dispersion function match of the entire cell.

imposed, that means the dispersion outside the DB cell is zero such as $D = 0, D' = 0$. The DA cell is called a double-bend achromat (DBA). Fig. 2.6 shows a triplet DBA cell. The zero dispersion region, used for insertion devices such as the undulator, the wiggler, and rf cavities, will generally reduce the beam emittance.

The horizontal (natural) emittance of an electron beam in an electron storage ring is determined by the equilibrium between quantum fluctuations and radiation damping. The minimum emittances for the DBA and the TME lattice are [6]

$$\epsilon_{MEDBA} = \frac{1}{4\sqrt{15} J_x} C_q \gamma^2 \theta^3, \quad (2.45)$$

$$\epsilon_{TME} = \frac{1}{3} \epsilon_{MEDBA} = \frac{1}{12\sqrt{15} J_x} C_q \gamma^2 \theta^3 \quad (2.46)$$

Where $C_q = 3.83 \times 10^{-13}$ m, $J_x \approx 1$ is the horizontal damping-partition number, and $\theta$ is the bending angle of dipoles in one double-bend cell. Relaxing the achromatic condition, the emittance can generally be reduced by a factor of 3. For example, The DBA lattice of Taiwan Photon Source (TPS) [9] consists of 24 double-bend cells with 6-fold symmetry and a circumference of 518.4 m. The natural emittance of this DBA lattice is $5.2 \, \text{mm} \cdot \text{mrad}$ in the achromatic mode. If we operate in the low emittance...
Figure 2.7: Optical functions of the DBA superperiod for the Taiwan Photon Source. Top, operating in achromatic mode. Bottom, operating in the low emittance mode (the nonachromatic mode).
mode (the nonachromatic mode) in which some finite dispersion in the straights, the
designed natural emittance is $1.7 \, \text{mm} \cdot \text{mrad}$ which produce very high brilliance
beams of synchrotron radiation. Fig. 2.7 depicts the lattice functions of a super-cell
in two operating modes. For the 3 GeV storage ring, high field wigglers are needed
to produce photon energies larger than about 20 keV. If insertion devices (ID) were
installed, the effective emittance for the low emittance mode would increase from
its original value of $1.7 \, \text{mm} \cdot \text{mrad}$ to $2.12 \, \text{mm} \cdot \text{mrad}$. On the other hand, the
emittance of the achromatic mode decreases from $5.2 \, \text{mm} \cdot \text{mrad}$ to $3.11 \, \text{mm} \cdot \text{mrad}$.
The emittance ratio is not 3 but 1.47 [9].

2.5.2 The Quadrupole-Bend Achromatic Lattice

![QBA cell](image)

**Figure 2.8:** Schematic of a QBA cell, where the quadrupole triplets
and doublets are arranged to attain betatron and dispersion function
match of the entire cell.

The quadrupole-bend achromat (QBA) lattice [9] is defined as a supercell com-
posed of two double-bend cells with unequal bending strengths for the outer and
inner dipoles shown in Fig. 2.8. Assume the lengths of outer and inner dipoles be
$L_1$ and $L_2$, respectively. Based on the small angle approximation, the QBA lattice
requires a necessary condition $L_2/L_1 = 3^{1/3} \approx 1.44$ to match the dispersion function.
The QBA lattice has an advantage over the double-bend nonachromat in having some
zero-dispersion straight sections and has an emittance much smaller than that of a
double-bend achromat. A QBA lattice with a circumference of 486 m and 12 QBA cells for the TPS design was studied last year [9]. Fig. 2.9 shows the optical functions of one QBA cell. The emittance of this lattice is $3.0 \, \text{mm} \cdot \text{mrad}$. Although the QBA emittance is larger than the emittance of the nonachromatic DB lattice ($1.7 \, \text{mm} \cdot \text{mrad}$), the ID loaded emittance of the QBA lattice in the achromatic straight section is smaller than that of the nonachromatic DB lattice while keeping some zero-dispersion sections for high field damping wigglers. Also the emittance of the QBA lattice is smaller than that of the corresponding DBA lattice ($5.2 \, \text{mm} \cdot \text{mrad}$) by nearly a factor of 2, and yet the QBA lattice retains the same flexibility as that of the DBA lattice.

Figure 2.9: Optical functions of the QBA cell for the Taiwan Photon Source.
Chapter 3

Independent Component Analysis for beam measurement and modeling

This chapter introduces Independent Component Analysis (ICA), a new source separation technique exploiting the time coherence of the source signals. The ICA method is based on a joint diagonalization of a set of un-equal time correlation matrices. The de-mixing matrix, which transform the sample data to the source signals is found as the joint diagonalizer of the un-equal time correlation matrices of the sample data with selected time-lag constants. This method was applied to decompose sample signals into its underlying source signals, and was first used to analyze experimental turn-by-turn BPM data of the Fermilab Booster in 2005 [4]. The method enables us to measure beta functions, phase advance, linear coupling angle, and dispersion function, and resolve coupled modes, thus the ICA method is more robust in mode separation.

In most accelerators, beam position monitors (BPMs) are used to record the
transverse position, or displacement, of the centroid motion of the particle beam. The measured displacement is a superposition of the unperturbed displacement and contributions arising from variables affecting the motion of the beam centroid. Using multiple BPMs where the number exceeds the number of changing physical variables affecting the beam, the ability to identify these variables is greatly enhanced by taking advantage of the inherent correlations between BPM readings.

This chapter is intended to study the applicability and error analysis of the ICA method in accelerators. It is organized as follows. The ICA algorithm based on a joint diagonalization of a set of covariance matrices is given in section 3.1. The application of ICA to turn-by-turn BPM data analysis are presented in section 3.2. Section 3.3 provides error analysis for the ICA method. The summary is given in section 3.4.

3.1 The ICA Algorithm

Turn-by-Turn BPM data reflect the beam transverse motion, that is composed of betatron motion, synchrotron motion (coupled through dispersion) and perturbations from other sources, such as noise, wake field, etc. The sampled data can be considered as a linear combination of contributions of a few physical source signals.

\[ x(t) = y(t) = As(t) + n(t) \]  \[ \text{[4]} \]

(3.1)

\( x(t) \) is a noisy instantaneous linear mixture of source signals. This model is commonplace in the field of narrow band array processing. Vector \( s(t) = [s_1(t), \ldots, s_n(t)]^T \) contains the signals emitted by \( n \) narrow band sources; vector \( y(t) = [y_1(t), \ldots, y_n(t)]^T \) contains the array output sampled at time \( t \); matrix \( A \) is the mixing matrix between sources and sensors. \( n(t) \) is modeled as a stationary, temporally white, zero-mean random process independent of the source signals. The fundamental restriction in
ICA is that the independent components must be nongaussian for ICA to be possible, because the mixing matrix $A$ is not identifiable for gaussian independent components.

### 3.1.1 Constructing the BPM Data Matrix

The data sampled by BPMs around the ring are put into a data matrix.

$$X = \begin{pmatrix} x_1(1) & x_1(2) & \cdots & x_1(N) \\ x_2(1) & x_2(2) & \cdots & x_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ x_M(1) & x_M(2) & \cdots & x_M(N) \end{pmatrix},$$

where $M$ is the total number of BPMs, $N$ is the number of turns, $x_i(j)$ is the reading of the $i$th BPM on the $j$th turn. Then we can apply the ICA algorithm to extract the mixing matrix $A$ and source signals $s$ from the data matrix $X$. The $i$th column $A_i$ of the mixing matrix $A$ is the spatial function of the $i$th mode. Each source signal $s_i$, the $i$th row of the source signal matrix $s$, is the temporal function of the $i$th mode. The physical origin of a mode can be identified by its spatial and temporal functions.

### 3.1.2 Preprocessing

Before applying the ICA algorithm on the BPM data, it is usually very useful to do some preprocessing. We discuss some preprocessing techniques that make the problem of ICA estimation simpler and better conditioned.

**Centering** The most basic and necessary preprocessing is to center, that is to subtract its mean vector $m_i = \bar{x}_i$ so as to make a zero-mean variable. This implies that the new $x_i$ is zero-mean as well, as can be seen by taking expectations on both sides of Eq. (3.1).
This preprocessing is made solely to simplify the ICA algorithms: It does not mean that the mean could not be estimated. After estimating the mixing matrix $A$ with centered data, we can complete the estimation by adding the mean vector of $s$ back to the centered estimates of $s$. The mean vector of $s$ is given by $A^{-1}m$, where $m$ is the mean that was subtracted in the preprocessing.

**Whitening** The important preprocessing strategy in ICA is to white the observed BPMs data matrix. This means that before the application of the ICA algorithm (and after centering) the observed matrix $X$ is linearly transformed to a new matrix $\xi$ which is white, i.e. its components are uncorrelated and their variances equal unity. In other words, the covariance matrix of $\xi$ equals the identity matrix:

$$C_{\xi}(0) = \langle \xi(t)\xi(t)^T \rangle = I. \quad (3.2)$$

The whitening transformation is always possible. We use the eigen-value decomposition(EVD) of the covariance matrix for whitening. First, compute the $m \times m$ sample covariance matrix $C_X(0) = \langle X(t)X(t)^T \rangle$. Perform eigenvalue decomposition on $C_X(0)$ to obtain

$$C_X(0) = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix}, \quad (3.3)$$

where $U_1, U_2$ are the orthogonal matrices of eigenvectors of $C_X(0)$ and $\Lambda_1, \Lambda_2$ are the diagonal matrices of its eigenvalues, and $\min(diag(\Lambda_1)) \geq \lambda_c \geq \max(diag(\Lambda_2))$. $\lambda_c$ is a cut-off threshold set to remove the singularity of the data matrix, and $\Lambda_1$ is $n \times n$ diagonal matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_c$. Whitening can now be done by

$$\xi = \Lambda_1^{-1/2}U_1^TX = \tilde{A}s. \quad (3.4)$$
Whitening transforms the mixing matrix into a new one, $\tilde{A}$, and the new mixing matrix $\tilde{A}$ is orthogonal. This can be seen from

$$C_\xi(0) = \langle \xi(t)\xi(t)^T \rangle = \xi\xi^T = \tilde{A}s\tilde{s}^T \tilde{A}^T = \tilde{A}\tilde{A}^T = I. \quad (3.5)$$

Here we see that whitening reduces the number of parameters to be estimated. Instead of estimating the $n^2$ parameters of the original $n \times n$ $A$ matrix, we only need to estimate the new orthogonal mixing matrix $\tilde{A}$ with $n(n - 1)/2$ degrees of freedom. For example in two dimensions, a single angle parameter an orthogonal transformation. Thus whitening solves half of the problems of ICA. Because compared with any ICA algorithms, whitening is a very simple and standard procedure, it is a good idea to reduce the complexity of the problem this way.

It is also quite useful to reduce the dimension of the data at the same time as we do the whitening. Then we look at the eigenvalues of $C_X(0)$ and discard those that are smaller than the cut-off threshold $\lambda_c$. This has often the effect of reducing white gaussian noise. However, a large cut-off threshold $\lambda_c$ carries the risk of introducing errors from unknown sources, which can be observed in models shown in Section 3.3.

### 3.1.3 Joint Diagonalization

The simple form of time-lag structure is given by covariances. This means simply use some lag constant $\tau$ so that the time-lagged covariance matrix

$$C_\xi(\tau) = \langle \xi(t)\xi(t-\tau)^T \rangle. \quad (3.6)$$

Let us consider a slightly modified version of the lagged covariance matrix as defined in eqn. 3.6, given by

$$\bar{C}_\xi(\tau) = (C_\xi(\tau) + C_\xi(\tau)^T)/2. \quad (3.7)$$
3.1 The ICA Algorithm

The reason why we considered this modified matrix instead of $C_\xi(\tau)$ is that we want to have a symmetric matrix, because then the following eigenvalue decomposition is well defined and simple to compute.

