CHARACTERIZATION OF THE PROTON ION SOURCE BEAM FOR THE HIGH INTENSITY NEUTRINO SOURCE AT FERMILAB

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To my parents.
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CHARACTERIZATION OF THE PROTON ION SOURCE BEAM FOR THE HIGH INTENSITY NEUTRINO SOURCE AT FERMILAB

Fermilab is considering an 8 GeV superconducting H\textsuperscript{−} linac with the primary mission of enabling 2 MW beam power from the 120 GeV Fermilab Main Injector for a neutrino program. The High Intensity Neutrino Source (HINS) R&D program is underway to demonstrate the technical feasibility in a 30 MeV prototype linac. The HINS Linac Front-end is composed of an ion source, a radio frequency quadrupole (RFQ), a medium energy beam transport and 16 room temperature Crossbar H-type cavities that accelerate the beam to 10 MeV. The cavities are separated by superconducting solenoids enclosed in individual cryostats. Beyond 10 MeV, the design uses superconducting spoke resonators.

Recently, the HINS proton ion source has been successfully commissioned. It produces a 50 keV, 3 msec pulsed beam with a peak current greater than 20 mA at 2.5 Hz. The beam is transported to the RFQ by a low energy beam transport (LEBT) that consists of two focusing solenoids, four steering dipole magnets and a beam current transformer. To understand beam transmission through the RFQ, it is important to characterize the 50 keV beam before connecting the LEBT to the RFQ. A wire scanner and a Faraday cup are temporarily installed at the exit of the LEBT to study the beam parameters.

All beam studies are based on data taken using the wire scanner. We start with interpreting the signal measured by the wire scanner. Then, we performed a beam-calibration to the steering dipole magnets. We then study transverse motion coupling
due to solenoidal field by measuring beam rotation through solenoid. Analysis to these measurements is accompanied with beam physics modeling and particle tracking simulation. Also, transverse emittance was measured using two different methods and results are compared.

Finally, a bunch shape monitor will be introduced. It is a high bandwidth instrumentation device that measures the longitudinal profile of a bunched proton/H⁻ beam. HINS will use it for its 2.5 MeV beam. Operational principle and commissioning of this monitor are discussed.
Contents

Acceptance ii

Acknowledgments v

Abstract viii

1 Introduction 1

1.1 Project X ......................................................... 2
  1.1.1 ICD-1 ...................................................... 3
  1.1.2 ICD-2 ...................................................... 4

1.2 HINS ............................................................. 8

2 Ion Source and Low Energy Beam Transport 11

2.1 Proton Source ..................................................... 11
  2.1.1 Duo-plasmatron ............................................. 11
  2.1.2 HINS Proton Source ....................................... 13
  2.1.3 High Voltage Supplies .................................... 14

2.2 Low Energy Beam Transport ..................................... 16
  2.2.1 Focusing Solenoids ......................................... 17
  2.2.2 Solenoid Replacement ...................................... 20
  2.2.3 Steering Dipole Magnets .................................. 25
### Contents

2.3 Summary ............................................. 25

3 Beam Characterization for the Proton Ion Source ............................................. 29

3.1 Transverse Beam Profile Measurement ............................................. 30
3.2 Beam Steering by the Dipole Magnets ............................................. 33
  3.2.1 Coupling due to Solenoidal Field ............................................. 33
  3.2.2 Beam Deflection ............................................. 34
3.3 Beam Rotation by Focusing Solenoid ............................................. 39
  3.3.1 Solenoid as a Lens ............................................. 39
  3.3.2 Particle Tracking ............................................. 44
  3.3.3 Beam Rotation by SOL-D ............................................. 47
  3.3.4 Beam Rotation by SOL-PET ............................................. 52
3.4 Tranverse Emittance ............................................. 54
  3.4.1 Measurement using Solenoid Variation Method ............................................. 59
  3.4.2 Measurement using Slit-Wire Scanner Method ............................................. 61
  3.4.3 Result Analysis ............................................. 70
3.5 Summary ............................................. 72

4 Bunch Shape Monitor ............................................. 73

4.1 Principle of Operation ............................................. 75
4.2 Resolution of BSM ............................................. 75
4.3 RF Deflector ............................................. 80
  4.3.1 Resonant Frequency ............................................. 81
  4.3.2 Quality Factor and Shunt Impedance ............................................. 85
4.4 Summary ............................................. 89

5 Conclusion ............................................. 91
CONTENTS

Appendix 93

A Magnetic Scalar Potential and Solenoidal Field 93

B Beam Dynamics of Linearly Coupled Systems 97

C The Kapchinskiy-Vladimirskiy Envelope Equation 101

D Beam Tranverse Emittance 105
   D.1 Phase Space and The Courant-Snyder Invariant 105
   D.2 RMS Emittance 106
   D.3 The Sigma Matrix 108

E Methods for Transverse Emittance Measurement 111
   E.1 Solenoid Variation 111
   E.2 Slit-Wire Scanner Method 113

Bibliography 119
List of Tables

3.1 Beam rotation per solenoid field - SOL-D . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 52
3.2 Beam rotation per solenoid field - SOL-PET . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 54
List of Figures

1.1 A schematic view of the Project X ICD-1 ........................................ 3
1.2 A schematic view of the Project X ICD-2 ........................................ 6

2.1 HINS duo-plasmatron proton source ........................................ 12
2.2 Layout of the HINS proton source and LEBT .............................. 15
2.3 The layout of LEBT-1 ............................................................... 16
2.4 The axial field profile of solenoid SOL-U and SOL-D ................. 18
2.5 Axial field excitation current for SOL-U and SOL-D ..................... 19
2.6 Comparison of the axial field profile of solenoid SOL-D and SOL-PET 21
2.7 Axial field excitation current for SOL-PET .................................. 22
2.8 The layout of LEBT-1 and LEBT-2 with solenoid field profile .......... 23
2.9 TRACK simulation of beam width versus solenoid current ............. 24
2.10 A schematic drawing of the steering dipole ................................ 26
2.11 The axial field profile of the corrector dipole magnets ................... 27

3.1 Transverse beam profile measured by wire scanner ....................... 31
3.2 The time structure of a 100 µs beam pulse ................................ 33
3.3 Beam profile measurement showing x-y coupling ......................... 35
3.4 Beam position response to change in dipole strength ................. 36
3.5 Beam deflection versus corrector dipole strengths ....................... 38
3.6 A lens-like solenoid .................................. 40
3.7 A schematic drawing of the solenoid lens model ................. 43
3.8 Vector field of SOL-D ................................ 45
3.9 Charged particle’s trajectory in solenoidal field ................. 46
3.10 Beam position measurement with different dipole settings .... 49
3.11 Beam rotation observed for different SOL-D settings ........... 50
3.12 Beam rotation by SOL-D: measurement and simulation ....... 51
3.13 Beam position measurement with different dipole settings .... 53
3.14 Beam rotation observed for different SOL-PET settings ....... 55
3.15 Beam rotation by SOL-PET: measurement and simulation .... 56
3.16 Phase space and Courant-Synder invariant .................... 58
3.17 RMS beam size as a function of the strength of SOL-PET ...... 60
3.18 Normalized RMS emittance versus beam current ............... 62
3.19 A schematic drawing of the slit-wire scanner setup ............ 63
3.20 3D plot of the measured beam distribution in $x - x'$ plane .... 64
3.21 3D plot of the measured beam distribution in $y - y'$ plane .... 65
3.22 A comparison of the emittance data before and after background subtraction ........................................ 67
3.23 Density plot in $x - x'$ phase space for different beam current 68
3.24 Density plot in $y - y'$ phase space for different beam current 69
3.25 A comparison of the emittance measured with different methods 71

4.1 Basic idea for operation of BSM ............................ 74
4.2 Electric field of a charged particle in a beam pipe ............... 76
4.3 Effect on BSM resolution due to strength of RF deflector ....... 77
4.4 BSM resolution as a function of length of deflector ............ 78
4.5 BSM resolution as a function of deflecting field ............... 79
LIST OF FIGURES

4.6 A picture of the RF deflector .................................................. 80
4.7 CST MWS model for BSM RF deflector - 1 .............................. 81
4.8 CST MWS model for BSM RF deflector - 2 .............................. 82
4.9 Frequency measurement of RF deflector using network analyzer ... 83
4.10 Resonant frequency versus endcap spacing ............................. 84
4.11 Unloaded quality factor measurement using network analyzer ...... 85
4.12 Quality factor as a function of deflector resonant frequency ...... 87
4.13 BSM deflector shunt impedance as a function of frequency ...... 88

D.1 Phase space and Courant-Synder invariant ............................ 107

E.1 A schematic drawing of the slit-wire scanner setup .................. 114
Chapter 1

Introduction

Fermilab is considering an 8 GeV superconducting $H^-$ linac with the primary mission of enabling 2 MW beam power from the 120 GeV Fermilab Main Injector for a neutrino program. New paradigms introduced into the front-end design include the adoption of short, high field superconducting solenoids as primary lattice focusing elements and a low energy transition at 10 MeV from room temperature to superconducting RF acceleration. The front-end linac in the energy range from 10 MeV to 400 MeV is foreseen as based on 325 MHz superconducting spoke resonators. This is considered as the Initial Conceptual Design (ICD-1) of Project X. The accelerator physics challenge presented by the ICD-1 in serving the long-baseline neutrino program and rare-decay program has motivated consideration of the ICD-2 alternative design, which is based on a 2 GeV Continuous Wave (CW) linac.

The High Intensity Neutrino Source (HINS) R&D program is underway to demonstrate the accelerator concepts specified in the ICD-1 in a 30 MeV prototype linac. The HINS Linac Front-end room temperature section is composed of an ion source, a low energy beam transport (LEBT), a radio frequency quadrupole (RFQ), a medium energy beam transport (MEBT) and 16 room temperature Crossbar H-type (RTCH)
cavities that accelerate the beam to 10 MeV. The RTCH cavities are separated by superconducting solenoids enclosed in individual cryostats. Beyond 10 MeV, the design uses superconducting spoke resonators.

As the design of Project X evolves, the elements in the HINS R&D program modifies to accommodate the needs by Project X. In the following, a briefly introduction to HINS and both the ICD-1 and ICD-2 of Project X is given. The introduction will follow the discussion in [1, 2, 3, 4].

1.1 Project X

Project X is a high intensity proton facility conceived to support a world-leading program in neutrino and flavor physics over the next two decades at Fermilab. The broad research program includes long-baseline neutrino experiments, neutrino interaction experiments, quark flavor Tevatron fixed target experiments, ultra-rare muon and kaon decay experiments and experiments driven by anti-protons from the Fermilab anti-proton complex. The long-baseline neutrino experiments and rare-decay experiments benefit most directly from the high proton beam power afforded by Project-X. This beam power presents simultaneously the promise of extraordinary physics reach and the substantial accelerator physics challenge of generating and handling enormous beam power.

The initial baseline design (ICD-1) of the Project X accelerator complex can generate and handle the beam power required for the long-baseline neutrino program, however the high beam power available at 8 GeV is not readily useable by the rare-decay experiments. Further, the auxiliary accelerator concepts developed to handle and condition 8 GeV beam power for near-term muon experiments does not scale well with increasing beam power and precludes the use of the Debuncher and Accumulator for the anti-proton research program.
1.1 Project X

Figure 1.1: A schematic view of the ICD-1 of Project X.

The accelerator physics challenge presented by the ICD-1 in serving the long-baseline neutrino program and rare-decay program has motivated consideration of the ICD-2 alternative design. The ICD-2 is based on a 2 GeV Continuous Wave (CW) linac. As with the ICD-1, the ICD-2 can readily drive the long-baseline neutrino program. The focus for the design of the ICD-2 is on driving the Project X rare-decay research program. In the following, we will briefly describe the design of ICD-1 and ICD-2.

1.1.1 ICD-1

The ICD-1 of Project X is comprised of an 8 GeV superconducting H- linac, paired with the existing (but modified) Recycler and Main Injector (MI) to provide in excess of 2 MW of beam power throughout the energy range 60 - 120 GeV, simultaneous with greater than 650 kW of beam power at 8 GeV. A schematic view is shown in Fig. 1.1.

The linac operates at 5 Hz with a total of $1.6 \times 10^{14}$ H- ions delivered per pulse.
Total available beam power from the linac at 8 GeV is thus 1.0 MW. The H- ions are stripped at injection into the Recycler in a manner that paints the beam both transversely and longitudinally to reduce space charge forces. Following the 1.25 ms injection, the proton beam is moved off the stripping foil and is transferred in a single turn into the Main Injector. These protons are then accelerated to 120 GeV and fast extracted to a neutrino target. The 120 GeV Main Injector cycle takes 1.4 seconds, producing 2.1 MW of beam power. At lower proton energies Main Injector cycle times can be shorter. Since loading the Recycler requires only one linac beam pulse, the remaining linac cycles (six for a 1.4 sec MI cycle) are available for distribution of 8 GeV protons from the Recycler. Total available 8 GeV beam power can be maintained above 650 kW with Main Injector operations at 2 MW for energies anywhere within the range 50 - 120 GeV.