For a set of time-lag constants $\tau(\tau = 1, 2, \cdots, \tau_k)$, then we have for the orthogonal separating matrix $W$ that diagonalizes all matrices $\tilde{C}_\xi(\tau)$ of this set, i.e.,

\[
\begin{align*}
\xi(t) & = Ws(t), \\
\xi(t - \tau) & = Ws(t - \tau), \\
C_\xi(\tau) & = (C_\xi(\tau) + C_\xi(\tau)^T)/2 \\
& = (\langle \xi(t)\xi(t - \tau)^T \rangle + \langle \xi(t - \tau)\xi(t)^T \rangle)/2 \\
& = \frac{1}{2}W(\langle s(t)s(t - \tau)^T \rangle + \langle s(t - \tau)s(t)^T \rangle)W^T \\
& = W\tilde{C}_s(\tau)W^T
\end{align*}
\]

Here the ICA method assumes that source signals have non-overlapping power spectra, which often holds because source signals $s_i(t)$ are usually harmonic oscillation with different tunes. Due to the independence of the $s_i(t)$, the time-lagged covariance matrix $C_s(\tau) = \langle s(t)s(t - \tau) \rangle$ is diagonal, we denote it by $D_\tau$, then the modified covariance matrix $\tilde{C}_s(\tau)$ equals to $D_\tau$. Thus we have

\[
\tilde{C}_\xi(\tau) = WD_\tau W^T.
\]

Finally, the source signals and the mixing matrix are

\[
s = W^TVX \\
A = V^{-1}W
\]

(3.8) (3.9)

where $V = \lambda_1^{-1/2}U_1^T$ and $V^{-1} = U_1\lambda_1^{1/2}$. In principle, using several time lags, we want to simultaneously diagonalize all the corresponding lagged covariance matrices. It must be noted that the diagonalization is not possible exactly, since the eigenvectors
of the different covariance matrices are unlikely to be identical, except in the theoretical case where the data is exactly generated by the ICA model. Here we use the joint approximate diagonalization algorithm to find a unitary matrix $W$ that diagonalizes all time-lagged covariance matrices $\tilde{C}_\xi(\tau)$, i.e., $\tilde{C}_\xi^* = W D_\tau W^T$, Where $D_\tau$ is diagonal. So here we formulate functions that express the degree of diagonalization obtained and find its maximum. One simple way to measure the diagonality of a matrix $M$ to use the operator

$$
off(M) = \sum_{i \neq j} m_{ij}^2,
$$

which gives the sum of squares of the off-diagonal elements $M$. Consider a set $\mathcal{C} = (\tilde{C}_\xi(1), \tilde{C}_\xi(2), \cdots, \tilde{C}_\xi(\tau_K))$ of $\tau_K$ matrices of size $n \times n$. The joint diagonality (JD) criterion is defined

$$
\mathcal{J}_1(W) = \sum_{\tau=1}^{\tau_K} off(W\tilde{C}_\xi(\tau)W^T)
$$

Minimizing $\mathcal{J}_1$ under the constraint that $W$ is orthogonal gives us the estimation of source signals. We can simplify the criterion $\mathcal{J}_1$. The sum of the squares of the elements of $W\tilde{C}_\xi(\tau)W^T$ is constant due to an orthogonal transformation $W$, thus the JD criterion could be expressed as the difference of the total sum of squares minus the sum of the squares on the diagonal. We can formulate JD criterion

$$
\mathcal{J}_2(W) = -\sum_{\tau=1}^{\tau_K} \sum_{i=1}^{n} (w_i\tilde{C}_\xi(\tau)w_i^T)^2.
$$

Thus minimizing $\mathcal{J}_2$ is equivalent to minimizing $\mathcal{J}_1$.

### 3.2 ICA Analysis for Synchrotron Beam Diagnosis

The independent component analysis (ICA) method isolates the independent source signals from the samples using unequal time correlations. Those identified source
signals can provide information on the betatron motion, synchrotron motion and other perturbation modes according to their spatial and temporal functions. The ICA method is essentially a time-correlation based method, which is efficient in isolating the source signals without overlapping power spectra, which often hold because of the harmonic oscillation with different tunes. To test the new method, simulations have been done for the Taiwan Photo Source (TPS) in the low emittance mode using beam history measurements at 168 BPMs. The ring consists of 24 double-bend cells with 6-fold symmetry and the circumference is 518.4 m. The designed natural emittance with slightly positive dispersion in the straight sections is less than 2 nm-rad. This low emittance lattice structure needs strong quadrupoles and sextupoles and the closed orbit distortions are sensitive to the alignment errors in the quadrupoles and sextupoles as well.

### 3.2.1 Betatron Modes

Turn-by-Turn BPM data are normally composed of coupled betatron, synchrotron oscillations contaminated with noises. If we assume that the transfer function of BPM system is linear, the data sampled by BPMs can be considered as a linear combination of source signals. The physical origin of ICA modes can be identified by their spatial and temporal functions. The betatron oscillation has a different phase at each BPM and will appear as two ICA modes with identical frequency spectrum. Thus the betatron function and phase advance can be derived according to their spatial and temporal functions [4]. The fractional part of the betatron tune can be obtained from the FFT on the temporal function. The betatron motion will appear as two modes with identical fractional tune. Let \( u(t) \) be the betatron component of the transverse motion, then

\[
u(t) = A_1 s_1(t) + A_2 s_2(t).
\] (3.10)
where $A_1$ and $A_2$ are spatial functions, $s_1$ and $s_2$ are sinelike and cosinelike modes. Then the betatron function and phase advance can be obtained \[4\]:

\[
\beta_i = a^2 (A_{1,i}^2 + A_{2,i}^2). \tag{3.11}
\]

\[
\psi_i = \tan^{-1} \left( \frac{A_{1,i}}{A_{2,i}} \right). \tag{3.12}
\]

To fully exploit the ICA method performance of the photon source, the study of beam dynamics is of great importance. At the Taiwan Photo Source (TPS), we use 168 BPMs (7 BPMs in each DBA cell). The betatron tunes are about $\nu_x = 26.22$ and $\nu_y = 12.28$. We applied the ICA analysis to turn-by-turn BPM data, from which we obtain the beta function, phase advances and betatron tunes. The betatron tune is an important signature to identify betatron modes. Fig. 3.1 shows that using the spatial function of two horizontal modes, one can derive the betatron amplitude function and phase advance, which are consistent with results from MAD modeling. The rms errors of beta function and phase advance are $\sigma_{\beta_x}/\beta_x = 0.0005$ and $\sigma_{\psi_x} = 0.001$.

### 3.2.2 Synchrotron Modes

For TPS, with an RF voltage of 5.0 MeV and the harmonic number of 864, the synchrotron tune ($\nu_s$) is 0.0067. One synchrotron signal can also be separated out. This mode can be identified by the synchrotron tune obtained from the FFT on the temporal function. The spatial pattern of the synchrotron mode enables us to obtain the dispersion function. Let $w(t)$ be the synchrotron oscillation component in the transverse motion,

\[
w(t) = A_s s_s(t). \tag{3.13}
\]

The dispersion $D$ is estimated as \[4\]

\[
D = bA_s, \tag{3.14}
\]
3.2 ICA Analysis for Synchrotron Beam Diagnosis

![Graph showing βx (m) vs BPM index]

**Figure 3.1:** Using the spatial function of two horizontal modes, one can derive the betatron amplitude function and phase advance, which are consistent with results from MAD modeling based on a realistic Taiwan Photon Source. Betatron oscillations are excited at amplitude 1 mm lattice.

Where $b$ is a constant. Fig. 3.2 shows results of a numerical experiment to check the Eq. (3.14). The dispersion function $D$ is in good agreement with results from MAD modeling. The rms error of dispersion is 0.0007 m.

### 3.2.3 Chromaticity

The momentum spread gives rise to tune spread in the beam, Machine chromaticities can be derived from tunes of betatron temporal functions vs beam momentum. Because of the high accuracy of betatron tune measurements the ICA method provides a convenient way to measure the chromaticity. This can be done by changing the momentum deviation and measuring the corresponding change of betatron tunes. For
3.2 ICA Analysis for Synchrotron Beam Diagnosis

Figure 3.2: The dispersion function $D$ vs results from MAD modeling based on a realistic Taiwan Photon Source lattice. Betatron oscillations are excited at amplitude 1 mm.

TPS, the natural chromaticities are $C_{x,\text{nat}}/C_{z,\text{nat}} = -78.2/ -32.5$. Fig. 3.3 shows a dependence of the betatron tunes on the fractional momentum deviation. The slope of betatron tune over momentum deviation can be obtained by a linear fitting. The measured natural chromaticities is about $C_{x,\text{nat}}/C_{z,\text{nat}} = -79.0/ -33.3$, which can lead to a natural tune spread of about $\Delta \nu_x/\Delta \nu_y = 0.2/0.3$ for a beam with an rms spread of $\delta = 0.002$. The error of the chromaticity measurements is about $\pm 0.8$. The chromaticity sextupoles can compensate the loss of focusing in quadrupoles. The reduced chromaticities (Fig. 3.4) are $C_{x,\text{nat}}/C_{z,\text{nat}} = -0.5/ -1.0$, which agree with results from MAD modeling $C_{x,\text{nat}}/C_{z,\text{nat}} = -0.3/ -1.4$. Fig. 3.5 show the measured chromaticities are almost independent of random noise level, so the ICA method achieves the high accuracy of chromaticies.
Figure 3.3: A dependence of the betatron tunes on the fractional momentum deviation without chromaticity sextupole compensation

Figure 3.4: A dependence of the betatron tunes on the fractional momentum deviation with chromaticity sextupole compensation
3.2 ICA Analysis for Synchrotron Beam Diagnosis

3.2.4 Linear Coupling

The main sources of linear coupling in the light source are quadrupole roll errors, closed orbit vertical offset in sextupoles and coupling from solenoids. In synchrotron storage rings, the coupling from solenoids is negligible [7]. Here we evaluate the impact of quadrupole roll errors on linear coupling in the 3 GeV Taiwan Photo Source (TPS) in low emittance mode, which consists of 240 quadrupoles. We modulate the magnitudes of quadrupole roll errors each time in the numerical experiments. And the random modulation are chosen so that all quadrupoles are rolled by a random small angle.

I The demixing

There are several approaches to tackle the linear coupling. Since we are interested in the two betatron tunes response to the random quadrupole roll errors, perturbation theory [5] based on the Hamitonian mechanism is more straightforward and simple,
and therefore is adopted here. The two eigentunes $Q_1, Q_2$ are given by:

$$Q_1 = Q_{x,0} - \frac{\delta}{2} + \frac{1}{2} \sqrt{\delta^2 + |G|^2}, \quad (3.15)$$

$$Q_2 = Q_{y,0} + \frac{\delta}{2} - \frac{1}{2} \sqrt{\delta^2 + |G|^2}, \quad (3.16)$$

$$\delta = Q_{x,0} - Q_{y,0} - l. \quad (3.17)$$

where $Q_{x,0}, Q_{y,0}$ are the uncoupled tunes. $\delta$ is the distance to the linear coupling resonance line in the $Q_x - Q_y$ plot. So it is the decimal split of the two uncoupled tunes. For TPS, $Q_{x,0} = 26.22$, $Q_{y,0} = 12.28$, $l = 26 - 12 = 14$. $G$ is the coupling strength.

In our simulation studies we apply ICA methods to separate normal modes. We find the ICA method can always separate those two normal modes. Those two eigentunes can be derived from the temporal functions shown in Fig. 3.7 and 3.9. The spatial functions of those two modes are presented in Fig. 3.6 and 3.8. Assume all quadrupoles rolled by a small angle $\phi \cdot h$, $\phi$ is the magnitude of quadrupole roll errors, $h$ is a random variable uniformly distributed in $[-1, 1]$. We can define $\Delta = |(Q_{x,0} - Q_{y,0}) - (Q_1 - Q_2)|$ as tune split shift. Fig. 3.10 shows tune split shift $\Delta$ vs the magnitude of quadrupoles roll errors $\phi$. As the magnitude of quadrupole roll errors increases across the linear coupling resonance, the two eigentunes of the machine approach each other. For TPS, $\phi = 2$ mrad will lead to a tune split shift about 0.001.