Modifications to the Recycler Ring to support Project X include integration of an H- injection system, a new RF system, a new extraction system, and measures to mitigate electron cloud effects. The Main Injector would need a new RF system, measures to preserve beam stability through transition, and measures to mitigate electron cloud effects.

The upgrade path to a high intensity beam supporting a possible future neutrino factory or a muon collider is the doubling the repetition rate and increasing the linac pulse length, with the potential to achieve a beam power of approximately 4 MW at 8 GeV. The linac hardware, conventional facilities, cryogenic plant, and utilities will be designed to accommodate these upgrades.

1.1.2 ICD-2

The initial Project X ICD-1 design and goals were mainly driven by the Project X synergy with the ILC and the 2 MW operation of the MI for neutrino program. The
details of operation with a slow extracted beam at 8 GeV were not considered in the Initial Conceptual Design. While the ICD-1 has evolved, it still follows the same path as the initial Project X proposal but with an increased beam current. A preliminary study of slow beam extraction with the ICD-1 indicates intrinsic problems and lack of flexibility. The ICD-2 is designed to address the deficiencies found in the ICD-1.

The main concept of the ICD-2 is to replace the slow extracted beam at 8 GeV with the beam accelerated in a Continuous Wave (CW) linac operating with a bunching frequency of 162.5 MHz. This concept has a number of notable advantages. First, the RF separation of the beam after acceleration allows simultaneous operation of several experiments. The time structure and the intensity of each beam can be varied independently by using a bunch chopper at low beam energy (∼2 MeV). Second, the beam quality of a CW linac is significantly better than for slow-extracted beams; in particular, the linac beam intensity does not have fluctuations inherent to slow extracted beam from a synchrotron. Third, the power of beam accelerated by a CW linac is set by high energy physics requirements (ability to use this power by experiment) rather than by technical or accelerator physics requirements. Fourth, the bunch length in a linac (<10 ps rms) is much smaller than can be reasonably achieved in a ring which enables unprecedented time-of-flight resolution that will be invaluable to next generation rare-decay experiments.

The energy of the linac is determined by the threshold of particle production. The linac energy of 1 GeV would be sufficient for muon production but the threshold of kaon production is slightly below 2 GeV. This sets the linac energy to about 2 GeV. Note that this energy is below the threshold for anti-proton production which results in a reduced background for stopping muon experiments.

The 2 MW Main Injector operation for the long-baseline neutrino program requires 8 GeV beam injection into the MI. Therefore, an additional acceleration stage from 2 to 8 GeV is required in the ICD-2. This can be achieved with a synchrotron or a
Figure 1.2: A schematic view of the ICD-2 of Project X.
1.1 Project X

pulsed linac. Both choices require the linac beam current to be 1 mA or above. For the ICD-2 a beam current of 1 mA is chosen. This sets the total power of the CW linac to 2 MW. Fig. 1.2 presents a schematic layout of the ICD-2 accelerator complex. The beam originates in a DC H- ion source capable of producing 1 - 10 mA beam currents. The ion source is followed by a CW radio-frequency quadrupole (RFQ) accelerator, which bunches the beam at 162.5 MHz and increases the beam energy to 2 - 2.5 MeV. Following the RFQ the ICD-2 concept assumes a CW bunch-by-bunch chopper, capable of removing individual bunches in a pre-programmed manner. The role of the chopper is to adjust the bunch time structure from a continuous stream of 162.5 MHz bunches to a bunch structure requested by beam users, while keeping the beam average current at 1 mA. Most of the time the beam accelerated to 2 GeV is directed to the experiments. However every 100 ms it is diverted to the Rapid Cycling Synchrotron (RCS) by a pulsed magnet for further acceleration and consecutive injection to Recycler and Main Injector. The required pulse duration is about 5 ms at a 10 Hz rate, thus representing about 5% duty cycle of the CW linac. The remaining 95% of the duty cycle are available for the 2-GeV programs.

Presently, the ICD-2 studies devoted to 2 - 8 GeV acceleration are concentrating on a rapid cycling synchrotron (RCS). A pulsed linac (2 - 8 GeV) is also possible; such a linac would be comparable to the ICD-1’s 2 - 8 GeV portion. The circumference of the synchrotron should be kept sufficiently small to mitigate effects of the beam space charge and instabilities. It is chosen to be 1/6 of MI circumference, which is 3319 m. This sets the repetition rate to be 10 Hz so that 6 injections need to be delivered to the Recycler. For 60 GeV MI operation the cycle duration is 0.8 s leaving 2 injections per MI cycle available for low energy problem. The 6 beam transfers are stored in the Recycler and then are transferred to the MI in a single transfer. Two out of eight injections sent to the Recycler in one MI cycle are available for a fast extraction 8-GeV program. Note that for the 120 GeV MI operation the cycle time is increased
to 1.4 s resulting in eight injections per MI cycle available for the fast extraction.

1.2 HINS

The Fermilab High Intensity Neutrino Source (HINS) Linac R&D program is building a first-of-a-kind 60 MeV superconducting H- linac. The HINS Linac incorporates superconducting solenoids for transverse focusing, high power RF vector modulators for independent control of multiple cavities powered from a single klystron, and superconducting spoke-type accelerating cavities starting at 10 MeV. This will be the first application and demonstration of any of these technologies in a low energy, high-intensity proton/H- linear accelerator. The HINS effort is relevant to a high intensity, superconducting H- linac (Project X) that might serve the next generation of neutrino physics and muon storage ring/collider experiments. The accelerator physics design philosophy of the HINS Linac is described in depth in [5].

In the HINS Linac, a 50 keV ion source provides beam to a 2.5 MeV RFQ with radial matching sections at the input and output ends that form the axially-symmetric beam. The RFQ is followed by a Medium Energy Beam Transport (MEBT) section that provides space for the beam chopper [6] while maintaining desired transverse and longitudinal beam characteristics with two RF buncher cavities and three SC solenoid magnets. The subsequent accelerating section up to 10 MeV comprises 16 RT crossbar H-type (RTCH) cavities [7] and 16 SC solenoid magnets [8]. The first four RTCH cavities are made with three spokes; the remaining 12 have four spokes. As a matter of economics, a subsequent design modification reduced the number of unique RT cavity designs required; nine unique designs fill the complement of 16 RT cavities. The solenoids, even in the RT section of the machine, will be superconducting. Field strengths of 6 Tesla are necessary to produce the short focal lengths required by the beam optics and for matching into the SC cavity section.
From 10 to 30 MeV, eighteen SC single-spoke cavities of a common $\beta = 0.2$ design are employed [9]. They are foreseen to be divided between two cryomodules. The cavities alternate with eighteen SC solenoids. The final 30 MeV is achieved with a single cryomodule of six solenoids and eleven SC spoke cavities of $\beta = 0.4$ design.

The HINS RF power distribution and control system design is particularly demanding in light of the goal to employ a single high-power klystron to drive multiple cavities accelerating a non-relativistic beam. The physics design calls for each cavity to operate at an individual gradient and synchronous phase and thus each experiences different beam loading. To meet tight amplitude and phase tolerances under these dynamic conditions, each cavity requires an individual amplitude and phase control element operating at the full power level of the cavity. This function is served by high power RF vector modulators [10]. Performance simulations of the vector modulator control under realistic RF and beam conditions are on-going. Further complicating the situation, the original HINS concept includes combining both RT and SC cavities on a single klystron. Adding a second klystron to the same modulator to separate the RT from the SC cavities remains a fall-back position.

In the meantime, the HINS proton ion source and low energy beam transport (LEBT) was successfully commissioned. It is capable to produce a 50 keV, 3 msec pulsed beam with a peak current greater than 20 mA (less than 50% of proton, to be discussed in Chapter 2) at 2.5 Hz (0.7% duty factor). The beam is transported to the radio frequency quadrupole (RFQ) by the LEBT that consists of two normal conducting focusing solenoids and four steering dipole magnets. To understand beam transmission through the RFQ, it is important to characterize the 50 keV beam before connecting the LEBT to the RFQ. A wire scanner and a Faraday cup are temporarily installed at the exit of the LEBT to study the beam parameters. Beam profile measurements are made for different LEBT settings and results are compared to those from computer simulations. Beam emittance is measured both directly (slit-wire
scanner method) and indirectly (solenoid variation method).

In the following, we will introduce the proton ion source, the LEBT and its elements, the beam instrumentation devices, and finally, a thorough discussion on the beam measurement techniques, simulations, results and their comparison.
Chapter 2

Ion Source and Low Energy Beam Transport

The HINS proton ion source and low energy beam transport (LEBT) has been successfully commissioned. It is capable to produce a 50 keV, 3 msec pulsed beam with a peak current greater than 20 mA (less than 50% of proton, discussed below) at 2.5 Hz (0.7% duty factor). The beam is transported to the radio frequency quadrupole (RFQ) by the LEBT that consists of two focusing solenoids and four steering dipole magnets. In the following, we will discuss the design of the proton source and its high voltage configuration. We will also discuss the LEBT and its elements, namely, the solenoid and the steering dipoles.

2.1 Proton Source

2.1.1 Duo-plasmatron

The HINS proton source is a duo-plasmatron. The duo-plasmatron became popular for use in particle accelerators some 30 - 40 years ago. Its technology is well-developed
Figure 2.1: A schematic drawing of the HINS proton source. It is a duo-plasmatron type of proton source.
2.1 Proton Source

and matured. Fig. 2.1 shows a schematic drawing of the HINS duo-plasmatron proton source. A combination of a hot cathode emitting electrons and a pulsed arc discharge in a low pressure $\text{H}_2$ gas creates plasma — a free mixture of ions and electrons. High mobility of electrons allows them to reach a higher temperature than the ions, which causes more hydrogen atoms to ionize and raises ions temperature. When the ions temperature rises to more than half of that of the electrons, a stable positive ion sheath is formed. The drifting electrons give rise to an arc current. Magnetic field produced by the solenoid confines plasma to a space away from the wall of the chamber. In the constriction (bottle neck), electrons spiral around field lines gaining more energy and ionizing more hydrogen atoms. Magnetic mirror in the constriction helps bipolar current to flow-out of the anode area. A negatively charged extractor pulls positive ions from the expanded plasma.

2.1.2 HINS Proton Source

The HINS proton source and LEBT assembly is located in the Meson Detector Building (MDB) at Fermilab. A schematic drawing of the Proton Source and LEBT arrangement is shown in Fig. 2.2. The ultra-high purity hydrogen gas is fed at about 50 - 150 mTorr pressure into the source plasma chamber where it is heated by a filament and ionized in an arc voltage of 50 - 150 Volts. The arc voltage is modulated to 1 - 5 msec pulse and 0.5 - 2.5 Hz repetition rate. The hydrogen ions are compressed by the magnetic field produced by the solenoid magnet into a small orifice and extracted to a vacuum chamber by the 40 kV negative potential at the extraction electrode. The $\text{H}^+$ ions are then accelerated to a maximum energy of 50 keV in the 18 mm wide gap between the extraction and acceleration electrodes.

The major drawback of this ion source is that the arc voltage sends a good number of $\text{H}_2^+$ ions from the plasma formation chamber to the plasma expansion chamber,
causing a mixture of proton and $\text{H}_2^+$ being accelerated to 50 keV and extracted as a particle beam. In fact, from measurement shown in Chapter 3.1, it is estimated that the proportion of proton in the beam is less than 50%.

Although, at the same kinetic energy 50 keV, proton moves faster than $\text{H}_2^+$, the possibility that the faster moving proton being separated from the slower $\text{H}_2^+$ is excluded. Consider a total length of 1 meter from the ion source to the wire scanner. Proton is half the weight of $\text{H}_2^+$ therefore at the same kinetic energy proton moves faster by a factor of $\sqrt{2}$. Suppose proton and $\text{H}_2^+$ leave the ion source at the same time. After 1 meter of travel, their time separation is about 0.134 $\mu$s, which is much small the the pulse length of 100 $\mu$s. So, at the wire scanner proton and $\text{H}_2^+$ are approximately at the same phase. The specification for the HINS proton ion source is described in detail in [11].

2.1.3 High Voltage Supplies

The proton source and its associated electronics operate at 50 kV potential above ground. Consequently, all the proton source associated electronics is placed inside an isolated cabinet. Ground shieldings around the proton source area and the cables connecting the proton source to the electronics cabinet are used. The LEBT beam transport and other sub-systems are at zero potential.

The proton source electronics cabinet is located inside a grounded relay rack. The front and rear door of the relay rack have magnetic switches interlocked to the high voltage power supply and a ground arm. The ground arm shuts off the supply and grounds the inner isolated high voltage part if the doors are moved slightly. The power for the electronics is provided by a 60 kV isolation transformer located in a relay rack next to the isolated high voltage (HV) cabinet.