II The coupling strength estimation

Experimental measurement of linear coupling is traditionally to measure the minimum tune splitting of those two normal modes. As the strength of quadrupole is varied across the linear coupling resonance, the normalized tunes of the machine approach each other, reaching a minimum value $|G|$ [6]. But we need to use a set of quadrupoles
Figure 3.6: The spatial function of linear coupling mode 1

Figure 3.7: FFT spectrum of the temporal function $s_1$
Figure 3.8: The spatial function of linear coupling mode 5

Figure 3.9: FFT spectrum of the temporal function $s_5$
3.2 ICA Analysis for Synchrotron Beam Diagnosis

Figure 3.10: The measured tune split shift $\Delta$ in the MAD modeling vs the magnitude of quadrupoles roll errors $\phi$.

Figure 3.11: The estimated coupling strength $|G|$ (blue line) is compared with the MAD modeling based on a realistic Taiwan Photon Source lattice.
for achieving independent horizontal or vertical tune change. Since the ICA method provides much information on the beta function and phase advance, we can find a new approach to estimate the coupling strength $|G|$ due to quadrupole roll errors. The details will be discussed in the following.

The coupling contribution from quadrupole roll errors in TPS is

$$G e^{i\chi} = \frac{1}{2\pi} \int \sqrt{\beta_x \beta_y} K_N \cdot \phi h e^{i[\psi_x - \psi_y - \frac{2\pi}{L} \delta]} ds. \quad (3.18)$$

Where $\beta_x$ and $\beta_y$, $\psi_x$ and $\psi_y$ are the beta functions and betatron phase advances at quadrupoles. $K_N \cdot \phi h ds$ is the integrated quadrupole roll errors, $L$ is the ring circumference. $G$ and $\chi$ are the coupling strength and the coupling phase. And all quadrupoles of the length $l$ rolled by a small angle $\phi \cdot h$, $\phi$ is the magnitude of quadrupole roll errors, $h$ is a random variable uniformly distributed in $[-1, 1]$.

The coupling can be approximated by:

$$G e^{i\chi} \simeq \frac{1}{2\pi} \sum \sqrt{\beta_x \beta_y} K_N l \cdot \phi h e^{i[\psi_x - \psi_y - \frac{2\pi}{L} \delta]} . \quad (3.19)$$

The fractional parts of the two betatron tunes are very close for most circular accelerators, so the maximum coupling angle difference $\frac{2\pi}{L} \delta$ is usually small, we can usually ignore this part. But for TPS, $\delta \approx 0.06$, the maximum $\frac{2\pi}{L} \delta$ is about $360^\circ \times 0.06 = 21.6^\circ$. We need to consider this part in this model. Once we know the distribution of quadrupole roll error, the coupling strength can be estimated as

$$|G| \simeq \frac{\phi}{2\pi} \sum \sqrt{\beta_x \beta_y} K_N l \cdot \phi h e^{i[\psi_x - \psi_y - \frac{2\pi}{L} \delta]} . \quad (3.20)$$

Now it is clear from Eq. (3.20) that the coupling strength is proportional to the magnitude $\phi$ of quadrupoles roll errors, the slope is related to the beta functions and phase advance, which can be derived form the ICA modes. Once the magnitude of quadrupoles roll errors is measured in experiments, we can estimate the coupling strength. Fig. 3.11 shows results of a numerical experiment to test the above predictions. The coupling strength $|G|$ is in good agreement with those predicted by
Eq. (3.20). The ICA method for the coupling strength measurement remarkably reduces the need of human interaction with the console and increases the accuracy. Once the ICA method identifies the betatron mode, the coupling strength will be obtained, which saves a tremendous amount of time.

### 3.2.5 Measurement of Vertical Dispersion

Vertical dispersion in photon sources is undesirable because it degrades the properties of photon beam. In an ideal lattice there is no vertical dispersion. However, due to the presence of quadrupole roll errors, a finite vertical dispersion is always present. Here we evaluate the impacts of quadrupole roll errors in the 3 GeV Taiwan Photon Source (TPS), which consists of 240 quadrupoles. We modulate the magnitudes of quadrupole roll errors each time in the numerical experiments. And the random
Figure 3.13: The rms errors of the vertical dispersion increase with the magnitude of quadrupoles roll errors $\phi$.

Figure 3.14: The sensitivity of dispersion to quadrupole roll errors $\phi$. 
modulation are chosen so that all quadrupoles are rolled by a random small angle. Assume all quadrupoles rolled by a small angle $\phi \cdot h$, $\phi$ is the magnitude of quadrupole roll errors, $h$ is a random variable uniformly distributed in $[-1, 1]$. By changing the magnitude of quadrupoles roll errors $\phi$, we observe the rms errors of the vertical dispersion will increase with quadrupoles roll errors shown in Fig. 3.13. Fig. 3.14 shows the sensitivity of dispersion to quadrupole roll. The rms vertical dispersion increases almost linearly with the magnitude of quadrupoles roll errors.

### 3.3 Error Analysis of the ICA method

#### 3.3.1 The effects of the cut-off threshold $\lambda_c$

The important step of preprocessing for the ICA method is to whiten the sample signals $X$. This is achieved by applying to $X$ a whitening matrix shown in section 3.1.2. We note that this whitening procedure reduces the dimensions of the $M \times N$ data matrix $X$ to that of a unitary $n \times N$ unitary matrix $\xi$ such as $\xi = \Lambda^{-1/2} U^T X$, here $n$, the dimension of the new mixing matrix $x$, is determined by the cut-off threshold $\lambda_c$, which needs to be large enough to have a meaningful eigenvalue decomposition (EVD) yet small enough so that there is little mixture of source signals with additional degrees of freedom.

The number of EVD eigenmodes above the cut-off threshold determines the number of significant physical variables that are changing and affecting the beam centroid motion. We need to focus on the physical meaning of the EVD results in order to illustrate their usefulness and limitations for beam dynamics analysis. For example, consider an ideal betatron oscillation recorded at the $n$-th turn can be written as

$$x_n = A \sin(2\pi n \nu_x + \chi)$$

(3.21)

Where $A$ is the oscillation amplitude, $\nu_x$ is the horizontal tune, and $\chi$ is the phase.
Figure 3.15: Top, the SVD eigenvalue distribution for an ideal betatron oscillation of the model in Eq. (3.21). Bottom, the typical SVD eigenvalue distribution for the horizontal motion data of the QBA lattice, the DBA lattice, and the low emittance lattice. Betatron oscillations are excited at amplitude 5 mm with $\Delta p/p = 0.001$. 
Figure 3.16: The rms errors of the resulting oscillation amplitude $A$ of the ICA method are estimated.

The horizontal beam data consist of $N_{BPM} = 168$ sequential BPMs and $N_t = 1024$ turns. A eigenvalue decomposition of the covariance matrix $C_x(0) = <XX^T>$ can be invoked to aid in the identification of dominated modes. The top plot of Fig. 3.15 shows the eigenvalue distribution. The curve connect the eigenvalues in order of decreasing eigenvalue amplitude. We see that the eigenvalues go down quickly from about 0.1 to $10^{-12}$. Most of the eigenvalues are small and about the same size, the long flat part is called the noise floor. Therefore above the noise-floor level, all the beam motions observed would be a linear combination of the first two modes. With knowledge of only two variables affecting the beam motion for the ideal betatron oscillation, one can identify the first two eigenmodes are mixtures of source signals. We set the cut-off threshold $\lambda_c$ a little bit above the noise-floor level, that means only the first two eigenmodes are kept such as $n = 2$. Then we perform ICA on the whitened data matrix $\xi$. The rms errors of the resulting oscillation amplitude $A$ of
the ICA method are estimated as shown in Fig. 3.16. The relative error $\sigma_A/A$ in calculation of the oscillation amplitude is less than $2 \times 10^{-4}$ at $n = 2$. For $n > 2$, the random noise becomes a secondary contribution to the rms error $\sigma_A/A$ of the ICA method, and it increased the rms error from $2 \times 10^{-4}$ to 0.1 for $n = 20$. Note that the eigenmodes in whitening are often a mixture of various physical patterns and therefore hard to interpret. However each of the orthogonal bases obtained from ICA correspond uniquely to the physical pattern in Eq. (3.9). Since the the resulting oscillation amplitude $A$ of the ICA method is sensitive to the cut-off threshold, one has to find the noise level and set up $\lambda_c$ to determine which eigenvalues are significant.

In typical applications, betatron oscillations are excited at amplitude 5 mm with $\Delta p/p = 0.001$ and turn-by-turn BPM data are recorded. There are about 18 SVD eigenvalues above noise floor shown in the bottom plot of Fig. 3.15. The simulation data are horizontal BPM reading from the QBA lattice for 1024 turns and 480 BPMs. The top two modes are significantly larger than the rest. They are mainly due to the two degrees of freedom in the horizontal betatron motion. Although we set up $\lambda_c$ above the noise floor to separate noise from signals, the two largest EVD eigenmodes principally from betatron motion can still be mixed with additional degrees of freedom such as a single degree of freedom in the horizontal synchrotron motion and etc.

Additional degrees of freedom will introduce errors into the time-lagged covariance matrices of the ICA method. For example, Fig. 3.18 shows the rms errors of the beta function and phase advance of the ICA method based on the QBA lattice depend on the number of eigenmodes ($n$) we kept in whitening. Up to $n = 5$, the measurement error $\sigma_{\beta_x}/\beta_x$ is estimated to be $\sigma_{\beta_x}/\beta_x = 0.0027$. That is the first two eigenmodes are mixtures of the two betatron modes. ICA can easily separate those mixtures shown in the top plot of Fig. 3.17. At about $n = 6$, an additional degree begins to affect the beam, the synchro-betatron coupling is coupled with the two degrees of freedom in the horizontal betatron motion. We can observe the synchro-betatron coupling in the
Figure 3.17: Top, FFT spectrum of the temporal function $s_1$ at $n=2$ and $\tau_k = 3$. Bottom, FFT spectrum of the temporal function $s_1$ at $n=6$ and $\tau_k = 3$. Betatron oscillations are excited at amplitude 5 mm with the fractional momentum deviation $\delta = \Delta P/P_0 = 0.001$. 
3.3 Error Analysis of the ICA method

temporal mode of ICA shown in the bottom of Fig. 3.17. An added complication in
demixing the EVD eigenmodes is that the transverse motion of the beam contain the
synchrotron motion which is coupled through the dispersion function. This shows up
as two synchrotron sidebands around the betatron tune in the bottom of Fig. 3.17.
These two sideband has nearly the same structure. The half distance between the
two synchrotron sidebands is called the synchrotron tune. For QBA, the synchrotron
tune is 0.008. Thus the rms error of the beta function $\sigma_{\beta_x}/\beta_x$ goes up to 0.1 at
$n = 10$. Thus the resulting error in beta function is very sensitive to the number of
eigenmodes we kept. Additional degrees of freedom is also largely attributable to the
rms error of the phase advance at $n > 5$ shown in Fig. 3.18. In order to suppress
the errors due to additional degrees of freedom, we may increase the set size $\tau_k$ of
covariance matrices. We will discuss it in the next section 3.3.2. Sometimes the large
$n$ will yield a better estimate of the synchrotron pattern. That’s because we need to
keep enough EVD eigenmodes to have meaningful ICA decomposition. For example,
the horizontal dispersion error is estimated to be $\sigma_{D_x}/D_X = 0.048$ at $n = 3$ shown in
Fig. 3.19. The value of the horizontal dispersion function $D_x$ with a precision better
than 4% can be achieved at $n = 5$.