The proton source itself is enclosed in a half-inch thick Lexian box covered by
Figure 2.2: A schematic drawing of the HINS proton source and LEBT arrangement.
2. Ion Source and Low Energy Beam Transport

**Figure 2.3:** The layout of LEBT-1. LEBT-1 consists of two identical focusing solenoids (SOL-U and SOL-D) and four corrector dipole magnets (DIP-UH, UH, DIP-DV). The toroid is a beam current monitor. The wire scanner and the Faraday cup are temporary beam diagnostics devices.

a grounded copper mesh. All connecting cables between the proton source and the HV relay rack are enclosed within a double walled PVC pipe with an outer ground shielding. This protection system and shielding has been tested up to 55 kV.

We estimated the upper limit of the capacitance of the HV system to be $< 1000$ pf (power supply $- 300$ pF, equipment rack $- 350$ pF, HV conductor line $- 220$ pF, proton source $- 20$ pF). So, for an operating voltage below 50 kV the stored energy is less than 1.25 Joule. A more detailed discussion can be found in [12].

### 2.2 Low Energy Beam Transport

The LEBT transports the 50 keV proton beam produced by the ion source to the radio frequency quadrupole (RFQ). It comprises two focusing solenoids and four steering dipole magnets.

The layout of the low energy beam transport is shown in Fig. 2.3. In the figure, the proton source is the duo-plasmatron described in the previous section. The two
focusing solenoids, labeled as SOL-U and SOL-D, are identical. There are four steering dipole magnets, two for horizontal and two for vertical, located between the two focusing solenoids. Except for the orientation, they are identical. They are labeled DIP-UH, DIP-UV, DIP-DH and DIP-DV. Beam diagnostics devices are installed in the LEBT. Toroid and Faraday cup are both beam current monitor and they serve to measure the beam transmission rate. The wire scanner is installed near the planned entrance of the RFQ to study the transverse beam properties. These devices and the measurement made using them will be discussed later in this report.

In the rest of the section, the specifications of the magnets will be presented. Also, due to a damage to one of the two focusing solenoids, solenoid replacement has been taken place. The considerations in the process of the replacement will also be discussed.

\[ 2.2 \text{ Low Energy Beam Transport} \]

2.2.1 Focusing Solenoids

The two focusing solenoids, SOL-U and SOL-D, are manufactured based on the design of the solenoid made for the Loma Linda project [13]. The axial field profile, \( B_{z0}(z) \), was measured both for the solenoid for Loma Linda project (1989) and for HINS (2007). A CST MWS model is also built to study the field profile. They are plotted together in Fig. 2.4. The field covers a region of about 25.4 cm long whereas the physical length of the solenoid is 16 cm. The effective length, given by \( \frac{1}{B_{peak}} \int B_{z0}(z)dz \), is calculated to be 12.8 cm. The axial field versus excitation current is also measured (see Fig. 2.5). The solenoids are capable to provide a magnetic field of up to about 0.8 Tesla. The excitation current gives 9.43 Gauss/Amp. The result of these measurements will be used in Chapter 3 for particle tracking simulation and beam measurement data analysis.
Figure 2.4: The axial field profile, $B_{z0}(z)$, of solenoid SOL-U and SOL-D. A CST MWS model of the solenoid is built. The field from the model are compared with Hall probe measurement made in 1989 and 2007. The field covers a region of about 25.4 cm long whereas the physical length of the solenoid is 16 cm. The effective length is calculated to be 12.8 cm.
Figure 2.5: The axial field versus excitation current was measured for SOL-U (and SOL-D). A linear fit shows that the excitation gives 9.43 Gauss/Amp.
2. Ion Source and Low Energy Beam Transport

2.2.2 Solenoid Replacement

Due to an incident, SOL-D was damaged and could be re-used only after re-built. Solenoid replacement with another kind was needed. We have located a solenoid used by the PET Project at Fermilab which is available. The solenoid, labeled SOL-PET, has a physical length of 22.35 cm. Due to a larger bore, it has a much longer field profile, 55.9 cm, compared to that of SOL-D, which is about 25.4 cm. The effective length, given by \( \frac{1}{B_{\text{peak}}} \int B_0(z)dz \), is calculated to be 20.9 cm. The field profiles of SOL-D and SOL-PET are compared in Fig. 2.6. The axial field versus excitation current is also measured and is shown in Fig. 2.7. The excitation is 12.8 Gauss/Amp.

In Fig. 2.6, it shows that the upper face of SOL-PET is placed at the same place as that of SOL-D. Since the physical length of SOL-PET is 6.35 cm longer, the RFQ will be moved downstream by the same length. Attention is needed to the fact that for this configuration, which is the simplest modification, the field from SOL-PET enter the RFQ region. The field at the RFQ vanes is 5% of the peak field. Particle tracking has shown that this field is insignificant to beam dynamics to the linac as a whole.

The old and new lattice, LEBT-1 and LEBT-2, are summarized in Fig. 2.8. On the two sides of the solenoid SOL-PET (or SOL-D), the two layouts are identical. The only difference between the two layouts is that SOL-PET is 6.35 cm longer than SOL-D.

To be sure that SOL-PET is an appropriate substitute to SOL-D, the particle tracking code TRACK is used to simulate and compare particle dynamics under SOL-D and SOL-PET. The beamlines used for tracking are those in Fig. 2.8. It can be shown that the beam parameters required by the RFQ, which can be provided by LEBT-1, can also be produced by LEBT-2. Fig. 2.9 shows simulation result of beam width versus solenoid excitation current. The setting on SOL-U remains unchanged.
Figure 2.6: A comparison of the axial field profile of solenoid SOL-D and SOL-PET. The replacement solenoid SOL-PET has a field profile of 55.9 cm long, a physical length of 22.35 cm, and an effective length of 20.9 cm. No modification was made upstream of the upper face of SOL-D. The physical length of SOL-PET is 6.35 cm longer, so the RFQ is moved downstream accordingly.
Figure 2.7: The axial field versus excitation current was measured for SOL-PET. A linear fit shows that the excitation gives 12.8 Gauss/Amp.
Figure 2.8: The layout of the two versions of low energy beam transport. LEBT-1 has SOL-D installed. LEBT-2 has SOL-PET as a replacement of SOL-D. The field profile of the solenoids are shown.
Figure 2.9: TRACK simulation of RMS beam width versus solenoid current. SOL-U remains unchanged and is the same for both sets of simulation. Within operating range, SOL-PET can reproduce the desire beam width output by SOL-D. Beam divergence can be controlled by adjusting SOL-U.
in the simulation. Within the operating range of SOL-PET, it can reproduce the desire beam width output by SOL-D. One can use SOL-U to adjust the divergence of the beam. It is proved that SOL-PET is an appropriate substitute to SOL-D.

2.2.3 Steering Dipole Magnets

Fig. 2.10 is a schematic drawing of the steering dipole installed in the LEBT. The four dipoles (DIP-UH, UV, DH, DV) share the same design. It has a length of 1 inch and a height and width of 3.375 inches. The figure shows only the windings for vertical beam steering. Full magnet would have identical coils wound on the other two legs. Steering in both the $x$ and $y$ planes is given by superimposing horizontal and vertical fields.

Fig. 2.11 shows the measured axial field profile of the dipole. For the measurement, an excitation current of 3 Amps was used. The effective length of the dipole is 4.012 inches. This information will be used in Chapter 3 for beam measurement and analysis.

2.3 Summary

In this Chapter, the principle of operation of the HINS duo-plasmatron proton source is introduced. It is capable to produce a 50 keV, 3 msec pulsed beam with a peak current greater than 20 mA (less than 50% proton) at 2.5 Hz (0.7% duty factor).

The low energy beam transport and its elements are also described. The original version of the LEBT, LEBT-1, comprises two identical focusing solenoids, SOL-U and SOL-D, and four identical steering dipole magnets, DIP-UH, UV, DH, DV. The solenoid is capable to produce a peak field of up to 8 kGauss on the axis at 850 Amps of excitation current. The dipole is capable to produce up to 100 Gauss of magnet
Figure 2.10: A schematic drawing of the steering dipole. The four dipoles (DIP-UH, UV, DH, DV) share the same design. It is 1 inch long and has a height and width of 3.375 inches. The picture shows only the windings for vertical beam steering.
Figure 2.11: The axial field profile of the corrector dipole magnets with 3 Amps of current. The four magnets have almost identical field profile. The field integral of this field profile is 1.019 kG-cm (or 401.2 G-in).
field at 3 Amps of current.

In an incident, SOL-D is damaged and is needed to be re-built. A solenoid from the Fermilab PET project, SOL-PET, is used as a substitute of SOL-D, giving rise to LEBT-2. SOL-PET was tested extensively and is proved to be able to transport the proton beam as done by SOL-D. The tests includes comparing field profile and particle tracking simulations. Proton beam transported by both LEBT-1 and LEBT-2 was measured and analyzed and is discussed in Chapter 3.
Chapter 3

Beam Characterization for the Proton Ion Source

In November 2008, the HINS proton ion source and low energy beam transport (LEBT) was successfully commissioned. It is capable to produce a 50 keV, 3 msec pulsed beam with a peak current greater than 20 mA (less than 50% proton, discussed below) at 2.5 Hz (0.7% duty factor). The beam is transported to the radio frequency quadrupole (RFQ) by the LEBT that consists of two focusing solenoids and four steering dipole magnets. To understand beam transmission through the RFQ, it is important to characterize the proton beam before connecting the LEBT to the RFQ. A wire scanner and a Faraday cup are temporarily installed at the exit of the LEBT to study the beam parameters.

In this Chapter, all data were taken using the wire scanner. We start with interpreting the signal measured by the wire scanner and the information that can be obtained. Then, we performed a beam-calibration to the steering dipole magnets. We then study transverse motion coupling due to solenoidal field by measuring beam rotation through solenoid. Analysis to these measurements will be accompanied with
beam physics modeling and particle tracking simulations. Finally, together with a slit the wire scanner will form an emittance probe (slit-wire scanner assembly) for direct emittance measurement. Result from direct emittance measurement is compared to indirect measurement using solenoid variation method. Note that, due to high voltage supplies limitation in the early stage of operation, some of the measurements were made for beam below 50 keV, which is specified in the context. Unless specified, the beam energy is assumed to be 50 keV.

3.1 Transverse Beam Profile Measurement

Horizontal and vertical beam profiles are measured using the wire scanner. Fig. 3.1 shows the results of a vertical beam profile measurement for a 48 keV proton beam with a pulse length of 100 µs at a repetition rate of 0.5 Hz. The two solenoids were optimized to locate the beam waist at the wire scanner. The data points reflect the integrated electric charge as seen by the wire at every step across the beam pipe. The narrow peak is the profile of the proton beam. The background signal comes from other particle species (mainly H⁺_2) being extracted from the source (see Chapter 2).

As shown in the figure, the background has an integrated signal that is larger than that of the proton. In fact, a peak current of 8 mA is recorded by the current transformer while the reading at the Faraday cup is only 3.5 mA, suggesting a beam loss of more than 50%. In contrast, the results coming from TRACK simulation shows zero beam loss under the same solenoid settings. This suggests that more than 50% of the beam is composed of non-proton ion species and that contribute to most of detected beam loss. Studies on other ion species in the beam will be discussed later.

The possibility that the faster proton being separated from the slower H⁺_2 is excluded. Consider a total length of 1 meter from the ion source to the wire scanner.
3.1 Transverse Beam Profile Measurement

(a) Raw signal from the wire scanner. The red line is a double Gaussian fit. The fit has a $R^2$ value of greater than 0.99.

(b) The fit successfully extracts the profile of the proton beam from raw data.

**Figure 3.1:** Transverse beam profile measurement using wire scanner. Due to the presence of $\text{H}_2^+$ ions, two Gaussians are used to fit the data. The profile of the proton beam is nicely extracted from the raw data.
Proton is half the weight of $\text{H}_2^+$ therefore at the same kinetic energy proton moves faster by a factor of $\sqrt{2}$. Suppose proton and $\text{H}_2^+$ leave the ion source at the same time. After 1 meter of travel, their time separation is about 0.134 $\mu$s, which is much smaller than the pulse length of 100 $\mu$s. So, at the wire scanner proton and $\text{H}_2^+$ are approximately at the same phase.

To extract the profile of the proton beam from the combined signal, two Gaussian distributions, one for the proton and one for the $\text{H}_2^+$, are used to fit the raw data. The expression used for fitting is

$$ signal(x) = g(A_1, \sigma_1, x_01; x) + g(A_2, \sigma_2, x_02; x) \cdot \text{pipe}(x) + k, \quad (3.1) $$

where

$$ g(A, \sigma, x_0, x) = A \cdot \exp\left(\frac{-(x - x_0)^2}{2\sigma^2}\right), \quad (3.2) $$

$$ \text{pipe}(x) = \frac{1}{r} \sqrt{r^2 - x^2}, \quad (3.3) $$

where $r$ is the radius of the beam pipe and $k$ is a constant. The normalization factor, $\text{pipe}(x)$, is applied to the second Gaussian since, as shown in Fig 3.1, the $\text{H}_2^+$ ions fill out the whole beam pipe, whose radius, $r$, is 17 mm.