3.3.2 The effects of $\tau_k$

The joint diagonalization of several covariance matrices is intended to reduce the
probability that an unfortunate choice of time lag $\tau$ results in un-identifiability of the
mixing matrix $\hat{A}$ from covariance matrix $C_\xi(\tau)$. Then un-identifiability of $\hat{A}$ arises
in the case of degenerate eigenvalue $[3]$. It does not seem possible to determine a
time lag $\tau$ such that the eigenvalues of $C_\xi(\tau)$ are distinct. It is to be expected that
when an eigenvalue of $\hat{C}_\xi(\tau)$ comes close to degeneracy, the robustness of determining
the mixing matrix $\hat{A}$ from an eigendecomposition is seriously affected. The situation
Figure 3.18: The rms errors of the beta function and phase advance of the ICA method depend on the number of eigenmodes\((n)\) we kept in whitening. Betatron oscillations are excited at amplitude 5 mm with the fractional momentum deviation \(\delta = \Delta P/P_0 = 0.001\).
3.3 Error Analysis of the ICA method

Figure 3.19: The rms errors of the dispersion function of the ICA method depends on the number of eigenmodes \( n \) we kept in whitening. Betatron oscillations are excited at amplitude 5 mm with the fractional momentum deviation \( \delta = \Delta P/P_0 = 0.001 \).
3.3 Error Analysis of the ICA method

Figure 3.20: Top, the spatial function of the synchro-betatron coupling mode $A_1$ at $n=16$ and $\tau_k=10$. Bottom, FFT spectrum of the temporal function $s_1$ at $n=16$ and $\tau_k=10$. Betatron oscillations are excited at amplitude 5 mm with the fractional momentum deviation $\delta = \Delta P/P_0 = 0.001$. 
Figure 3.21: Top, the spatial function of the synchro-betatron coupling mode $A_6$ at $n=16$ and $\tau_k=10$. Bottom, FFT spectrum of the temporal function $s_6$ at $n=16$ and $\tau_k=10$. Betatron oscillations are excited at amplitude 5 mm with the fractional momentum deviation $\delta = \Delta P/P_0 = 0.001$. 


3.3 Error Analysis of the ICA method

Figure 3.22: The rms errors of the resulting optical functions of the ICA method with different sampling turns $N_t$ in the QBA lattice. Left top: $\sigma_{\beta_x}$ vs $N_t$. Right top: $\sigma_{\psi_x}$ vs $N_t$. Left bottom: $\sigma_{D_x}$ vs $N_t$. Betatron oscillations are excited at amplitude 5 mm with the fractional momentum deviation $\delta = \Delta P/P_0 = 0.001$. The estimation is made by taking 16 eigenmodes $n = 16$ and a set of $\tau_k = 10$ whitened covariance matrices.
is more favorable if we consider simultaneous diagonalization of a set \( \{ \tilde{C}_\xi(\tau) | \tau = 1, 2, \ldots, \tau_k \} \) of \( \tau_k \) whitened covariance matrices.

One can improve the ICA accuracy by taking a large set of covariance matrices. It is always possible to find a set of covariance matrices to suppress the rms errors of the beta function and phase advance due to additional degrees of freedom. Consider the synchro-betatron coupling in section 3.3.1, Fig. 3.18 demonstrates the improvement in the \( \beta \) measurement that can be achieved by setting \( \tau_k \geq 10 \), the rms deviation around the ring is only 0.27%. Now the two synchrotron sidebands are invisible in the betatron mode \( s_1 \) depicted in Fig. 3.20, even though they do exist and show up in the mode \( s_6 \) depicted in Fig. 3.21. The improvement in beta function accuracy may then allow studies of subtle beam dynamics issues and provide better control of the beam.

Because the approximate joint diagonalization allows the information contained in a set of covariance matrices to be integrated in a single mixing matrix \( \hat{A} \), this approach generally increases that statistical efficiency of the procedure by inferring the value of the mixing matrix \( \hat{A} \) from a large set of statistics. Thus the ICA method is able to identified source signals with only a few hundred sampling turns. That is the ICA method is less affected by the number of sampling turns \( N_t \) because its results are based on the joint diagonalization of a set, e.g. \( \tau_k = 10 \), of several covariance matrices instead of only one. Fig. 3.22 shows the dependence of optical functions on the number of sampling turns \( N_t \).

### 3.3.3 The effects of the BPM reading noise

In the following we discuss the results of the optical functions with the various random BPM reading noise. Without BPM noise we find that the minimum rms errors of the horizontal beta function and phase advances of the ICA method are estimated to be
\[ \sigma_{\beta_x}/\beta_x = 0.27\% \] and \[ \sigma_{\psi_x} = 1.7\%. \] In reality BPM readings always contain random noises which affect the results of the ICA analysis. We insert white Gaussian noise into the BPM data matrix \( X \). Fig. 3.23 shows the rms errors of the resulting beta functions and phase advances of the ICA method for different levels of BPM noise. The BPM reading error is 10 \( \mu m \) typically. When we add 500 \( \mu m \) reading error, the resulting errors only increase to \[ \sigma_{\beta_x}/\beta_x = 0.5\% \] and \[ \sigma_{\psi_x} = 1.8\%, \] respectively. Because the ICA method removes white noise as its first step, the rms errors of beta function and phase advances are still very small even with the large BPM reading noise \[ \sigma_{\text{noise}} = 1 \text{ mm}. \] Fig. 3.23 also shows the rms errors of the resulting dispersion function slowly increase with random noise level. Thus the ICA method is more robust in mode separation and is less affected by the BPM reading noises.

### 3.4 Summary

We extended the ICA method to study beam dynamics, chromatic aberration and linear coupling motion in the DBA and QBA lattices for the Taiwan Photon Source. We find that the ICA methods enable us to achieve the high accuracy of optical functions, tunes, and chromaticities. And the ICA method is efficient in mode separating. The betatron and synchrotron modes can be identified by their tunes. Their spatial patterns are used to calculate the dispersion function. Because of chromaticity, the momentum spread gives rise to tune spread in the beam. Machine chromaticities can be derived from tunes of betatron temporal functions vs beam momentum. In the presence of linear coupling, betatron modes have two separate eigenfrequencies. The fractional part of the betatron tunes can be separately measured according to two temporal functions. This section offers a new approach to estimate the coupling strength \( |G| \), which remarkably reduces the need of human interaction with the console and saves a tremendous amount of time.
Figure 3.23: The rms errors of the resulting optical functions of the ICA method with various random noise levels are estimated in the QBA lattice. Data of 1024-turn and 480-BPM are used to calculate $\sigma_{\beta x}/\beta x$ (left top), $\sigma_{\psi x}$ (right top), and $\sigma_{Dx}$ (left bottom). Betatron oscillations are excited at amplitude 5 mm with the fractional momentum deviation $\delta = \Delta P/P_0 = 0.001$. The estimation is made by taking 16 eigenmodes $n = 16$ and a set of $\tau_k = 10$ whitened covariance matrices.
The simulation of optical functions measurements suggests that the prior demixing of synchro-betatron coupling is essential for obtaining relevant results. The ICA method is not affected by the sampling number. The anticipated BPM noise of 100 $\mu$m yields beta-function measurement errors comparable to the specified tolerances. Measuring the dispersion may require keeping enough eigenmodes in whitening.
Chapter 4

Equilibrium Emittance with Coupling and Radiation

In circular accelerators and colliders of particles, the presence of coupling may indeed induce transfer of oscillations from one plane to the other, tilt of the normal directions in which the betatron motion is decoupled, generating beating of the $\beta$ function and modifying the nominal equilibrium emittances. The analytical tools that have been developed to help in understanding the physics of the mechanisms involved in the betatron coupling and related impact of radiation\[8\], the theory developed allows one to estimate the two borderline cases. The one extreme case is to evaluate the emittance ratio for vanishing linear coupling $g = \epsilon_z/\epsilon_x = \langle H_z \rangle/\langle H_x \rangle$, the coupling coefficient can be so large that the contribution of the vertical dispersion to the emittance becomes negligible in the other extreme case $g = (\| G \|/\Delta)^2/(2 + (\| G \|/\Delta)^2)$, where the coupling strength $\| G \|$ is defined in Eq. (4.1), and $\Delta$ is the distance from the resonance. In general, both the vertical dispersion and the linear coupling due to the unavoidable sources of residual imperfections all around the ring has much effect on the vertical emittance. We need to find new approach to evaluate
the emittance ratio in the limit $|G| \ll \Delta$, that is the weak coupling condition.

In section 4.1, we review the betatron coupling motion [8]. In section 4.2, we find that the vertical emittance is determined by two factors: the coupling of the vertical and horizontal betatron oscillation and spurious vertical dispersion generated by the magnet errors. We evaluate explicitly these two processes, the analytical solutions show the two contributions are independent and uncorrelated in the weak coupling condition, so the final vertical emittance will be given by the sum of them. Applications of the derived formulas to the quadruple-bend achromatic low emittance lattice are presented in section 4.3. Simulations show the vertical emittance due to quadrupole roll errors are mainly contributed by the vertical dispersion other than betatron coupling. The summary is given in section 4.4.

4.1 Betatron Coupling motion

Betatron motions are coupled through solenoidal and skew-quadrupole fields. The skew quadrupole field arises from quadrupole rolls, vertical closed-orbit error in sextupoles or horizontal closed-orbit error in skew sextupoles. The solenoidal field exists in electron storage rings, and in high-energy detectors at the interaction point. A three-dimensional field couples the equations of motion in the following way [8]:

$$
\ddot{x} + K_x(\theta)x = -(K + \dot{S})z - 2S\dot{z},
$$

$$
\ddot{z} + K_z(\theta)z = -(K - \dot{S})z + 2S\dot{x},
$$

all the functions depend on the variable $\theta$, that is the angle at the accelerator center, with $s$ as the distance along the beam axis and $\rho$ as the average radius of the accelerator, and

$$
K(\theta) = \frac{\rho^2}{2B\rho} \left[ \frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z} \right], \quad S(\theta) = \frac{\rho}{2B\rho} B_\theta,
$$
are effective solenoid and skew quadrupole strengths. $K_x(\theta) = \rho^2(1/\rho^2 - B_1(\theta)/B \rho)$, $K_z(\theta) = B_1(\theta)\rho^2/B \rho$ are focusing functions. The Hamiltonian $H$ was found by writing the subsequent canonical equations

$$
\dot{x} = -\frac{\partial H}{\partial p_x} = p_x - S z, \\
\dot{p}_x = -\frac{\partial H}{\partial x} = -K_x x - K_z - (p_z - S x) S, \\
\dot{z} = -\frac{\partial H}{\partial p_z} = p_z + S z, \\
\dot{p}_z = -\frac{\partial H}{\partial z} = -K_z z - K x - (p_x - S z) S.
$$

The form of the Hamiltonian $H$ associated can be shown to be

$$
H = \frac{1}{2} [K_x x^2 + K_z z^2 + 2 K_x z + (p_x - S z)^2 + (p_z + S x)^2],
$$

$$
H_0 = \frac{1}{2} [K_x x^2 + K_z z^2 + p_x^2 + p_z^2],
$$

$$
H_1 = [K x z + S x p_z - S z p_x + \frac{1}{2} S^2 (x^2 + z^2)].
$$

Where $H_1$, obtained by subtracting $H_0$ of betatron motion from $H$, is the perturbed Hamiltonian for linear coupling. The solutions of the unperturbed betatron motion are

$$
x = a_1 u e^{i \nu_x \theta} + \bar{a}_1 \bar{u} e^{-i \nu_x \theta},
$$

$$
p_x = a_1 (\dot{u} + i \nu_x u) e^{i \nu_x \theta} + \bar{a}_1 (\dot{\bar{u}} - i \nu_x \bar{u}) e^{-i \nu_x \theta},
$$

$$
z = a_2 \nu x e^{i \nu_x \theta} + \bar{a}_2 \bar{\nu} e^{-i \nu_x \theta},
$$

$$
p_z = a_2 (\dot{\nu} + i \nu_x \nu) e^{i \nu_z \theta} + \bar{a}_2 (\dot{\bar{\nu}} - i \nu_x \bar{\nu}) e^{-i \nu_z \theta},
$$