The fitting has a coefficient of determination, the $R^2$ value, larger than 0.99 (see Fig. 3.1(a)). The profile of the proton beam is nicely extracted from the fitting (see Fig. 3.1(b)). The RMS beam size, $\sigma_{\text{rms}}$, or $\sigma_1$ in Eq. (3.1), obtained from this measurement is 0.64 mm. This measurement has been repeated for ten times. The error for such a measurement is $\pm 2\%$. Fig. 3.2 is a plot showing the time structure of the proton beam pulse. This 100 $\mu$s pulse beam has a flat-top of about 50 $\mu$s.
3.2 Beam Steering by the Dipole Magnets

In addition to beam profile measurement, the wire scanner can also be used to determine the position of the beam relative to the beam pipe. Since there is no beam position monitor (BPM) installed in the beamline, it is appropriate to use the wire scanner as a BPM to examine the performance of the corrector dipole magnets. Beam deflection as a function of magnet strength is measured for a 45 keV proton beam. In the following we will first experience the x-y motion coupling due to the presence of focusing solenoids. Then, to study the beam deflection by the dipoles, the solenoid SOL-D is switched off to decouple the motion. The dipole field profile presented in Chapter 2.2.3 is used to estimate the amount of beam deflection. The result is compared to the measurement.

3.2.1 Coupling due to Solenoidal Field

The proton ion source is configured to produce a 45 keV 2 mA beam with a pulse length of 100 µs at 0.5 Hz. The focusing solenoids are set to focus the beam down
to a reasonably small size at the wire scanner. The strength of the corrector dipole magnets are varied and the profile of the beam are measured with the wire scanner.

Fig. 3.3 shows how the horizontal and vertical beam position response to changes in the strength of a horizontal dipole (DIP-DH). As the strength of DIP-UH varies, the beam is supposed to receive a horizontal kick. However, change in both horizontal and vertical beam position are detected. This is also true for the other three dipole magnets. This is due to the x-y coupling induced by the downstream solenoid (SOL-D). The coupling effect will be discussed in detail in Chapter 3.3. In the following section, we will show the measurement result of the beam deflection by the corrector dipole and compare it with theoretical estimation.

### 3.2.2 Beam Deflection

In order to measure the beam deflection as a function the strength of the corrector dipole magnets, the downstream solenoid, SOL-D, is switched off to avoid x-y coupling.

Fig. 3.4 shows the change in the beam profile (or position) as the strength of the dipoles is varied. Since the SOL-D is turned off, the size of the proton beam is much larger. Fortunately, the movement of the peak of the beam profile can still be clearly observed. The vertical position of the beam does not response to excitation in the horizontal dipole magnets (see Fig. 3.4(a) and 3.4(c)). Also, the horizontal position does not change with the vertical dipole magnets (see Fig. 3.4(b) and 3.4(d)). This suggests that the dipoles were well aligned such that x and y motion are not coupled up to our resolution. Note that the two horizontal dipoles bend the beam in different directions. This is also true for the two vertical dipoles.

The amount of beam deflection as a function of dipole strength can be estimated using the data plotted in Fig. 3.4. In the following we will make a theoretical estima-
3.2 Beam Steering by the Dipole Magnets

Figure 3.3: Plot of beam profile at different corrector dipole strength. As the strength of the horizontal dipole DIP-DH varies, both the horizontal and vertical beam position change. This is due to the x-y coupling induced by the solenoidal field (SOL-D).
Figure 3.4: Beam profile measurement under different dipole strengths. The solenoid SOL-D is switched off to avoid x-y coupling.
3.2 Beam Steering by the Dipole Magnets

...tion on the beam deflection and compare it with the measurement results.

When a charged particle beam passes through a dipole magnet, the beam orbit is deflected. The angle of deflection of can be obtained from the Lorentz force law and is given by [14]

\[ \theta_d = \frac{e}{p} \int_{z_1}^{z_2} Bdz = \frac{1}{B\rho} \int_{z_1}^{z_2} Bdz \tag{3.4} \]

where \( e \) is the charge of an electron, \( p = \gamma \beta mc \) is the momentum of the beam, and \( B\rho = p/e \) is the momentum rigidity of the beam. \( B \) is the transverse magnetic field along the beam axis. For a proton beam at 45 keV the momentum rigidity is 30.653 kG-cm. The field profile of the corrector dipole magnets is measured and is shown in Fig. 2.11.

Using the above numbers and Eq. (3.4), the spatial deflection of the beam at the wire scanner can be calculated and is plotted in Fig. 3.5. As a comparison, it is plotted together with measurement result derived from data shown in Fig. 3.4. For the downstream dipoles, result from calculation was 3.3 mm/Amp. Measurement result for DIP-DH was 3.1 mm/Amp and that for DIP-DV was 2.8 mm/Amp. However, the result for the upstream dipoles, DIP-UH and DIP-UV, shows significantly smaller deflection compared to calculation. Result from calculation was 7.3 mm/Amp while measurement result for DIP-UH was 4.0 mm/Amp and that for DIP-UV was 3.5 mm/Amp. This is due to the fact that the two upstream dipoles are located too close to the upstream solenoid, SOL-U, such that the dipole field was “absorbed” into the iron shielding of the solenoid, which shortened span of the dipole field and lowered the field amplitude. Another implication from this result is that the alignment (roll angle) of the dipoles is acceptable up to the resolution of beam position measurement using the wire scanner. This step is crucial in the measurement of beam rotation due to solenoidal field, which is discussed in next section.
Figure 3.5: Beam spatial deflection at the location of the wire scanner. Comparison is made between measurement result shown in Fig. 3.4 and calculation using Eq. (3.4).
When a beam travels through a solenoid, its horizontal and vertical dynamics are coupled due to the beam rotation by the solenoidal field, as discussed in Chapter 3.2.1. This complicates the beam orbit control system for a solenoidal transport lattice. Since solenoid is the primary focusing elements for the HINS linac, knowledge on beam rotation by solenoidal field is valuable.

In the following, we study the beam rotation due to solenoidal field by analyzing the motion of a particle when it passes through a solenoid. Then we will show the measurement results for beam rotation by the downstream solenoid (both SOL-D in LEBT-1 and SOL-PET in LEBT-2). Results from analytical calculation and measurement will be compared.

3.3.1 Solenoid as a Lens

When a beam travels through a solenoid, its horizontal and vertical dynamics are coupled due to the beam rotation by the solenoidal field. To understand the beam rotation through a solenoid, one can consider the trajectory of a charged particle which enters a solenoid off-axis or off-angle. This is treated with two different methods: solenoid as a lens and 3-D field particle tracking. In this section we study the beam rotation by a solenoid by considering a solenoid as a lens. The treatment follows the discussion in [15].

We can model a solenoid as a lens by considering that the magnetic field is uniform inside the solenoid and zero otherwise. A schematic drawing is shown in Fig. 3.6. The solenoid has a length of $L$. Inside the solenoid, for $0 < z < L$, the magnetic field is uniform ($\vec{B} = B_0 \hat{z}$). The magnetic field is zero outside the solenoid. For region near $z = 0$ and $z = L$, the field rises rapidly from zero to $B_0$ (or drops from $B_0$ to zero).
Figure 3.6: A schematic drawing of a lens-like solenoid. The solenoid has a length of \( L \). Inside the solenoid, for \( 0 < z < L \), the magnetic field is uniform (\( \vec{B} = B_0 \hat{z} \)). Outside the solenoid, the field is zero.
I Inside the solenoid

We first consider the motion of the particle inside the solenoid ($0 < z < L$). In a uniform solenoidal magnetic field, the trajectory of a charged particle is in general a helix. The projection of the helix on the x-y plane is a circle of radius $R$. From $\vec{F} = q(\vec{v} \times \vec{B})$, the radius of the circle, $R$, is given by

$$\gamma m v^2 \over R = q v_\perp B_0,$$

so that

$$R = \gamma m v_\perp \over q B_0,$$

where $\gamma$ is the relativistic Lorentz factor, $m$ is the rest mass of the particle, $q$ is the charge of the particle, and $v_\perp$ is the particle’s velocity projected onto the x-y plane.

The cyclotron frequency, $\omega$, is given by

$$\omega = v_\perp \over R = q B_0 \over \gamma m.$$

(3.7)

Using the cyclotron frequency, we can calculate the azimuthal angle, $\phi_h$, rotated by the particle’s trajectory about the axis of the helix. This is given by the product of the cyclotron and the time it takes for the particle to traverse the field:

$$|\phi_h| = \omega t = {q B_0 L \over \gamma m v_z} \approx {q B_0 L \over p},$$

(3.8)

where $p = \gamma \beta mc$ is the momentum of the particle, and $v_z \gg v_\perp$ is assumed.

II At the two ends of the solenoid

Near the two ends of the solenoid, $z = 0$ and $z = L$, where the axial magnetic field is changing, Maxwell equation suggests that there exists a radial component of the magnetic field, given by

$$\nabla \cdot \vec{B} = \frac{1}{r} \frac{\partial (r B_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0,$$

(3.9)
so that
\[ B_r \approx -\frac{r}{2} \frac{\partial B_z}{\partial z}. \] (3.10)
This radial field component gives an azimuthal kick to the particle when it enters or exits the solenoid. The change in the azimuthal momentum can be obtained from the Lorentz force law
\[ \frac{dp_\phi}{dt} = v_z \frac{dp_\phi}{dz} = qv_z B_r \approx -qv_z \frac{r}{2} \frac{\partial B_z}{\partial z} \] (3.11)
so that
\[ \Delta p_\phi \approx -\frac{qr}{2} \Delta B_z, \] (3.12)
where \( \Delta B(z = 0) = B_0 \) and \( \Delta B(z = L) = -B_0 \). A schematic drawing is shown in Fig. 3.7. A charged particle enters the solenoid \((z = 0)\) with an initial radial component of momentum. It receives an azimuthal kick from the solenoid radial field component \( B_r(z = 0) \), resulting in a total transverse momentum that defines the helical path. The particle traverses the solenoid along this helical path and receives another azimuthal kick from \( B_r(z = L) \) at the exit of the solenoid.

An important implication of the model is that the azimuthal kick received by the particle when it enters the solenoid \((z = 0)\) cancels that at the exit \((z = L)\), as expected since the conservation of canonical angular momentum is zero \([15]\). Another important result is that the angle rotated about the axis of the solenoid by the particle’s trajectory, \( \phi_{lens} \), is equal to one-half of the angle rotated about the axis of the helix, \( \phi_h \) as defined in Eq. (3.8) (see \([15]\)). Hence (compared to Eq. (B.9)),
\[ \phi_{lens} = \frac{1}{2} \phi_h = -\frac{qB_0 L}{2p}. \] (3.13)
This can be interpreted as the angle of rotation of a beam when it traverses a solenoid. Note that the angle of rotation is in the \(-\hat{\phi}\) direction if the longitudinal velocity \( v_z \) is in the same direction of the axial solenoid field \( B_0 \). In Chapter 3.3.3 and 3.3.4, we use this result to compare to the result for the beam rotation measurement. Since the
3.3 Beam Rotation by Focusing Solenoid

Figure 3.7: A schematic drawing of the solenoid lens model. A charged particle enters the solenoid \((z = 0)\) with an initial radial component of momentum. It receives an azimuthal kick from the solenoid radial field component \(B_r(z = 0)\), resulting in a total transverse momentum that defines the helical path. The particle traverses the solenoid along this helical path and receives another azimuthal kick from \(B_r(z = L)\) at the exit of the solenoid, resulting in zero azimuthal momentum. The angle rotated about the axis of the helix by the particle trajectory is labeled as \(\phi_h\). The angle rotated about the axis of the solenoid, \(\phi_{lens}\), is one-half of \(\phi_h\).
measured axial field profile is non-uniform, in Eq. (3.13) we use the effective length, 
$L_{\text{eff}}$, calculated from the measured axial field profile shown in Fig. 2.6, resulting in

$$
\phi_{\text{lens}} = -\frac{q}{2p}B_0L_{\text{eff}},
$$

(3.14)

where $B_0$ is the peak magnetic field.

### 3.3.2 Particle Tracking

In this section we will find the beam rotation by tracking a charged particle’s trajectory when it passes through a solenoid. To track a charged particle traveling through a solenoid, we need to know the magnetic field produced by the solenoid. The 3-D magnetic field array can be measured directly, or can be calculated using the measured axial field profile, $B_{z_0}(z)$, of the solenoid.