They are linear combinations of the four constants $a_1, \bar{a}_1, a_2, \bar{a}_1$, and contain oscillatory terms with betatron tunes $\nu_x$ and $\nu_z$. Introducing these general solutions into
4.1 Betatron Coupling motion

the perturbed Hamiltonian $H_1$ for linear coupling, the entire solution for the vector $Y = (x, p_x, z, P_z)$ can be written as follows in presence of linear coupling:

$$Y_j = \sum_{k=1}^{4} \omega_{jk}(\theta) A_k, j = 1, \ldots, 4,$$

$$\omega_{11,13} = \frac{G}{2\omega_{2,1}} \left[ \frac{\beta_x}{2\rho} \right]^{1/2} e^{i(\mu_x + \omega_{2,1} \theta)},$$

$$\omega_{21,23} = \frac{G}{2\omega_{2,1}} \left[ \frac{\rho}{2\beta_x} \right]^{1/2} (i - \alpha_x) e^{i(\mu_x + \omega_{2,1} \theta)},$$

$$\omega_{31,33} = \left[ \frac{\beta_z}{2\rho} \right]^{1/2} e^{i(\mu_z - \omega_{1,2} \theta)},$$

$$\omega_{41,43} = \left[ \frac{\rho}{2\beta_z} \right]^{1/2} (i - \alpha_z) e^{i(\mu_z - \omega_{1,2} \theta)},$$

Where $j$ is numbering the four components of $Y$, $A_k$ is the actual complex constant. And $A_1 = \bar{A}_2$, $A_3 = \bar{A}_4$. $\omega_{j2} = \bar{\omega}_{j1}, \omega_{j4} = \bar{\omega}_{j3}$, $\Delta = \nu_x - \nu_z - l$ is the distance from resonance. $\omega_{1,2} = \frac{1}{2} [\Delta \pm \sqrt{\Delta^2 + |G|^2}]$ are tunes of two normal modes. This means that the betatron tunes are separated by $\lambda = \sqrt{\Delta^2 + |G|^2}$, and the minimum separation between the normal mode tunes is $|G|$, which is the coupling resonance strength:

$$G = \frac{1}{2\pi} \int \sqrt{\beta_x \beta_z} A_{tc} e^{i(\psi_x - \psi_z - \Delta \theta)} ds,$$  \hspace{1cm} (4.1)

$$A_{tc} = K \rho ^2 + \frac{S(\theta)}{\rho} \left( \frac{\alpha_x}{\beta_x} - \frac{\alpha_z}{\beta_z} \right) + j \frac{S(\theta)}{\rho} \left( \frac{1}{\beta_x} + \frac{1}{\beta_z} \right)$$  \hspace{1cm} (4.2)

These solutions are strictly valid in the absence of radiation, but coupled betatron oscillations enhanced by photo emission and damped by the longitudinal acceleration as well as by the average energy loss in the presence of the focusing component of the
magnetic field. The followings show the variations of the constants $A_k$ due to these effects\[8\].

\[
\frac{d|A_k^2|}{dt} = -\alpha_k |A_k^2| + Q_k,
\]

Where $Q_k$ are the transverse beam amplitude coefficients

\[
Q_k = \left\langle 1 \frac{C_q \gamma^2 < P_\gamma >}{2\rho} \left| \frac{[D_x \omega_{2k} - \dot{D}_x \omega_{1k} + D_z \omega_{4k} - \dot{D}_z \omega_{3k}]^2}{Im^2(\omega_{1k} \omega_{2k} + \omega_{3k} \omega_{4k})} \right| \right\rangle,
\]

$C_q = 3.83 \times 10^{-13} m$. $\alpha_k$ are the damping coefficients, which are proportional to the damping partition numbers $J_k$ such as $\alpha_k = \frac{(P)}{2E_0} J_k$. $P$ is the instantaneous power radiated by a relativistic electron.

A stationary state will occur after a few damping times and it corresponds to the condition $\frac{d|A_k|^2}{dt} = 0$. Thus the Equilibrium amplitudes are

\[
|A_k^2| = \frac{Q_k}{\alpha_k},
\]

The emittances $\epsilon_{x,z}$ is defined as the invariant mean-square amplitudes of the transverse oscillations, we have

\[
\epsilon_x = \frac{2\langle x^2 \rangle}{\beta_x} = \frac{4}{\beta_x} \langle |A_1|^2 |\omega_{1}^2| + |A_3|^2 |\omega_{3}^2| \rangle [8],
\]

\[
\epsilon_z = \frac{2\langle z^2 \rangle}{\beta_z} = \frac{4}{\beta_z} \langle |A_1|^2 |\omega_{3}^2| + |A_3|^2 |\omega_{3}^2| \rangle [8],
\]

\[
\epsilon_x = C_q \frac{\gamma^2}{J_x \rho} \left[ \frac{4\omega_x^2 |G|^2}{(4\omega_x^2 + |G|^2)^2} \langle H_1 \rangle + \frac{4\omega_z^2 |G|^2}{(4\omega_z^2 + |G|^2)^2} \langle H_3 \rangle \right] [8] \tag{4.3}
\]

\[
\epsilon_z = C_q \frac{\gamma^2}{J_z \rho} \left[ \frac{16\omega_x^4 |G|^2}{(4\omega_x^2 + |G|^2)^2} \langle H_1 \rangle + \frac{16\omega_z^4 |G|^2}{(4\omega_z^2 + |G|^2)^2} \langle H_3 \rangle \right] [8] \tag{4.4}
\]

Here the $H$ functions are defined as

\[
H_1 = \frac{|G|^2}{\omega_x^2} \frac{1}{4} (\eta_x^2 + \zeta_x^2) + (\eta_z^2 + \zeta_z^2) + \frac{|G|}{\omega_x} Re \left[ (\eta_x + i\zeta_x)(\eta_z - i\zeta_z)e^{i\omega t} \right]
\]

\[
H_3 = \frac{|G|^2}{\omega_z^2} \frac{1}{4} (\eta_x^2 + \zeta_x^2) + (\eta_z^2 + \zeta_z^2) + \frac{|G|}{\omega_1} Re \left[ (\eta_x + i\zeta_x)(\eta_z - i\zeta_z)e^{i\omega t} \right]
\]
\[ \eta_x = \frac{D_x}{\sqrt{\beta_x}} \]

\[ \zeta_x = \frac{1}{\rho \sqrt{\beta_x}} (\beta_x \dot{D}_x + \rho \alpha_x D_x) = \frac{1}{\sqrt{\beta_x}} (\beta_x D'_x + \alpha_x D_x) \]

\[ \eta_z = \frac{D_z}{\sqrt{\beta_z}} \]

\[ \zeta_z = \frac{1}{\rho \sqrt{\beta_z}} (\beta_z \dot{D}_z + \rho \alpha_z D_z) = \frac{1}{\sqrt{\beta_z}} (\beta_z D'_z + \alpha_z D_z) \]

### 4.2 Equilibrium Emittance Approximation

The equilibrium emittances in Eq. 4.3 4.4 appear to be fairly complicated. Consider three cases,

(a) For vanishing linear coupling \((G \rightarrow 0)\), the two terms in the square brackets of Eq. 4.3 are equal to zeros, the two remaining terms become

\[
\frac{4 \omega_2^2 \left| G \right|^2}{(4 \omega_2^2 + \left| G \right|^2)^2} \langle \mathcal{H}_1 \rangle = \eta_x^2 + \zeta_x^2 = \langle \mathcal{H}_x \rangle
\]

\[
\frac{16 \omega_1^4}{(4 \omega_1^2 + \left| G \right|^2)^2} \langle \mathcal{H}_3 \rangle = \eta_z^2 + \zeta_z^2 = \langle \mathcal{H}_z \rangle
\]

In this condition, the transverse equilibrium emittances are \(\epsilon_x = \frac{\epsilon_x}{\epsilon_z} = \frac{\eta_x}{\eta_z}\). Their ratio \(g = \frac{\epsilon_x}{\epsilon_z}\) is given by \(g = \frac{\eta_x}{\eta_z}\). As expected, the vertical emittance is equal to zero if the vertical dispersion vanishes in addition to the coupling strength.

(b) For vanishing vertical dispersion \((G \neq 0)\), the equilibrium emittance become equivalent to

\[
\langle \mathcal{H}_{1,3} \rangle = \frac{\left| G \right|^2}{4 \omega_{2,1}^2} \langle \eta_x^2 + \zeta_x^2 \rangle = \frac{\left| G \right|^2}{\omega_{2,1}^2} \langle \mathcal{H}_x \rangle
\]
Introducing Eq. 4.5 into Eq. 4.4, the emittance ratio $g$ is given by

$$g = (|G|/\Delta)^2/[2 + (|G|/\Delta)^2].$$

Consider the limit $|G| \gg \Delta$, the transverse emittances are equal. And the vertical emittance takes half the value of the horizontal emittance at $|G| = 0$.

(c) For the weak coupling, coupling and vertical dispersion are not vanishing, so that not only $H_x, H_z$ are contributing, but also the products $D_x D_z, \dot{D}_x D_z$, $D_x \dot{D}_z$, and $\dot{D}_x D_z$. When in the weak coupling condition, that means the coupling strength $|G| \ll \Delta$. Here $\Delta = Q_x - Q_z - l$ is the separation of betatron tunes. Employing the Taylor Expansion, the emittances are written as:

$$\epsilon_x = C_q \gamma^2 J_x \rho \left[ \langle H_x \rangle + \frac{|G|^2}{2\Delta^2} \langle H_z \rangle + \frac{|G|}{\Delta} \text{Re} \left[ (\xi_x + i\xi_z)(\eta_x - i\eta_z) e^{i\phi} \right] \right]$$

$$\epsilon_z = C_q \gamma^2 J_z \rho \left[ \langle H_z \rangle + \frac{|G|^2}{2\Delta^2} \langle H_x \rangle + \frac{|G|}{\Delta} \text{Re} \left[ (\xi_x + i\xi_z)(\eta_x - i\eta_z) e^{i\phi} \right] \right]$$

So the emittances consist of the three terms, the first term is contributed by the coupling of the vertical and horizontal betatron oscillation, the second term is contributed by the vertical dispersion generated by the magnet errors, the last one represents the correlation between the betatron coupling and the vertical dispersion. Simulations show the last term has less impact on the vertical emittance. Neglecting the correlated term make it possible to write the emittance as

$$\epsilon_x = C_q \gamma^2 J_x \rho \left[ \langle H_x \rangle + \frac{|G|^2}{2\Delta^2} \langle H_z \rangle \right] = C_q \gamma^2 \frac{I_5}{I_2 - I_4} \frac{G^2}{2\Delta^2} \frac{I_2}{I_2 - I_4}$$

$$\epsilon_z = C_q \gamma^2 J_z \rho \left[ \langle H_z \rangle + \frac{|G|^2}{2\Delta^2} \langle H_x \rangle \right] = C_q \gamma^2 \frac{I_5}{I_2} \frac{G^2}{2\Delta^2} \frac{I_2}{I_2 - I_4}$$
In order to simplify the following calculations, let us now assume that the
damping partition numbers are equal to 1, that means that the radius of cur-
vature $\rho$ is large and there is no gradient in the dipoles. Then the emittance
ratio is defined as

$$ g = \frac{\epsilon_z}{\epsilon_x} $$

$$ = \frac{\langle H_z \rangle}{\langle H_x \rangle} + \frac{|G|^2}{2\Delta^2} $$

$$ = \frac{I_{5z}}{I_5} + \frac{|G|^2}{2\Delta^2} $$

Where $I_5, I_{5z}, I_2$ and $I_4$ are radiation integrals.

$$ I_5 = \int \frac{H_x}{|\rho|^3} ds $$

$$ I_{5z} = \int \frac{H_z}{|\rho|^3} ds $$

$$ I_2 = \int \frac{1}{\rho^2} ds $$

$$ I_4 = \int \left( \frac{D}{\rho} \right) \left( \frac{1}{\rho^2} + 2K_1 \right) ds $$

$$ K_1 = (1/B\rho)(\partial B_z/\partial x) $$

### 4.3 Applications of Equilibrium Emittance Approximation

The beam phase-space areas (emittances) of an electron storage ring arise from the
equilibrium between quantum fluctuation and radiation damping. For a planer stor-
age ring, the vertical beam emittance arises essentially from the linear betatron cou-
pling, where the vertical betatron motion is coupled to the horizontal betatron motion,
and the residual vertical dispersion function. The vertical emittance in the ring is determined primarily by the betatron coupling induced by alignment errors and the vertical dispersion. In particular, vertical dipole errors and a non-zero vertical orbit in the quadrupoles will directly generate vertical dispersion. And a non-zero vertical closed orbit in the sextupoles, vertical sextupole misalignments will only couple the horizontal and vertical betatron motion. Furthermore, not only will quadrupole rotations introduce vertical dispersion, but also couple the x and z planes. Based on a Hamiltonian formalism, Guignard has developed an analytical treatment on the coupled betatron motions and its impact on the vertical emittance [8]. In the weak coupling approximation with $|G| \ll |\Delta|$, where $\Delta = \nu_x - \nu_z - l$ is the resonance proximity parameter, $\nu_x$ and $\nu_z$ are the betatron tunes, $l$ is the integer part of the difference between the betatron tunes, $G = |G| \exp(i\varphi)$ is the linear coupling resonance strength:

$$G = \frac{1}{2\pi B \rho} \int \frac{\partial B_x}{\partial x} \sqrt{\beta_x \beta_z} \exp\left[i(\psi_x - \psi_z - \Delta \theta)\right] ds,$$

(4.5)

where $\psi_x$ and $\psi_z$ are the betatron phase advances, and $\theta = s/\rho$ is the orbiting angle. Employing the Taylor expansion, we obtain the vertical emittance:

$$\epsilon_z = \epsilon_d + \epsilon_c + \epsilon_{dc},$$

(4.6)

$$\epsilon_d = \frac{C_q \gamma^2}{\mathcal{J}_z \rho} \langle \mathcal{H}_z \rangle,$$

$$\epsilon_c = \frac{C_q \gamma^2 |G|^2}{\mathcal{J}_x \rho} \frac{2\Delta^2}{\langle \mathcal{H}_x \rangle},$$

$$\epsilon_{dc} = \frac{C_q \gamma^2 |G|}{\mathcal{J}_x \rho} \frac{|\Delta|}{\mathcal{J}_s} \text{Re}[(\eta_x + i\zeta_x)(\eta_z - i\zeta_z)e^{i\phi}].$$

where $C_q = 3.83 \times 10^{-13}$ m, $\gamma$ is the Lorentz factor, $\rho$ is the dipole bending radius, $\mathcal{J}_x$ and $\mathcal{J}_z$ are the horizontal and vertical damping partition numbers, $\phi = \psi_x - \psi_z -$
4.3 Applications of Equilibrium Emittance Approximation

\( \Delta \cdot \theta + \varphi \), and the normalized dispersion functions are

\[
\eta_x = \frac{D_x}{\sqrt{\beta_x}}, \quad \zeta_x = \frac{1}{\sqrt{\beta_x}}(\beta_x D'_x + \alpha_x D_x),
\]
\[
\eta_z = \frac{D_z}{\sqrt{\beta_z}}, \quad \zeta_z = \frac{1}{\sqrt{\beta_z}}(\beta_z D'_z + \alpha_z D_z),
\]
and the \( \mathcal{H} \)-functions are \( \mathcal{H}_x = \eta_x^2 + \zeta_x^2 \) and \( \mathcal{H}_z = \eta_z^2 + \zeta_z^2 \).

The equilibrium vertical emittance \( \epsilon_z \) consists of the three terms: the first term \( \epsilon_d \) arising from the vertical dispersion, the second term \( \epsilon_c \) of the coupling between the vertical and horizontal betatron oscillations, and the correlation between the horizontal and vertical dispersion functions \( \epsilon_{dc} \). We carry out systematic numerical simulations with many random seeds to study the contributions of these three terms, and find that the correlation term between the horizontal and vertical dispersion functions is small (see Fig. 4.1 below). Neglecting the correlation term, the vertical emittance is

\[
\epsilon_z = C_q \frac{\gamma^2}{J_z \rho} \left[ \langle \mathcal{H}_z \rangle + \frac{|G|^2 J_z}{2 \Delta^2 J_x} \langle \mathcal{H}_x \rangle \right].
\]

Here the equilibrium emittances are determined by the coupling of the vertical and horizontal betatron oscillations and the spurious vertical dispersion function generated by the magnet errors. These two contributions are independent and uncorrelated in the weak coupling condition, and the final emittance is the sum of them. Since the horizontal and vertical damping partition numbers are approximately equal to 1, the equilibrium emittance ratio is

\[
\frac{\epsilon_z}{\epsilon_x} = \frac{\langle \mathcal{H}_z \rangle}{\langle \mathcal{H}_x \rangle} + \frac{|G|^2}{2 \Delta^2}.
\]

Among the expected sources of vertical emittance in accelerator lattices, the random rolls of quadrupoles and dipoles are most important. Quadrupole rolls couple the horizontal \( x \) and the vertical \( z \) planes. Figure 4.1 shows the three vertical emittance contributions generated by all quadrupoles with random rolls with uniform distribution vs the coupling strength \( G \) for a low emittance lattice [9].
Figure 4.1: The dispersion contribution $\epsilon_d$, the coupling contribution $\epsilon_c$ and the correlation $\epsilon_{dc}$ for a low emittance QBA lattice [9] are plotted vs the linear coupling strength $G$. The linear coupling strength is varied by random roll of all quadrupoles in uniform distribution. Note that the correlation term is negligible. The analytical solution $\epsilon$ agrees well with that obtained in the MAD modeling for $|G| \leq 0.05$, where the betatron functions are not strongly perturbed.
Notice that the contribution from correlation term $\epsilon_{dc}$ is small. The vertical emittance is dominated by the vertical dispersion $\epsilon_d$ and the betatron coupling $\epsilon_c$. In the full coupling condition, the theory developed above allows one to estimate the emittance ratio, which can be reached for different amplitudes of the vertical dispersion. The analytical solution $\epsilon_z = \epsilon_d + \epsilon_c + \epsilon_{dc}$ in Fig. 4.1 agrees very well with the MAD modeling based on the QBA lattice. The analytical result is a good approximation up to the linear betatron-coupling coefficient $G$ of 0.05.

The ratio $\epsilon_d/\epsilon_c$ is shown in Fig. 4.2, where the error bar shows the variation of 10 random seeds used in the calculation. We note that $\epsilon_d$ is roughly four times as large as $\epsilon_c$ for the QBA lattice. The emittance ratio $\epsilon_d/\epsilon_c$ significantly depends on lattice structure and betatron tunes shown in the Fig. 4.3. For the DBA lattice, the vertical dispersion and betatron coupling offer the same contributions to the vertical emittance. When the parameter $G$ is small, the ratio is more sensitive to the dispersion function stopband width to be discussed in the next chapter.

### 4.4 Summary

In general, both of the vertical dispersion and the linear coupling due to the unavoidable sources of residual imperfections determine the vertical emittance. The analytical treatment of the coupled motions based on a Hamiltonian formalism was developed to help in understanding the physics of the mechanisms involved in the betatron coupling and related impact of radiation. We evaluated the emittance in the limit of the difference coupling strength $| G | \ll \Delta$, that is the weak coupling
Figure 4.2: The vertical emittance arises essentially from the vertical dispersion $\epsilon_d$ and the betatron coupling $\epsilon_c$. Our numerical simulations for a QBA lattice show that their ratio $\epsilon_d/\epsilon_c$ is about four.
Figure 4.3: Emittance ratio comparison in the QBA, the DBA, the low emittance lattice, and SPEAR3.
condition. The equilibrium emittances are

\[
\varepsilon_x = C_q \frac{\gamma^2}{J_x} \left[ \langle H_x \rangle + \frac{|G|^2}{2\Delta^2} \langle H_z \rangle \right] = \frac{C_q \gamma^2 I_5}{I_2 - I_4} + \frac{|G|^2 C_q \gamma^2 I_{5z}}{2\Delta^2 (I_2 - I_4)}
\]

\[
\varepsilon_z = C_q \frac{\gamma^2}{J_z} \left[ \langle H_z \rangle + \frac{|G|^2}{2\Delta^2} \langle H_x \rangle \right] = \frac{C_q \gamma^2 I_{5z}}{I_2} + \frac{|G|^2 C_q \gamma^2 I_5}{2\Delta^2 I_2}
\]

So the vertical equilibrium emittance \( \varepsilon_z \) consists of the three terms, the first term \( \varepsilon_d \) is contributed by the vertical dispersion generated by the magnet errors, the second term \( \varepsilon_c \) is contributed by the coupling of the vertical and horizontal betatron oscillation, the last contribution \( \varepsilon_{dc} \) represents the correlation between the betatron coupling and the vertical dispersion. Simulations show the last term has little impact on the vertical emittance, and that \( \varepsilon_d \) is roughly four times as large as \( \varepsilon_c \) for the QBA lattice.
Chapter 5

Vertical Emittance Correction

In recent years, many low emittance electron storage rings around 3 GeV beam energy have been proposed and constructed in the world. These storage rings are intended to produce high brilliance X-ray. Most of these low emittance lattices were designed based on the double-bend achromatic (DBA) cells, which is composed of two dipoles plus 5 to 6 pairs of matching quadrupoles to minimize the horizontal beam emittance. However, the achromatic condition is usually relaxed to further reduce the beam emittance by a factor of 3. Unfortunately, high field insertion devices (IDs) located in a dispersive straight section can spoil the beam emittances. Recently, a quadruple-bend achromatic (QBA) lattice concept has been proposed [9] to alleviate this difficulty. A QBA-cell is a supercell made of 2 DBA cells, where 2 long (inner) and 2 short (outer) dipoles with matching quadrupoles are used to minimize the beam emittance [6]. For the same number of dipoles, quadrupoles and straight sections, the emittance of a QBA-lattice is a factor of 2 smaller than that of a DBA-lattice. Although the non-achromatic DB-lattice provides a slightly smaller emittance than that of the corresponding QBA-lattice, the effective emittances of the bare DBA and QBA lattices are about equal. When strong-field IDs are installed in the storage ring,
the QBA lattice becomes superior than the non-achromatic DB lattice, particularly for beam energies lower than 3-4 GeV.

The photon brilliance of synchrotron light source is inversely proportional to the product of the horizontal and vertical emittances. The vertical beam emittance in a storage ring arises essentially from the linear betatron coupling and the quantum fluctuation due to the vertical dispersion function as derived in Ref. [8]. So far, there is no systematic analysis and correction methods on the vertical emittance.

This chapter is intended to study the main source of the vertical emittance and explore effective correction methods. Effective correction depends on precise beam measurements. Recently, the independent component analysis (ICA) [4] has been developed for beam measurements. We study the effectiveness of employing the ICA method to measure the vertical dispersion function and examine the effectiveness of the vertical emittance correction.

In order to minimize the vertical emittance, we need precise measurement and correction of betatron coupling and vertical dispersion function. Data analysis of turn-by-turn data at many beam position monitors (BPMs) will provide independent modes of particle motion [4]. The ICA method can practically sample turn-by-turn data of beam motion at all BPMs in minutes, i.e. the measurement time is short. Analyzing the BPM data in independent modes, one can separate the betatron modes from the synchrotron modes, where the synchrotron mode can provide accurate measurement of the dispersion function. The resonance strengths (stopband widths) of major resonances can be derived from the beam measurement. We will explore the stopband correction with several families of skew quadrupole correctors. This method can correct the vertical dispersion function without identifying the source of errors. Since the vertical dispersion function is essentially generated by random errors, it would be very difficult to isolate each of these errors. Fortunately, we can effectively correct the vertical dispersion function with finite families of skew quadrupoles.
Since the QBA lattice has an advantage over the non-achromatic double-bend lattice in having zero-dispersion straight sections and has an emittance smaller than that of a double-bend achromatic lattice, we demonstrate our correction method with a QBA lattice. Furthermore, the QBA has another advantage over the double bend non-achromatic lattice that the correction of the linear betatron coupling will not affect the vertical dispersion function.