The axial field profile, $B_{z_0}(z)$, of the downstream solenoid, SOL-D, and its replacement, SOL-PET, are measured and shown in Chapter 2.2.1 (see Fig. 2.4 and 2.6). Using $B_{z_0}(z)$, the magnetic field, up to the third order, near the axis of the solenoid can be calculated as (see Appendix A)

$$
B_x(x, y, z) = -\frac{1}{2}xB_z'(z) + \frac{1}{16}x(x^2 + y^2)B_z''(z) - \ldots,
$$

$$
B_y(x, y, z) = -\frac{1}{2}yB_z'(z) + \frac{1}{16}y(x^2 + y^2)B_z''(z) - \ldots,
$$

$$
B_z(x, y, z) = B_{z_0}(z) - \frac{1}{4}(x^2 + y^2)B_{z_0}''(z) + \ldots,
$$

(3.15)

where the “prime” is the derivative with respect to “$z$”. A plot of the vector field on the using the axial field profile of SOL-D and Eq. (3.15) is shown in Fig. 3.8.

The motion of a charged particle obeys the Lorentz force law

$$
\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}).
$$

(3.16)

Assuming change in the energy and in the longitudinal velocity $v_z$ of the particle is
3.3 Beam Rotation by Focusing Solenoid

Figure 3.8: A plot of the vector field of the solenoid SOL-D. The measured axial field profile, \( B_0(z) \), shown in Fig. 2.4 and Eq. 3.15 are used to generate the plot.
Figure 3.9: Protons at 48 keV are tracked through the magnetic field produced by SOL-D. (a) The trajectories of 20 protons are simulated. These trajectories represent the beam envelope of a beam. Rotation of the envelope is observed. (b) The proton enters the fields at \( x = 1 \) mm, \( x' = 5 \) mrad, and \( y = y' = 0 \).
negligible along the solenoid, we have
\[ \frac{dp_x}{dt} \rightarrow \gamma m v_z^2 \frac{d^2 x}{dz^2} \quad \text{and} \quad \frac{dp_y}{dt} \rightarrow \gamma m v_z^2 \frac{d^2 y}{dz^2}. \]

The equations of motion of a particle in a solenoidal field obtained from the Lorentz force law using the above substitution read
\[
\begin{align*}
x'' &= -\frac{B_y(x, y, z)}{B \rho} + y \frac{B_z(x, y, z)}{B \rho}, \\
y'' &= \frac{B_y(x, y, z)}{B \rho} - x \frac{B_z(x, y, z)}{B \rho},
\end{align*}
\]
(3.17)
where \( B \rho = p/q = \gamma m v/q \approx \gamma m v z/q \) is the momentum rigidity of the particle.

Using Eq. (3.15), Eq. (3.17) and the axial field profile of SOL-D, a proton with a kinetic energy of 48 keV is tracked and its trajectory is plotted in Fig. 3.9(b). The proton enters the fields at \( x(z_i) = 1 \text{ mm}, \ x'(z_i) = 5 \text{ mrad}, \) and \( y(z_i) = y'(z_i) = 0. \)
Since the proton enters the fields at zero azimuthal angle on the x-y plane (\( \phi = 0 \)), the proton displaces on the plane an angle
\[ \phi_{\text{track}} = \tan^{-1} \left( \frac{y(z_f)}{x(z_f)} \right), \]
(3.18)
where \( z_f \) is the longitudinal location where the solenoid field vanishes. This angle is also the angle of the rotation of a beam when it travels through a solenoid. We note that \( \phi_{\text{track}} \) is not sensitive to \( x'(z_i) \) and \( y'(z_i) \) provided that they are \( \ll 1. \)

In Chapter 3.3.3 and 3.3.4, we use this result to compare to the result for the beam rotation measurement.

### 3.3.3 Beam Rotation by SOL-D

The beam rotation by solenoidal field can be measured using the wire scanner with the aid from the corrector dipole magnets. In this section, we discuss the way of measurement and its results.
The two focusing solenoids, SOL-U and SOL-D, are optimized to locate the proton beam waist near the wire scanner to produce decent profile measurement. Beam position can also be measured from the profile measurement. For example, in Fig. 3.1, the vertical beam position is 11.8 mm measured from the bottom of the beam pipe. Together with a horizontal scan, the beam position can be determined.

For a certain solenoid setting, the beam position is measured for some particular corrector dipole magnets settings. The result is plotted in Fig. 3.10. In the figure, measurements are made for nine different dipole settings. These settings span out a rotated square on the x-y plane. Since we have shown that there is no x-y coupling when the downstream solenoid SOL-D is turned off (see Fig. 3.5), the presence of the solenoid field produced by SOL-D is the only contribution to the rotation of the square. The angle of rotation of the square is the angle of rotation of the beam. In this example, the beam rotation is \((-55\pm2)\) degrees. The angle of rotation is in the \(-\hat{\phi}\) direction because the magnetic field points in the \(+\hat{z}\) direction (Lenz’s Law). This set of scan is repeated for different SOL-D settings (see Fig. 3.11). One can observe that the amount of beam rotation increases with the strength of the solenoid SOL-D.

The angle of beam rotation as a function of solenoid strength is obtained from these measurements and is plotted together with the results from the analyses discussed in Chapter 3.3.1 and 3.3.2. This is shown in Fig. 3.12. The error for each of the measurements is \(\pm2\) degrees. As shown in the figure, three sets of measurement have been made for different SOL-U settings. The amount of beam rotation has no obvious dependence on the strength of SOL-U, as expected, since the corrector dipole magnets are located downstream of it. The two approaches discussed in Chapter 3.3.1 and 3.3.2 on estimating the amount of beam rotation produce almost identical results in the region of solenoid strength where the measurement was made. Linear fit was performed to data and result for beam rotation in terms of degree per kGauss is summarized in Table 3.1.
Figure 3.10: Beam position measurement with different dipole settings. These settings span out a square on the x-y plane. The angle of rotation of the square is the beam rotation due to the solenoid SOL-D. In this example the beam rotation is -55 degrees.
Figure 3.11: The beam position measurement is made for different SOL-D settings. The amount of beam rotation increases with the strength of SOL-D.
Figure 3.12: A comparison of the measurement and simulation results for the beam rotation by SOL-D. The amount of beam rotation has no obvious dependence on the strength of SOL-U, as expected. The two approaches of simulation, $\phi_{\text{lens}}$ and $\phi_{\text{track}}$, produce identical results. The measurement result is in good agreement with the simulation.
3. Beam Characterization for the Proton Ion Source

<table>
<thead>
<tr>
<th></th>
<th>degrees/Amp</th>
<th>degrees/kGauss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solenoid Lens</td>
<td>0.108</td>
<td>11.6</td>
</tr>
<tr>
<td>Tracking</td>
<td>0.107</td>
<td>11.5</td>
</tr>
<tr>
<td>SOL-U 420</td>
<td>0.0660</td>
<td>7.10</td>
</tr>
<tr>
<td>SOU-U 440</td>
<td>0.0847</td>
<td>9.14</td>
</tr>
<tr>
<td>SOL-U 460</td>
<td>0.0997</td>
<td>10.8</td>
</tr>
</tbody>
</table>

Table 3.1: Beam rotation per solenoid field — SOL-D.

3.3.4 Beam Rotation by SOL-PET

Beam rotation measurement was also made after the replacement of SOL-D with SOL-PET. The two focusing solenoids, SOL-U and SOL-PET, are optimized to locate the proton beam waist near the wire scanner to produce decent profile measurement. The beam position relative to the beam pipe are measured using the wire scanner.

In the Fig. 3.13, measurements are made for nine different dipole settings. These settings span out a rotated square on the x-y plane. Since we have shown that there is no x-y coupling when the downstream solenoid is turned off (see Fig. 3.5), the presence of the solenoid field produced by SOL-PET is the only contribution to the rotation of the square. The angle of rotation of the square is the angle of rotation of the beam. In this example, the beam rotation is (+75±2) degrees. The angle of rotation is in the +\(\hat{\phi}\) direction because the magnetic field points in the −\(\hat{z}\) direction (Len’s Law). This set of scan is repeated for different SOL-PET settings (see Fig. 3.14). One can observe that the amount of beam rotation increases with the strength of the solenoid SOL-PET.

The angle of beam rotation as a function of solenoid strength is obtained from these measurements and is plotted together with the results from the analyses discussed
Figure 3.13: Beam position measurement with different dipole settings. These settings span out a square on the x-y plane. The angle of rotation of the square is the beam rotation due to the solenoid SOL-PET. In this example the beam rotation is +75 degrees.
in Chapter 3.3.1 and 3.3.2. This is shown in Fig. 3.15. The error for each of the measurements is ±2 degrees. As shown in the figure, three sets of measurement have been made for different SOL-U settings. The amount of beam rotation has no obvious dependence on the strength of SOL-U, as expected, since the corrector dipole magnets are located downstream of it. The two approaches discussed in Chapter 3.3.1 and 3.3.2 on estimating the amount of beam rotation produce similar results in the region of solenoid strength where the measurement was made. The discrepancy on the two results stems from the fact that $\phi_{\text{lens}}$ calculation is sensitive to the effective length of the solenoid, which is determined by the solenoid field profile measurement (see Fig. 2.6) and is subject to an uncertainty. The measurement result is in good agreement with the simulation. Linear fit was performed to data and result for beam rotation in terms of degree per kGauss is summarized in Table 3.2.

### Table 3.2: Beam rotation per solenoid field – SOL-PET.

<table>
<thead>
<tr>
<th></th>
<th>degrees/Amp</th>
<th>degrees/kGauss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solenoid Lens</td>
<td>0.242</td>
<td>18.9</td>
</tr>
<tr>
<td>Tracking</td>
<td>0.241</td>
<td>18.8</td>
</tr>
<tr>
<td>SOL-U 350</td>
<td>0.242</td>
<td>18.9</td>
</tr>
<tr>
<td>SOL-U 400</td>
<td>0.245</td>
<td>19.14</td>
</tr>
<tr>
<td>SOL-U 450</td>
<td>0.226</td>
<td>17.7</td>
</tr>
</tbody>
</table>

3.4 **Tranverse Emittance**

Beam emittance is an important quantifier of beam quality. It is a measure to the coherence property of the beam: the degree to which the beam particles have nearly
Figure 3.14: The beam position measurement is made for different SOL-PET settings. The amount of beam rotation increases with the strength of SOL-PET.
Figure 3.15: A comparison of the measurement and simulation results for the beam rotation by SOL-PET. The amount of beam rotation has no obvious dependence on the strength of SOL-U, as expected. The two approaches of simulation, $\phi_{\text{lens}}$ and $\phi_{\text{track}}$, produce similar results. The measurement result is in good agreement with the simulation.
the same coordinates as the reference particle [16]. Consider a particle beam. Each of
the particles in a beam is described by three pairs of position-momentum coordinates.
At every instance, each particle is represented by a single point in the six dimensional
phase space volume. A beam can then be expressed as a collection of points in
the three projection of the phase space volume, namely, \( x - p_x, y - p_y \) and \( z - p_z \),
where \( x, y \) and \( z \) are the position coordinates, \( p_x, p_y \) and \( p_z \) are the momentum
coordinates. In practice, it is more convenient to measure the divergence of a beam.
So the unnormalized phase space projection, \( x - x', y - y' \) and \( \phi - \Delta W \) are used
instead. Here \( x' = dx/dz \) and \( y' = dy/dz \) are the divergence angles, \( \phi \) is the phase
and \( \Delta W \) is the energy variable.

With linear focusing, elliptical distributions in phase space remain elliptical [16].
The trajectory of each particle in the phase space projection traces out an ellipse,
called trajectory ellipse, defined by

\[
\epsilon_x = \gamma x^2 + 2\alpha xx' + \beta x'^2. \tag{3.19}
\]

This is sometimes called the Courant-Snyder invariant. Here \( \gamma, \alpha \) and \( \beta \) are the
Courant-Snyder parameters with the normalization \( \gamma\beta - \alpha^2 = 1 \). The phase space
area enclosed by the ellipse is equal to \( \pi \epsilon_x \).

For a matched beam, the phase space isodensity contours of the beam are con-
centric and geometrically similar to the trajectory ellipses of the particles [16]. By
choosing a particular density contour that encloses a certain portion of the total
number of particles (see Fig. 3.16), one can define the beam emittance to be the
Courant-Snyder invariant, \( \epsilon_x \), for the chosen ellipse, as defined in Eq. (3.19).

The concept of RMS emittance, based on statistical moments of any particle
distribution, is a quantity that can effectively reflect the degradation of the beam
quality under the presence of nonlinear forces. For an arbitrary particle distribution
Figure 3.16: The $x - x'$ projection of the phase space of a particle beam with Gaussian Distribution.
3.4 Tranverse Emittance

\( \rho(x, x') \), the RMS emittance is defined as

\[
\epsilon_{x, \text{rms}} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}. \tag{3.20}
\]

Here \( \langle x^2 \rangle = \beta \epsilon_{x, \text{rms}} \) and \( \langle x'^2 \rangle = \gamma \epsilon_{x, \text{rms}} \) are the second moments, and \( \langle xx' \rangle = -\alpha \epsilon_{x, \text{rms}} \) is the covariance of the distribution \( \rho(x, x') \). For a beam with RMS emittance \( \epsilon_{x, \text{rms}} \), the RMS beam width is \( \sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \epsilon_{x, \text{rms}}} \) and the RMS beam divergence is \( \sigma'_x = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \epsilon_{x, \text{rms}}} \).