This chapter is intended to study the applicability of the stopband correction schemes in storage rings. We organize this chapter as follows. In section 5.1, we examine the vertical dispersion function due to skew quadrupole errors, and discuss its measurement with the ICA method. In section 5.2, we discuss the stopbands for the linear betatron coupling and the vertical dispersion function. In section 5.3, we apply the stopband correction system for both the linear coupling and the vertical dispersion function correction. The summary is given in section 5.4.

Since the vertical dispersion function plays a major role in the vertical emittance for synchrotron light source and damping rings of linear colliders, it is important to be able to correct the spurious vertical dispersion. This allows one to maximize the beam brilliance.

### 5.1 Vertical dispersion due to residual imperfections

Since $\epsilon_d > \epsilon_c$, we need to examine some possible methods of vertical dispersion correction. The equation of motion for the vertical betatron motion is [6]

$$z'' + \frac{K_z(s)}{1 + \delta} z = -\frac{\Delta B_x}{B\rho(1 + \delta)},$$

(5.1)
where \( \delta = (p - p_0)/p_0 \) is the fractional off-momentum deviation, \( K_z(s) \) is the focusing function, \( B \rho \) is the momentum-rigidity of the on-momentum particle, and

\[
\frac{\Delta B_z}{B \rho} = \frac{1}{\rho} (a_0 + b_1 z + a_1 x + 2b_2 xz + \cdots).
\]

Here \( a_0 \) arises essentially from the dipole roll, \( b_1 \) is the gradient error, \( a_1 \) is the skew quadrupole field, and \( b_2 \) is the sextupole field. Substituting \( x = x_{co} + D_x \delta + x_\beta \), we obtain

\[
z'' + \frac{\tilde{K}_z(s)}{1 + \delta} z = -\frac{1}{(1 + \delta) \rho} \left[ a_0 + a_1 (x_{co} + x_\beta + D_x \delta) + 2b_2 (x_\beta + D_x \delta) z \right], \tag{5.2}
\]

where the effective focusing function is \( \tilde{K}_z(s) = K_z(s) + b_1/\rho + 2b_2 x_{co} \). Expanding the vertical coordinate in \( z = z_{co} + D_z \delta + z_\beta \), we find

\[
\begin{align*}
\dot{z}_{co}'' + \tilde{K}_z(s) z_{co} &= -\frac{a_0}{\rho} - \frac{a_1}{\rho} x_{co} \\
\dot{z}_\beta'' + \tilde{K}_z(s) z_\beta &= -\frac{a_1}{\rho} x_{co} x_\beta - \frac{2b_2}{\rho} x_\beta z_\beta \\
D_z'' + \tilde{K}_z(s) D_z &= h_z(s), \tag{5.3}
\end{align*}
\]

where

\[
h_z(s) = +\tilde{K}_z(s) z_{co} + \frac{a_0 + a_1 x_{co}}{\rho} - \frac{a_1 + 2b_2 z_{co}}{\rho} D_x(s). \tag{5.4}
\]

The vertical emittance in an accelerator is primarily determined by vertical dispersion function, which is generated by the inhomogeneous term \( h_z(s) \) in Eq. (5.3). Major sources of \( h_z \) are vertical closed orbit error in quadrupoles \( \tilde{K}_z z_{co} \), dipole roll \( a_0/\rho \), and skew quadrupole field error in locations with nonzero horizontal dispersion function. The skew quadrupole field produced by the quadrupole roll can produce both linear betatron coupling and vertical dispersion. In fact, the horizontal and vertical closed orbit, betatron functions, and dispersion functions are all coupled. When coupling is small and the horizontal betatron tune is sufficiently far away from an
integer, the normalized vertical dispersion functions are
\[
\frac{D_z(s)}{\sqrt{\beta_z(s)}} = \frac{1}{2 \sin \pi \nu_z} \int_s^{s+C} \sqrt{\beta_z(t)} h_z(s) \cos(\pi \nu_z + \psi_z(s) - \psi_z(t)) dt, \quad (5.5)
\]
\[
\frac{\alpha_z}{\sqrt{\beta_z}} D_z + \sqrt{\beta_z} D_z' = -\frac{1}{2 \sin \pi \nu_z} \int_s^{s+C} \sqrt{\beta_z(t)} h_z(s) \sin(\pi \nu_z + \psi_z(s) - \psi_z(t)) dt, \quad (5.6)
\]
where \( \nu_z, \beta_z, \alpha_z = -\beta_z'/2, \psi_z \) are the “unperturbed” vertical betatron tune, the vertical betatron functions, and the vertical betatron phase function respectively. The dispersion functions are obtained by closed orbit integrals.

5.1.1 The ICA measurement method

The ICA measurement method has recently been introduced to analyze beam motion in accelerators [4]. Using numerical algorithms to process the turn-by-turn data at many beam position monitor (BPM) positions, one can filter out white noise and attain independent components of the beam motion. Possible independent components of beam motion are the coupled betatron motion, synchrotron oscillations, and other dynamical noise sources. This method has been successfully applied to a rapid cycling accelerator at the Fermilab Booster [4].

If we assume that the transfer function of BPM system is linear, the data sampled by BPMs can be considered as a linear combination of source signals. The physical origin of ICA modes can be identified by their spatial and temporal functions. The ICA method is essentially a time-correlation based method, which is efficient in isolating the source signals without overlapping power spectra, which often hold because the harmonic oscillation with different tunes. The synchrotron mode identified by the synchrotron tune obtained from the fast Fourier transform (FFT) on the temporal function enables us to obtain the vertical dispersion function.
5.1.2 Application of ICA to dispersion function measurement

Using particle tracking data of 1024-turn, we find good agreement between the ICA measurements and simulations as shown in Fig. 5.1, where the red dots are measured vertical dispersion function at BPM locations and the blue curve is the vertical dispersion obtained from MAD modeling. The dispersion function derived from the ICA method agrees with the vertical dispersion of MAD modeling to an rms error of $\sigma_D/D \approx 0.22\%$.

![Figure 5.1: The measured dispersion function is compared with the MAD modeling based on a QBA low emittance lattice.](image)

In reality BPM readings always contain random noise which limits the precision of the vertical dispersion measurement by the ICA method. We insert white Gaussian noise into data sampled by BPMs. The rms errors of the resulting vertical dispersion function are estimated as shown in Fig. 5.2.
5.2 The integer stopband Integrals

As an example, we consider the effect of skew quadrupoles on the vertical dispersion in Eq. (5.3):

\[ D_z'' + K_z(s)D_z = \frac{1}{B\rho} \frac{\partial B_x}{\partial x} D_x, \]  

(5.7)

where \( \frac{\partial B_x}{\partial x} \) is the skew quadrupole field strength, and \( D_x \) is the horizontal dispersion function at the skew quadrupole location. Performing the Floquet transformation to Eq. (5.7), with

\[ \eta_z = \frac{D_z}{\sqrt{\beta_z}}, \quad \phi = \frac{1}{\nu_z} \int_0^s \frac{ds}{\beta}, \]  

(5.8)

the equation for the normalized co-ordinates becomes

\[ \frac{d^2 \eta_z}{d^2 \phi^2} + \nu_z^2 \eta_z = \nu_z^2 \beta_z^{3/2} \frac{1}{B\rho} \frac{\partial B_x}{\partial x} D_x, \]  

(5.9)

Figure 5.2: Estimation of errors of ICA methods with various random noise levels. The estimation at each noise level \( \sigma_{noise} \) of BPMs is made by repeating the measurement of vertical dispersion 20 times with Gaussian random noise added to each BPM.
where $\beta_z$ is the vertical betatron function at the skew quadrupole location. Since the driving function in the right hand side of Eq. (5.9) is a periodic function of $2\pi$ in $\phi$, we expand it in a Fourier series:

$$ F(\phi) = \nu_z \beta_z^{3/2} \frac{1}{B\rho} \frac{\partial B_x}{\partial x} D_x = \sum_{k=-\infty}^{\infty} f_k e^{ik\phi}. \quad (5.10) $$

For the skew quadrupole arisen from quadrupole roll, $\frac{1}{B\rho} \frac{\partial B_x}{\partial x} = K_1 \sin(2\theta_{roll})$, where $K_1$ is the quadrupole strength, and $\theta_{roll}$ is the roll angle of the quadrupole. The Fourier amplitude $f_k$ is the integer stopband integral given by

$$ f_k = \frac{1}{2\pi} \int \nu_z \beta_z^{3/2}  D_x \frac{1}{B\rho} \frac{\partial B_x}{\partial x} e^{-ik\phi} d\phi = \frac{1}{2\pi} \int \sqrt{\beta_z} D_x \frac{1}{B\rho} \frac{\partial B_x}{\partial x} e^{-ik\phi} ds. \quad (5.11) $$

With the Fourier expansion, the vertical dispersion function becomes

$$ D_z(s) = \nu_z \sqrt{\beta_z(s)} \sum_{k=-\infty}^{\infty} \frac{f_k e^{ik\phi}}{\nu_z^2 - k^2} = 2\nu_z \sqrt{\beta_z(s)} \sum_{k=0}^{\infty} \frac{|f_k| \cos(k\phi + \xi_k)}{\nu_z^2 - k^2}. \quad (5.12) $$

The vertical dispersion is most sensitive to the harmonics close to the vertical betatron tune. The resulting vertical dispersion is usually dominated by a few harmonics near $[\nu_z]$, which is an integer nearest to the betatron tune. For example, the normalized vertical dispersions are generated by one quadrupole roll error shown in Fig. 5.3. Here the rotation angle is 1 mrad. And the dispersions are dominated by the harmonics close to the vertical betatron tune (11.3) shown in Fig. 5.4.

We carried out systematic stopband analysis for the low emittance QBA lattice [9] by making quadrupole rolls of different quadrupoles. Figure 5.5 compares the stopband integrals derived from the Fourier analysis of the vertical dispersion function with those of Eq. (5.11).

In general, the vertical dispersion function can be produced by any horizontal dipole-field error $h_z(s)$ in Eq. (5.3). We can carry out Floquet transformation by changing $s$ to $\phi = \frac{1}{\nu_z} \int_0^{s} \frac{1}{\beta_z} ds$, and define the Fourier harmonics $f_k$ of the perturbation
5.2 The integer stopband Integrals

Figure 5.3: The vertical dispersion is generated by one quadrupole roll error (red star) based on the low emittance QBA lattice[9].

Figure 5.4: The vertical dispersion is most sensitive to the error harmonics close to the vertical betatron tune(11.3).
Figure 5.5: Comparison of $A_k(\nu_z^2 - k^2)/\nu_z$, where $A_k$ is the $k$th Fourier harmonics of the dispersion function, for the QBA lattice [9], with $f_k$ of Eq. (5.11). The vertical dispersion functions are generated by 1 mrad quadrupole roll to different quadrupoles in the QBA lattice.

as

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{\beta_z} h_z(s) e^{-ik\phi} ds,$$  \hspace{1cm} (5.13)

where $k$ is an integer. We carry out simulations with dipole errors. Figure 5.6 compares the $A_k(\nu_z^2 - k^2)/\nu_z$, where $A_k$ is the Fourier harmonic of the vertical dispersion function, with $f_k$ of Eq. (5.13) for a 1 mrad dipole roll for the QBA lattice [9].