In the next two sections, we will discuss emittance measurement using two different methods: solenoid variation method and slit-wire scanner method.

3.4.1 Measurement using Solenoid Variation Method

The RMS beam size \( \sigma \) changes with the strength of the focusing solenoids. This relation is measured for different beam current and is plotted in Fig. 3.17. The two solenoids, SOL-U and SOL-PET, are optimized to locate the beam waist to near the wire scanner. The strength of SOL-U remains unchanged for the measurements. Recall from Chapter 3.1 that the error for these beam width measurements is about \( \pm 2 \% \). These relations can be used to reconstruct the transverse beam emittance. The method is discussed in the following.

Consider the RMS K-V envelope equation:

\[
\sigma'' + k_x(z) \sigma_x = \frac{K_{sc}}{2(\sigma_x + \sigma_y)} - \frac{\epsilon_{x, \text{rms}}^2}{\sigma_x^3} = 0,
\]

\[
\sigma'' + k_y(z) \sigma_y = \frac{K_{sc}}{2(\sigma_x + \sigma_y)} - \frac{\epsilon_{y, \text{rms}}^2}{\sigma_y^3} = 0.
\tag{3.21}
\]

where \( k_x(z) = k_y(z) = g^2(z) = \left( \frac{B_z(z)}{2B_0} \right)^2 \) is the solenoid focusing strength, and

\[
K_{sc} = \frac{2I_b}{\gamma^3 \beta^3 I_0}
\tag{3.22}
\]

is the generalized perveance. Here \( I_b \) is the beam current, \( \gamma \) is the Lorentz factor, \( \beta = \sqrt{1 - 1/\gamma^2} \), and \( I_0 = 4\pi \epsilon_0 mc^3/q \) is the characteristic current.
(a) A beam current of 4 mA.

(b) A beam current of 8 mA.

(c) A beam current of 12 mA.

(d) A beam current of 16 mA.

**Figure 3.17:** The measured RMS beam width is plotted as a function of the strength of SOL-PET. The two solenoids, SOL-U and SOL-PET, are optimized to locate the beam waist to near the wire scanner. The strength of SOL-U remains unchanged for the measurements.
3.4 Tranverse Emittance

It implies that the analysis is assumed to be in the Larmor rotating frame. The use of decoupled transfer matrix can be justified by the fact that our 50 keV proton beam is approximately a round beam. So we can decoupled Eq. (3.21) by setting $\sigma_x = \sigma_y$ and $\epsilon_x = \epsilon_y$. It can be shown that for a round beam with $\sigma_x = \sigma_y$ and $\epsilon_x = \epsilon_y$, under solenoidal focusing channel, beam envelope analysis in the two transverse planes is identical and decoupled, provided that we interpret the results in the Larmor rotating frame. A more detailed discussion on this topic is given in [17].

The solution to Eq. (3.21), the RMS beam width $\sigma_x(z)$ and $\sigma_y(z)$, can be solved by knowing the Twiss parameter $\beta$, $\alpha$, and the RMS emittance $\epsilon_{rms}$. Fig. 3.17 shows RMS beam width as a function of solenoid strength. We can used the mentioned three parameters to fit a parabola to the data shown in Fig. 3.17. RMS beam emittance $\epsilon_{rms}$ can then be determined.

The RMS emittance is plotted as a function of the beam current in Fig. 3.18. Note that the current in the plot is the proton beam current, which is 40% of the total beam current. The major source of uncertainty to this measurement is the drift in SOL-PET excitation current. A drift of 1 Amps in SOL-PET results in up to $\pm0.03$ mm-mrad or error. Analysis will be given in the next section when we compare the results from the two measurement methods.

3.4.2 Measurement using Slit-Wire Scanner Method

Another method that has been used for the HINS proton ion source emittance measurement is the slit-wire scanner method. A schematic drawing of the setup is shown in Fig. 3.19. The spatial distribution (distribution in $x$) is scanned by the slit while the angular distribution (distribution in $x'$) is scanned by the wire scanner located a distance $L$ downstream in the beamline. At every position $x$, the slit is held fixed to allow a portion of the beam to pass through. The particles in the portion (beamlet)
Figure 3.18: The normalized RMS emittance is plotted as a function of the beam current. The beam current in the plot is the proton current, which is about 40% of the total beam current.
reaching to the plane of the wire scanner travel in a straight line [18]. So the angular distribution at every position \( x \) of the beam can be measured by the wire scanner, forming density distribution function \( \rho(x, x') \) on the \( x - x' \) projection of the phase space.

The measured density distribution functions in both the \( x - x' \) and \( y - y' \) phase space planes for different beam current are shown in the 3D density plots in Fig. 3.20 and 3.21. In the figures, the island is the signal from the proton while the strip is the signal from the \( H_2^+ \) particle population (in white because the signal is huge and is out of the plotting range). As shown in the figures, the beam is convergent at the location of the slit. The beam size is large and the full width of the beam is about 18 mm in both planes for all current. The divergence of the beam increases with the beam current. At larger beam current, the beam appears as an "S" shape in the phase space. This is a sign of non-linearity in focusing/defocusing forces, which includes solenoidal field aberration and space charge force.

To compute the RMS emittance from the measured data, signal from \( H_2^+ \) must be subtracted. A comparison of the signal before and after background subtraction is
3. Beam Characterization for the Proton Ion Source

(a) A beam current of 4 mA.

(b) A beam current of 8 mA.

(c) A beam current of 12 mA.

(d) A beam current of 16 mA.

Figure 3.20: 3D plot of the measured beam distribution $\rho(x, x')$ using the slit-wire scanner assembly. The strip in white is signal from the $H^+_2$ particle population, which is huge and is out of the plotting range.
3.4 Tranverse Emittance

(a) A beam current of 4 mA.

(b) A beam current of 8 mA.

(c) A beam current of 12 mA.

(d) A beam current of 16 mA.

Figure 3.21: 3D plot of the measured beam distribution $\rho(y, y')$ using the slit-wire scanner assembly. The strip in white is signal from the $H^+_2$ particle population, which is huge and is out of the plotting range.
shown in Fig. 3.22. Data near the region of the strip is erased. It is then filled up with 2-D interpolation using the data in the surroundings.

After background subtraction, it is ready to compute the RMS emittance from the data. The cleaned signal is the intensity distribution function $\rho(x_i, x'_j)$. It is a $m \times n$ array storing the density information. The second moments can be calculated as

$$\langle x^2 \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i - \langle x \rangle)^2 \rho_{ij}}{I_{tot}}$$

$$\langle x'^2 \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (x'_j - \langle x' \rangle)^2 \rho_{ij}}{I_{tot}}$$

$$\langle xx' \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i - \langle x \rangle)(x'_j - \langle x' \rangle) \rho_{ij}}{I_{tot}}$$

where

$$\langle x \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} x_i \rho_{ij}}{I_{tot}}$$

$$\langle x' \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} x'_j \rho_{ij}}{I_{tot}}$$

are the arithmetic mean of the density function $\rho(x_i, x'_j)$. Here $I_{tot} = \sum_{i=1}^{m} \sum_{j=1}^{n} \rho_{ij}$ is the normalization factor. From the second moments, we can construct the RMS emittance

$$\epsilon_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}.$$  \hspace{1cm} (3.28)

The Twiss parameters can also be calculated [19],

$$\beta = \frac{\langle x^2 \rangle}{\epsilon_{x,rms}}, \quad \alpha = -\frac{\langle xx' \rangle}{\epsilon_{x,rms}},$$

$$\gamma = \frac{\langle x'^2 \rangle}{\epsilon_{x,rms}} = \frac{1 + \alpha^2}{\beta}.$$  \hspace{1cm} (3.29)

The cleaned signal is presented as density plot in Fig. 3.23 and 3.24. RMS ellipse defined by

$$\epsilon_{x,rms} = \gamma x^2 + 2\alpha xx' + \beta x'^2$$  \hspace{1cm} (3.30)
3.4 Tranverse Emittance

Figure 3.22: A comparison of the measurement data before and after background subtraction. Signal near the region of the strip in white is erased. It is then filled up by 2-D interpolation.
3. Beam Characterization for the Proton Ion Source

(a) A beam current of 4 mA.

(b) A beam current of 8 mA.

(c) A beam current of 12 mA.

(d) A beam current of 16 mA.

Figure 3.23: Phase space plot in the $x-x'$ plane. With the cleaned distribution function, the RMS beam emittance and the Twiss parameters can be calculated. Ellipses defined by the Twiss parameters are drawn in the figures.
3.4 Tranverse Emittance

(a) A beam current of 4 mA.

(b) A beam current of 8 mA.

(c) A beam current of 12 mA.

(d) A beam current of 16 mA.

**Figure 3.24:** Phase space plot in the $y - y'$ plane. With the cleaned distribution function, the RMS beam emittance and the Twiss parameters can be calculated. Ellipses defined by the Twiss parameters are drawn in the figures.
is drawn on the top of the phase space density plot.

The RMS emittance as a function of the beam current is compared to the result discussed in Chapter 3.4.1, where the RMS emittance is deduced from solenoid variation method (Fig. 3.18). The two sets of results are plotted together in Fig. 3.25. In the plot, proton beam current, which is about 40% of total beam current, is used. The major source of error for the slit-wire scanner measurement comes from the signal cleaning process. By changing the parameters used in data interpolation, an error of \( \pm 0.05 \text{ mm-mrad} \) is recorded. Analysis is given in the next section.

### 3.4.3 Result Analysis

In Fig. 3.25, results for emittance measurement using two methods, solenoid variation and slit-wire scanner, are compared. We observe that, up to measurement error, the two sets of results agree at lower proton current but diverge at higher current. In this section, we try to explain this observation.

First of all, it is important to realize that, although we plot them together, the two sets of results describe the beam emittance at two different locations along the beamline. In the solenoid variation method, we used K-V envelope equation to determine the RMS beam emittance at the entrance of the beamline, i.e. the proton ion source. Since space charge effect has been taken into account, the result can be considered as the emittance of the proton source, where space charge does not have enough time to affect the emittance. In the slit-wire scanner method, emittance was measured at the slit-wire scanner assembly, which was the exit of the beamline, which is about 1 m or 320 ns away from the proton source. It is reasonable to interpret the result from the solenoid variation method as the emittance performance of the proton source while the result from the slit method as the effect of space charge on emittance. According to Fig. 3.25, space charge effect led to emittance growth at higher beam
3.4 Tranverse Emittance

Figure 3.25: The beam emittance is plotted as a function of proton beam current. Results from measurement using the slit-wire scanner assembly and using solenoid variation method are compared.
current, which is reasonable.

3.5 Summary

The HINS proton ion source and low energy beam transport (LEBT) was successfully commissioned. It is capable to produce a 50 keV, 3 msec pulsed beam with a peak current greater than 20 mA (less than 50% proton) at 2.5 Hz (0.7% duty factor).

Beam studies started with interpreting the signal measured by the wire scanner. We successfully extracted the proton signal from a mixture of proton and $\text{H}_2^+$. The RMS beam width of the proton up to 2% error. Then, we performed a beam-calibration to the steering dipole magnets. The dipole magnets were calibrated and aligned. We then study transverse motion coupling due to solenoidal field by measuring beam rotation through solenoid. Measurement result matches with prediction from beam physics modelings and particle tracking simulations. Finally, transverse beam emittance were measured both directly (slit-wire scanner) and indirectly (solenoid variation). The result from solenoid variation method can be interpret as the emittance performance of the proton source as a function of beam current, since the emittance was determined at the source with space charge effect taken into account. For the slit method, beam emittance was measured at the slit, which is about 1 m or 320 ns away from the source. Space charge effect has enough time to cause emittance growth, which can be observed from the comparison of the two sets of results.

The results for our beam studies provide valuable information on the quality of the proton beam. This information is important in beam matching for LEBT to RFQ, beam steering control under solenoidal focusing channel, and etc.
Chapter 4

Bunch Shape Monitor

The HINS radio frequency quadrupole (RFQ) accelerates 50 keV proton/H\(^-\) beam to 2.5 MeV and bunches it at 325 MHz. The longitudinal beam quality of the 2.5 MeV beam is one of the important indicators to the performance of the RFQ. It is essential to develop a way to monitor the longitudinal quality of the 2.5 MeV beam, especially during the commissioning of the HINS RFQ.