5.3 Correction Scheme

For synchrotron light source and for damping rings of linear colliders it is important to be able to minimize the vertical emittance and to correct the spurious vertical
5.3 Correction Scheme

Figure 5.6: Comparison of $A_k(\nu_z^2 - k^2)/\nu_z$ derived from the vertical dispersion function with the Fourier harmonics $f_k$ of Eq. (5.13). The vertical dispersion function is generated by a 1 mrad dipole roll to different dipoles in the QBA lattice [9].

dispersion. This allows one to maximize the luminosity. Figures 5.5 and 5.6 show that the dispersion function can be expanded in terms of major stopband integrals. An effective used method to make correction is to analysis the harmonics of the vertical dispersion and to eliminate the largest components of the stopband integral with harmonics near the vertical betatron tune. The coupling strength of the difference resonance will be corrected simultaneously without altering the machine tunes or introducing any unwanted harmonics.

To provide a concrete example for the stopband correction, we study a QBA lattice with a circumference of 486 m and 12 QBA cells for 3 GeV beam energy [9]. There are ten quadrupole families with reflection symmetry in a superperiod. The betatron tunes are $\nu_x = 26.25$ and $\nu_z = 11.37$, and the natural chromaticities are -64 and -30.
Figure 5.7 shows the optical functions. The emittance of this lattice is 2.7 nm rad. The effective emittance of the QBA lattice is about a factor of 2 smaller than that of the corresponding DBA lattice, while keeping some zero-dispersion sections for high field damping wigglers. High field wavelength shifters located in the dispersive free straight section can further decrease the beam emittance.

![Optical functions for a section of a quadruple-bend achromatic lattice](image)

**Figure 5.7:** Optical functions for a section of a quadruple-bend achromatic lattice [9].

### 5.3.1 The Correction Requirement Estimation

We would like to know typical size of the stopband integrals. Using the QBA lattice [9] example with random rolls in quadrupoles and dipoles at 1 mrad amplitude in uniform distribution, the typical stopband integrals are listed in Table 5.1. In practice, the perturbing field errors, due mainly to random rotation errors in the dipole magnets
and the quadrupoles, are not known a priori. Figure 5.10 shows the vertical dispersion generated by all quadrupoles, where the tilt angles vary between -1 mrad and 1 mrad with a uniform random distribution.

Using the ICA analysis for the BPM turn-by-turn data, we can estimate the stopband integrals by measuring the vertical dispersion. Similarly, we can measure the linear coupling resonance strength by measuring the minimum split between two betatron tunes. The resonance strengths for the vertical dispersion functions for random quadrupole and dipole rolls of ±1 mrad in uniform distribution are given in Table 5.1. The stopband strengths depend on the distribution of the skew quadrupole errors.

The vertical dispersion function produced by the perturbing horizontal magnetic field has a few dominant harmonics near the vertical betatron tune of 11.37. Figure 5.11 shows the amplitudes of Fourier harmonics of the vertical dispersion function. We note that the dominant harmonics are 11 and 12. Using four families of skew quadrupoles, we can compensate two harmonics of the vertical dispersion function.

### 5.3.2 Magnets used

Based on what we have discussed, the vertical emittance correction can be accomplished by correction to the linear betatron coupling and the vertical dispersion func-
5.3 Correction Scheme

Figure 5.8: Coupling correctors. The phase-advance differences (left) between the horizontal and vertical planes are about 80.5°. The two families of skew quadrupoles are placed near high beta points in the non-dispersion section of the QBA cells.

We would like to demonstrate the correction scheme by using 6 families of skew quadrupoles: a total of 38 skew quadrupoles distributed around the QBA ring.

For the purpose of coupling corrections, the 6 skew quadrupoles in the non-dispersion section of the QBA cells have been arranged to minimize the effect of synchrotron radiation. There are two families of skew quadrupoles, each with 3 units. Those two set steering magnets which are apart by 80.5° in the phase-advance difference between the horizontal and vertical planes are used to only make coupling correction shown in Fig. 5.8.

The stopband integrals with harmonic 11 and 12 are corrected using four families of the 32 skew quadrupole correctors. These skew quadrupoles are placed at straight sections in the dispersive region of QBA cells. We can attain a phase difference of nearly 90° on the sides of the triplet and doublet quadrupole sections in Fig 5.9.
5.3 Correction Scheme

Figure 5.9: Harmonic Correction Scheme
5.3 Correction Scheme

5.3.3 Excitation Scheme

Two different schemes are chosen for emittance correction in the ring. They are discussed in the following paragraphs. Define $\Delta = \frac{2\pi l}{\theta} = K_1 L/(2\pi)$, where $K_1$ is the strength of the skew quadrupole. $\theta = \pi/4$ is the rotation angle of the skew quadrupole, and the length $L$ is 0.1 m.

Case 1: The emittance coupling is corrected using the two families of correctors based on six skew quadrupoles only. An algorithm for the optimization is applied. This algorithm is given by the following system of linear equations:

$$
\begin{pmatrix}
ImG \\
ReG
\end{pmatrix} =
\begin{pmatrix}
-4.3596 & -0.8089 \\
-0.6849 & -4.2417
\end{pmatrix}
\begin{pmatrix}
\Delta_1 \\
\Delta_2
\end{pmatrix}
$$

Two set steering magnets with the phase-advance differences between two betatron planes close to an odd integer multiple of $90^\circ$ correct the real and imaginary parts of the coupling strength separately. After the coupling correction, the major contribution to the vertical emittance comes from photon emission due to tilted quadrupoles and so on. All these errors need to be eliminated or minimized in order to achieve a lower vertical emittance of the stored beam.

Case 2: The approach we adopted is to adjust the four families of the skew quadrupole strength to minimize the vertical dispersion produced by changing the tilt angles of quadrupole magnets, as well as to minimize the coupling. The coupling and the vertical dispersion correction are done simultaneously by solving a system of
5.3 Correction Scheme

linear equations.

\[
\begin{pmatrix}
ImG \\
ReG \\
Imf11 \\
Ref11 \\
Imf12 \\
Ref12
\end{pmatrix} =
\begin{pmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_3 \\
\Delta_4 \\
\Delta_5 \\
\Delta_6
\end{pmatrix}
\]

where the \( M \) matrix for the QBA lattice example is

\[
\begin{pmatrix}
-4.3596 & -0.8089 & 1.4399 & 2.6237 & -2.3801 & -0.2796 \\
-0.6849 & -4.2417 & 5.1403 & 4.8504 & -3.5524 & 0.8408 \\
-0.0045 & 0.0028 & -1.1571 & 0.1625 & 0.0046 & -0.0024 \\
-0.0001 & -0.0085 & 0.1629 & 1.1553 & -0.0197 & -0.0003 \\
0.0530 & -0.0443 & -0.0571 & -0.0644 & -9.1407 & -0.0044 \\
0.0234 & 0.0181 & 0.1539 & 0.1607 & 0.6799 & -3.2554
\end{pmatrix}
\]

5.3.4 Simulation results

Some of the results described in the previous sections could be explained in terms of the harmonics of the dispersion. For example, Fig 5.11 shows the harmonics of the vertical dispersion before dispersion correction for the QBA lattices. It can be seen that only those harmonics near 11 are dominant after correction, all of the dominant harmonics were reduced. produces a dispersion having only a few predominant harmonics, so that only a small number of skew quadrupole correctors will be needed for dispersion correction. The analytical and simulated results show the equilibrium emittances are reduced by the factor of 7 in Fig. 5.12.
Figure 5.10: The vertical dispersion generated by all quadrupoles with the uniformly distributed tilt angle 1 mrad before (red solid line) and after (blue dashed line) harmonic correction.

5.4 Summary

Detailed studies of potentially limiting effects in beam dynamics, and the development of high performance diagnostics, motivate efforts to reduce the vertical emittance would require very precise correction of betatron coupling and vertical dispersion. The stopband correction processes involve several skew quadrupole correctors and beam position monitors (BPMs) to control the transverse beam emittance. This method makes corrections to the vertical dispersion without identifying the source of errors, even though it is clear what kinds of magnet errors produce lattice function perturbations. First we apply the ICA analysis to turn-by-turn BPM data from which we can derive accurate vertical dispersion, and the linear coupling strength is measured by changing strength of steering magnets, then determine corrections to compensate for the magnet imperfection by eliminating the main harmonic components of vertical dispersion. The effectiveness of this correction was studied for some
Figure 5.11: Fourier Harmonics of the normalized vertical dispersion function before (red bars) and after (blue bars) stopband correction show that the harmonic correction is effective in reducing the vertical dispersion. The linear betatron coupling correction (green bars) does not alter Fourier harmonics of the vertical dispersion function.

alignment and field errors in the magnets and beam position monitors.

This idea is particularly applicable to the QBA lattice since the QBA lattice has an advantage over the nonachromatic double-bend lattice in having zero-dispersion straight sections and has an emittance much smaller than that of a double-bend achromatic lattice. We showed the coupling strength and the vertical dispersion can be controlled individually. The goal of improving machine performance can be attained by reducing the perturbation of magnet errors globally and not necessarily by making a complete correction at all points around the ring. Resulting improvement in the lattice function and machine performance is that the equilibrium emittances are reduced by the factor of 7.
Figure 5.12: The vertical emittance before (red line) and after (blue line) harmonic correction. The coupling correction (green line) have a little effect on the vertical emittance.

A practical aspect is the time required for collecting measurement data. Since lattice functions are physically derived quantities from BPM turn-by-turn data, the collecting time is determined by the sampling turn number. The ICA method is used to measure optical functions and write results to a file. It takes about one minute to make vertical dispersion function measurements at all BPMs.
Chapter 6

Conclusions

In this dissertation, I have presented the ICA method to analyze beam dynamics without resorting to any particular machine model. The main feature of ICA is a systematic statistical analysis of a BPM-reading matrix. It can easily identify the source signals with synchro-betatron and betatron linear couplings and significantly reduce the effects of BPM random noise. This topic is of fundamental importance to studies which require very precise measurement of optical functions. In addition to the study of ICA measurement, two chapters have been dedicated to develop vertical emittance correction in which the excitation of skew-quadrupole correctors is determined from the ICA measurement of the vertical dispersion.

In Chapter 3, the ICA technique is applied to measure the optical functions of the storage ring. The accuracy of optical functions are sensitive to the cutoff threshold or the number of eigenmodes kept in whitening (the first step of the ICA method). When more than five modes are kept, the betatron modes are contaminated by additional source signals, e.g. synchrotron motion, which decrease accuracy of beta function and phase advance. The improvement in optical function accuracy is achieved by increasing amount of the unequal time-correlation covariance matrices. The effects of
BPM noises and sampling points are studied. It is shown that the accuracy of optical function is less affected by the white noise and the number of sampling points.

In Chapter 4, detailed evaluation of the source of the vertical emittance have been carried out. The analytical and simulated results show the equilibrium emittances are determined by the coupling of the vertical and horizontal betatron oscillation and spurious vertical dispersion generated by the magnet errors. These two contributions are independent and uncorrelated in the weak coupling condition, so the final emittances will be given by the sum of them. The contribution from correlation almost has no effect on the vertical equilibrium emittance. The contribution of spurious vertical dispersion is much larger than that generated by the linear coupling based on the QBA lattice with randomly tilted quadrupoles.

Finally, in Chapter 5, the stopband correction scheme for the linear betatron coupling and the vertical dispersion function is studied. The QBA lattice is used as a concrete example for the demonstration of this correction scheme. Six families of skew quadrupoles can effectively minimize both the vertical dispersion and the linear betatron coupling. In this manner, the vertical emittance is reduced down to 5 pm, corresponding to a vertical to horizontal emittance ratio of less than 0.1%. The direct stopband matrix is calculated from the model of the QBA lattice. Simulations confirmed that the stopband correction system has the strength and flexibility to cope with the expected magnet errors.
Bibliography


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Professional Experience

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Publications

- F. Wang, S.Y. Lee, Vertical emittance stopband correction for the Quadruple-bend achromatic low emittance lattice, was submitted in December 2007.
- X. Pang, F. Wang, X. Wang, S.Y. Lee, K.Y. Ng, Emittance growth scaling laws for crossing systematic space charge 6th order resonances and random octupole driven 4th order resonances, was submitted in March 2008.