The bunch shape monitor (BSM) is a high bandwidth instrumentation device that measures the longitudinal profile of a bunched proton/H\(^-\) beam. The Fermilab linac has three of them manufactured and installed (at various energy from 116 MeV to 401 MeV) during the Fermilab linac update [20]. They have been used to make measurement [21]. Recently, one of them was taken out from the Fermilab linac tunnel for possible HINS usage. In the following the basic idea of the BSM is described. Elements of the BSM, especially the RF deflector, are studied. Modifications and tests on the monitor required will also be discussed.
Figure 4.1: The basic idea for the operation of BSM. A proton bunch hitting a target wire, held at -10 kV, produces secondary electron emission. Passing through a collimator, the electron beam has a longitudinal profile similar to that of the proton beam. The deflecting field in the RF deflector maps the longitudinal profile into a spatial profile, which is projected onto the plane of an electron detector.
4.1 Principle of Operation

Fig. 4.1 shows a schematic drawing of the basic idea of the operation of BSM. The BSM uses secondary electron emission to detect the proton/H\(^-\) bunch shape. To make a measurement, the proton/H\(^-\) beam collide with a fixed target wire (held at -10 kV) to produce electron. Electron beam bunch being accelerated to 10 keV has a longitudinal profile which is related to that of the proton/H\(^-\) bunch. The electron beam passes through a collimator and reaches an RF deflector. The deflector oscillates at a voltage equal to \(V(t) = A \cos(\omega t + \phi)\), where \(\omega\) is an integer multiple of the beam bunching frequency and \(\phi\) is a controllable phase angle. The RF field will map the temporal (longitudinal) profile into a spatial profile, projected onto the plane of an electron detector. An electron multiplier tube located at the far end detects the intensity of the electron along the spatial profile.

The main components of the BSM include the target wire (tungsten), the actuator that drives the wire, the RF deflector, and the electron multiplier. Some of these components require modifications before the BSM can accommodate the HINS Linac.

4.2 Resolution of BSM

The HISN RFQ output has an estimated bunch length of \(\sigma_z \approx 2.5\) mm. At 2.5 MeV, the relativistic \(\beta\) is 0.073. Using \(\sigma_z = \beta c \sigma_t\), the bunch length in time domain is \(\sigma_t \approx 300\) ps, which is about 35 degree at 325 MHz. To measure such a beam, a 5% resolution suggests a bandwidth of tens of gigahertz, which is not easy to achieve for typical non-destructive devices.

The principal limitation, however, is due to a spatial spread of the charge field [22]. Fig. 4.2 shows a non-relativistic charged particle and its electric field distribution in a beam pipe. The field spans a length of \(L \approx 2R/\gamma\) on the beam pipe. The time
duration is

\[ \Delta t = \frac{L}{\beta c} = \frac{2R}{\gamma \beta c}. \]  

(4.1)

For a beam pipe with radius of 2 cm, \( \Delta t \approx 2 \text{ ns} \). This is almost seven times the proton bunch length of 300 ps.

On the other hand, the resolution can be achieved by the Fermilab BSM is estimated to be around 10 - 30 ps [23]. Consideration on resolution can be summarized into four main factors:

1. the time necessary for the secondary electrons to be ejected from the wire
2. the emittance of the secondary electron beam
3. the path length differences among the electrons
4. the strength of the RF deflector

The effect of the first three items is small compare to the last one. Here we discuss in more detail the effect on resolution due to the strength of the RF deflector.
4.2 Resolution of BSM

![Diagram of electron and RF deflector with notation](image)

**Figure 4.3:** Effect on BSM resolution due to the strength of the RF deflector. The resolution is defined as $\Delta \sigma_\phi / \sigma_{\phi, full} = w_s / 2dx$. To the first order, the resolution improves with a stronger deflecting field $E_x$.

Fig. 4.3 shows a schematic drawing explaining the effect of deflecting field to the BSM resolution. The resolution is defined as

$$\frac{\Delta \sigma_\phi}{\sigma_{\phi, full}} = \frac{w_s}{2dx} = \frac{w_s}{2L_T \theta_x}. \quad (4.2)$$

For the $i^{th}$ electron in the beam, the angle of deflection is given by

$$\theta_{x,i} = \frac{1}{B \rho} \int_{t_0}^{t_f} E_x(z, t) dt$$

$$= \frac{1}{B \rho} E_{x0} \int_{t_0}^{t_f} \sin(\omega_{rf} t + \phi_i) dt, \quad (4.3)$$

where $B \rho$ is the momentum rigidity of the electron, $E_{x0}$ is the peak electric field, $\omega_{rf}$ is the RF frequency, and $\phi_i$ is the phase of the $i^{th}$ electron relative to the RF phase. The limit of integration is given by $t_0 = -L_D/2\beta_c$ and $t_f = L_D/2\beta_c$. To keep the center of the beam undeflected, we require

$$\phi_{center} = 0 \quad (4.4)$$

$$\phi_{head/tail} = \pm \frac{\sigma_{\phi, full}}{2}. \quad (4.5)$$
Thus, the resolution can be written as

$$\frac{\Delta \sigma_\phi}{\sigma_{\phi, full}} = \frac{w_s}{2L_T \theta_{x, head}}. \quad (4.6)$$

Using Eq. (4.6), we can estimate the resolution and how does it relate to different parameters of the RF deflector. Fig. 4.4 shows the resolution as a function of the length of deflecting plates $L_D$. Here $\sigma_{\phi, full} = 300$ ps, electron energy $KE_e = 10$ keV, slit width $w_s = 1$ mm, and $L_T = 13$ inches are used in the calculation. Due to the fact that the transit time factor is dictated by the electron energy $KE_e$, for all values of peak field $E_{x0}$ the optimum length of deflecting plates ranges from 4 cm to 6 cm.

In Fig. 4.5, the resolution is plotted as a function of peak deflecting field $E_{x0}$. 

**Figure 4.4:** BSM resolution as a function of length of deflector.

Here $\sigma_{\phi, full} = 300$ ps, electron energy $KE_e = 10$ keV, slit width $w_s = 1$ mm, and $L_T = 13$ inches are used in the calculation.
Figure 4.5: BSM resolution as a function of peak deflecting field $E_{x0}$. Here $\sigma_{\phi,\text{full}} = 300$ ps, deflector length $L_D = 5$ cm, slit width $w_s = 1$ mm, and $L_T = 13$ inches are used in the calculation. Below 1% resolution, the economy of higher field reduces. A 1% resolution, or 3 ps, can be achieved with a peak field of 400 kV/m.
Here $\sigma_{\phi,\text{full}} = 300$ ps, deflector length $L_D = 5$ cm, slit width $w_s = 1$ mm, and $L_T = 13$ inches are used in the calculation. These are values measured for the BSM in hand. Below 1% resolution, the economy of higher field reduces. A 1% resolution, or 3 ps, can be achieved with a peak field of 400 kV/m.

4.3 RF Deflector

The RF deflector resonates at $n \times f_0$, where $f_0$ is the bunching frequency of the beam. The Fermilab linac has a frequency of 201.25 MHz below 116.5 MeV and 805 MHz beyond. The RF deflector was designed to operate at 805 MHz.

The HINS linac bunching frequency is 325 MHz. The first step to proceed is to study the properties of the deflector to find out the required modifications. A CST Microwave Studio (MWS) model of the RF deflector is built based on the real one (see Fig. 4.6). Fig. 4.7 shows a CST MWS model for the deflector and its components. The deflector is a coaxial like, half wavelength resonator. The copper arms, floated by
4.3 RF Deflector

Figure 4.7: CST MWS model for BSM RF deflector showing its components. The deflector is a coaxial like resonator. The copper arm, held by nylon support ring, is about $\lambda/2$ long. The frequency can be tuned by the adjustable endcap spacing.

nylon support rings, serves as inner conductor whose two ends are electrically open. The spacing at the two ends of the deflector is adjustable for frequency tuning by changing capacitive load. DC voltage can be applied to the copper arms for electron beam steering and focusing. Fig. 4.8 shows a perspective view and a snapshot of the electric field pattern in deflector.

4.3.1 Resonant Frequency

A series of RF measurement has been made for the deflector. Fig. 4.9 shows frequency measurement when the endcaps are adjusted such that the $\pi$-mode of the deflector
(a) A perspective view.  
(b) Side view with snapshot of electric field.

**Figure 4.8:** CST MWS model for BSM RF deflector.
4.3 RF Deflector

Figure 4.9: Frequency measurement of RF deflector using network analyzer. The endcaps are adjusted such that the \( \pi \)-mode of the deflector resonates at 650 MHz. The mode at 750 MHz is the 0-mode at which the field cancels at the center.

The discrepancy of the measurement and the simulation results reaches 30 MHz. This might due to the inaccuracy in determining the size of endcap spacing in the network analyzer measurement. In any case, the tuning range covers 650 MHz \( \times \) 325 MHz \( \times \) 2. It suggests that the RF deflector can be used by HINS without machining.
Figure 4.10: Resonant frequency versus endcap spacing. The discrepancy of the measurement and the simulation results reaches 30 MHz. This might due to the inaccuracy in determining the size of endcap spacing. The tuning range covers $650 \text{ MHz} = 325 \text{ MHz} \times 2$. 
Figure 4.11: Unloaded quality factor $Q_0$ measurement using network analyzer. The input power loop is adjusted to critically couple. Marker 1 points to the resonant frequency $f_0$. Marker 2 and 3, pointing at $f_1$ and $f_2$, track the points where the real and imaginary part of the impedance are equal. $Q_0$ is given by $f_0/(f_1 - f_2)$.

### 4.3.2 Quality Factor and Shunt Impedance

As we can see in the previous section, in order to tune the deflector to 650 MHz, the endcap spacing has to reduce to less than 1 mm. The increase in capacitive load might worsen the quality factor and the shunt impedance of the deflector. As RF power is “expensive”, it worths studying the these quantities before we put the deflector in use for HINS. A sample of unloaded quality factor $Q_0$ measurement using network analyzer is shown in Fig. 4.11.
For a certain endcap spacing, the input RF power loop is adjusted to critically coupling the power source and the deflector, as shown in the Figure. Marker 1 points to the resonant frequency $f_0$. Marker 2 and 3, pointing at $f_1$ and $f_2$, track the points where the real and imaginary part of the impedance are equal. These are called the half-power points [24]. The unloaded quality factor is given by

$$Q_0 = \frac{f_0}{f_1 - f_2}. \quad (4.7)$$

Both the unloaded and loaded quality factor are measured as a function of the frequency. The result is shown in Fig. 4.12. The unloaded quality factor $Q_0$ is twice the loaded quality factor $Q_L$ for all frequencies. This can be seen by considering

$$Q_0 = Q_L(1 + \beta), \quad (4.8)$$

where $\beta$ is the coupling factor. It is set to one for all frequencies by re-orienting the input power loop to critically couple. At 650 MHz, the loaded quality factor $Q_L$ is measured to be around 650, which is even higher than that at the Fermilab linac frequency of 805 MHz.

The shunt impedance is another important quantifier to the deflector. In our measurement, it is defined as

$$R_{sh} = \frac{V_p^2}{2P_d}, \quad (4.9)$$

where $V_p$ is the peak voltage between the two deflecting plates, and $P_d$ is the RF power dissipated in the resonator. We measure $R_{sh}$ as a function of frequency. Again, for all frequencies, the coupling $\beta$ is set to one. The result is shown in Fig. 4.13. At 650 MHz, the shunt impedance is measured to be 120 kΩ. This is also higher than that at 805 MHz.
Figure 4.12: Quality factor as a function of resonant frequencies. The unloaded quality factor $Q_0$ is twice the loaded quality factor $Q_L$ for all frequency. This is expected since the input power loop is readjusted to critically couple for every measurement.
Figure 4.13: Shunt impedance as a function of resonant frequency.
The shunt impedance is measured to be about 120 kΩ at 650 MHz.
4.4 Summary

The bunch shape monitor (BSM) is a high bandwidth instrumentation device that measures the longitudinal profile of a bunched proton/H⁻ beam. The Fermilab linac has three of them manufactured and installed (at various energy from 116 MeV to 401 MeV) during the Fermilab linac update. One of them was taken out from the Fermilab linac tunnel for possible HINS usage. Since the bunching frequency of HINS, 325 MHz, is different from that of the Fermilab linac of 805 MHz, one of the major components of the BSM, RF deflector, is subject to modifications before HINS usage.

It has been shown that the RF deflector has a tuning range from 600 MHz to 820 MHz, which covers $650 \text{ MHz} = 325 \text{ MHz} \times 2$. It is also shown that tuning the deflector to 650 MHz does no harm to quality factor nor shunt impedance. Thus the deflector can be used for HINS without any machining.

Other components, including the tungsten target wire, the actuator that drives the wire, and the electron multiplier tube, should be tested before the BSM being installed in the beamline. Examples of tests include electrostatic beam steering and focusing, and RF beam sweeping. These tests involve thermionically emitted electrons emitted from the wire when it is heated by a current.
4. Bunch Shape Monitor
Fermilab is considering an 8 GeV superconducting H$^-$ linac with the primary mission of enabling 2 MW beam power from the 120 GeV Fermilab Main Injector for a neutrino program. The High Intensity Neutrino Source (HINS) R&D program will build a new 30 MeV test linac to demonstrate new technologies for application in this high intensity hadron linac front-end.

Recently, the HINS proton ion source and low energy beam transport (LEBT) have been successfully commissioned. It produces a 50 keV, 3 msec pulsed beam with a peak current greater than 20 mA at 2.5 Hz. A wire scanner and a Faraday cup are temporarily installed at the exit of the LEBT to study the beam parameters.

The wire scanner was used to measure transverse beam profile and position of 50 keV proton beam. Gaussian fit to measurement data gives RMS beam width with accuracy up to 2%. Beam position measurement was used to cross-calibrate the strength and aligning steering dipole magnets successfully. Transverse motion coupling due to solenoidal field was studied by measuring beam rotation by solenoid. Various models were built to analyze beam rotation. Results from measurement and models match well with each other. This is valuable experience to the HINS
program since solenoid is the primary focusing channel in the design. Beam transverse emittance was measured in two different ways: solenoid variation method and slit-wire scanner method. There is a consistent discrepancy between the results from the two methods. We were able to explain this discrepancy by considering the conservation of phase space but not emittance in the presence of non-linear forces (due to space charge and solenoid aberration). Also, relation between emittance and beam current was qualitatively explained by space charge and curvature of plasma surface in proton source. However, due to inaccurate measurement of beam current, we were not able to give this relation a quantitative explanation. The properties of the 50 keV proton beam was studied and characterized successfully. The proton source/LEBT is ready to mate with the RFQ, which is under commissioning.

A beam diagnostic device, bunch shape monitor (BSM), was introduced. It is a high bandwidth instrumentation device that measures the longitudinal profile of a bunched proton/H\(^{-}\) beam. HINS instrumentation effort has planned to adapt a BSM from the Fermilab linac, whose bunching frequency is 805 MHz. The major challenge of the adaptation is the modification needed by the RF deflector of the BSM. A CST MWS model was built to study the properties of the deflector. We found that no machining is required to make the deflector to work at 650 MHz, which is twice the HINS bunching frequency. The modification also brings no harm to quality factor and shunt impedance of the deflector. Further test has been planned for other components of the BSM, including the tungsten target wire, the actuator that drives the wire, and the electron multiplier tube. Examples of tests include electrostatic beam steering and focusing, and RF beam sweeping. These tests involve thermionically emitted electrons emitted from the wire when it is heated by a current. It is foreseeable that the adaptation of the BSM by HINS will be successful.
Appendix A

Magnetic Scalar Potential and Solenoidal Field

The basic differential laws of magnetostatics are
\begin{align*}
\nabla \times \vec{B} &= \mu_0 \vec{J} \\
\nabla \cdot \vec{B} &= 0
\end{align*} (A.1)

In a region where the current density $\vec{J}$ is zero, $\nabla \times \vec{B} = 0$ permits the expression of the vector magnetic induction $\vec{B}$ as the gradient of a magnetic scalar potential, $\Phi$, given by [25]
\begin{equation}
\vec{B} = -\nabla \Phi. \tag{A.2}
\end{equation}

Inside a solenoid, in cylindrical coordinates whose $\hat{z}$ axis aligned with the axis of the solenoid, the magnetic field and the scalar potential have no azimuthal dependence. The scalar potential in Eq. (A.2) can then be expressed as
\begin{align*}
\Phi(r, \phi, z) &= \Phi(r, z) \\
&= \Phi_0(z) + r\Phi_1(z) + r^2\Phi_2(z) + \ldots \tag{A.3} \\
&= \sum_{i=0}^{\infty} r^i \Phi_i(z).
\end{align*}
Using $\nabla \cdot \vec{B} = 0$, Eq. (A.2) and A.3, we obtain, in cylindrical coordinates,

$$\nabla^2 \Phi(r, z) = \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \right] \sum_{i=0}^{\infty} r^i \Phi_i(z) = 0 \quad (A.4)$$

so that

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \sum_{i=0}^{\infty} r^i \Phi_i(z) = -\frac{\partial^2}{\partial z^2} \sum_{i=0}^{\infty} r^i \Phi_i(z). \quad (A.5)$$

This suggests that all the odd terms vanish and gives rise to a recursive relation

$$\Phi_i(z) = -\frac{1}{i^2} \Phi''_{i-2}(z) \quad \text{for} \quad i = 2, 4, 6 \ldots \quad (A.6)$$

where “prime” is the derivative with respect to $z$. The scalar potential in Eq. (A.3) can then be rewritten as

$$\Phi(r, z) = \Phi_0(z) - \frac{1}{4} r^2 \Phi''_0(z) + \frac{1}{64} r^4 \Phi''''_0(z) - \ldots$$

$$= \sum_{i=0, 2, \ldots}^{\infty} (-1)^{i/2} \frac{1}{(i!!)^2} r^i \Phi^{(i)}_0(z). \quad (A.7)$$

The magnet field can be obtained by substituting Eq. (A.7) into Eq. (A.10). Up to the third order, the magnet field is given by

$$B_z(r, z) = -\frac{\partial}{\partial z} \Phi(r, z)$$

$$= -\Phi'_0(z) + \frac{1}{4} r^2 \Phi'''_0(z) - \ldots \quad (A.8)$$

$$= B_{z0}(z) - \frac{1}{4} r^2 B_0''(z) + \ldots$$

and

$$B_r(r, z) = -\frac{\partial}{\partial r} \Phi(r, z)$$

$$= \frac{1}{2} r \Phi'_0(z) - \frac{1}{16} r^3 \Phi'''_0(z) + \ldots \quad (A.9)$$

$$= -\frac{1}{2} r B'_{z0}(z) + \frac{1}{16} r^3 B''_{z0}(z) - \ldots$$
where $B_{z0}(z) \equiv -\Phi'_0(z)$ is the magnet field along the solenoid axis. In cartesian coordinates, the magnetic field is given by

$$
B_x(x, y, z) = -\frac{1}{2} x B'_z(z) + \frac{1}{16} x(x^2 + y^2) B''_{z0}(z) - \ldots,
$$

$$
B_y(x, y, z) = -\frac{1}{2} y B'_z(z) + \frac{1}{16} y(x^2 + y^2) B''_{z0}(z) - \ldots,
$$

$$
B_z(x, y, z) = B_{z0}(z) - \frac{1}{4}(x^2 + y^2) B''_{z0}(z) + \ldots.
$$

(A.10)

So, provided that the 1-D axial field profile $B_{z0}(z) = B_z(0, 0, z)$ of a solenoid is known, the magnetic field near the axis of the solenoid can be calculated using Eq. (A.10).
A. Magnetic Scalar Potential and Solenoidal Field
Appendix B

Beam Dynamics of Linearly Coupled Systems

The linearized equations of motion that contain only the linear and linear coupling focusing elements can be written as

\[ x'' + k(z)x = q(z)y + 2g(z)y' + g'(z)y, \]
\[ y'' - k(z)y = q(z)x - 2g(z)x' - g'(z)x, \]  

where

\[ k(z) = -\frac{1}{B\rho} \frac{\partial B_y}{\partial x}, \]
\[ q(z) = \frac{1}{2B\rho} \left( \frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x} \right), \]
\[ g(z) = \frac{1}{2B\rho} B_z(z) \]

are the effective quadrupole, skew quadrupole, and solenoid strengths, respectively. The “prime” is the derivative with respect to “z”. In the absence of quadrupole and
skew quadrupole, the equations of motion become

\[
x'' - 2g(z)y' - g'(z)y = 0, \\
y'' + 2g(z)x' + g'(z)x = 0,
\]

or (compared to Eq. (3.17))

\[
x'' = \frac{B_z}{B\rho} y' - \frac{B_y}{B\rho}, \\
y'' = -\frac{B_z}{B\rho} x' + \frac{B_x}{B\rho},
\]

where we have substituted \( B_x = -\frac{y}{2} \frac{\partial B_z}{\partial z} \) and \( B_y = -\frac{x}{2} \frac{\partial B_z}{\partial z} \).

The system can be decoupled in the Larmor (rotating) frame. Let \( s = x + jy \), where \( j \) is the complex imaginary number. The coupled equations of motion Eq. (B.5) become

\[
s'' + j2g(z)s' + jg'(z)s = 0.
\]

Transforming the system into the rotating frame by introducing

\[
\tilde{s} = \tilde{x} + j\tilde{y} = se^{-j\phi(z)},
\]

where

\[
\phi(z) = -\int_0^z g(z')dz'
\]

is the azimuthal angle rotated by the particle about the axis of the solenoidal field (or half of the angle rotated about the axis of the particle helical trajectory \([15]\), (see Eq. (3.13)). The system is decoupled and the equation of motion becomes

\[
\tilde{s}'' + g^2(z)\tilde{s} = 0.
\]

In the rotating frame, both the horizontal and vertical planes are being focused by
the solenoid. The transfer matrix is given by

\[
\tilde{M} = \begin{pmatrix}
\cos \phi & \frac{1}{g} \sin \phi & 0 & 0 \\
-g \sin \phi & \cos \phi & 0 & 0 \\
0 & 0 & \cos \phi & \frac{1}{g} \sin \phi \\
0 & 0 & -g \sin \phi & \cos \phi \\
\end{pmatrix},
\]

or, in the laboratory frame,

\[
M = \begin{pmatrix}
\cos^2 \phi & \frac{1}{g} \sin \phi \cos \phi & -\sin \phi \cos \phi & -\frac{1}{g} \sin^2 \phi \\
-g \sin \phi \cos \phi & \cos^2 \phi & g \sin^2 \phi & -\sin \phi \cos \phi \\
\sin \phi \cos \phi & \frac{1}{g} \sin^2 \phi & \cos^2 \phi & \frac{1}{g} \sin \phi \cos \phi \\
-g \sin^2 \phi & \sin \phi \cos \phi & -g \sin \phi \cos \phi & \cos^2 \phi \\
\end{pmatrix},
\]

where \( \phi = gL = B_2L/2B_\rho \) is the angle of rotation about the solenoid axis when the particle traverses the solenoid with a length \( L \).
B. Beam Dynamics of Linearly Coupled Systems
Appendix C

The Kapchinskiy-Vladimirskiy Envelope Equation

Here we derive the Kapchinskiy-Vladimirskiy (K-V) envelope equation following the work in Lee [14], Wangler [16] and Lund [17]. Consider a beam with a transverse beam cross section varying along the longitudinal coordinate $z$ that remains elliptical. The envelope of the beam has radii $r_x(z)$ and $r_y(z)$. The charge distribution is centered at $x = y = 0$ with uniform space charge within the ellipse $(x/r_x)^2 + (y/r_y)^2 = 1$, and zero outside. Individual particles within this uniform density elliptical beam follow the equations of motion

$$
x'' + k_x(z) x - \frac{2K_{sc}}{(r_x + r_y)r_x} x = 0, \\
y'' + k_y(z) y - \frac{2K_{sc}}{(r_x + r_y)r_y} y = 0,$$

where $k_x(z) = -k_y(z) = -\frac{1}{B_p} \frac{\partial B_z}{\partial x}$ is the quadrupole focusing strength, and

$$K_{sc} = \frac{2I_b}{\gamma^3 \beta^3 I_0}$$

is the generalized perveance. Here $I_b$ is the beam current, $\gamma$ is the Lorentz factor, $\beta = \sqrt{1 - 1/\gamma^2}$, and $I_0 = 4\pi \varepsilon_0 mc^3/q$ is the characteristic current. Performing Floquet
transformation of the linear K-V-Hill equation

\[ x = w_x(z)e^{j\psi_x(z)} \quad \text{and} \quad y = w_y(z)e^{j\psi_y(z)}, \quad (C.3) \]

we obtain

\[
\begin{align*}
    w''_x + k_x(z)w_x - \frac{2K_{sc}}{(r_x + r_y)r_x}w_x - \frac{1}{w_x^3} &= 0, \\
    w''_y + k_y(z)w_y - \frac{2K_{sc}}{(r_x + r_y)r_y}w_y - \frac{1}{w_y^3} &= 0,
\end{align*}
\]

(C.4)

here we have used \( \psi'_x = 1/w_x^2 \) and \( \psi'_y = 1/w_y^2 \). Relating the envelope radii, \( r_x \) and \( r_y \), to the betatron amplitude function, \( \beta_x \) and \( \beta_y \), and the beam emittance, \( \epsilon_x \) and \( \epsilon_y \),

\[
\begin{align*}
    r_x &= \sqrt{\beta_x \epsilon_x} = w_x\sqrt{\epsilon_x}, \\
    r_y &= \sqrt{\beta_y \epsilon_y} = w_y\sqrt{\epsilon_y},
\end{align*}
\]

(C.5)

the K-V envelope equations are given by

\[
\begin{align*}
    r''_x + k_x(z)r_x - \frac{2K_{sc}}{r_x + r_y} - \frac{\epsilon_x^2}{r_x^3} &= 0, \\
    r''_y + k_y(z)r_y - \frac{2K_{sc}}{r_x + r_y} - \frac{\epsilon_y^2}{r_y^3} &= 0.
\end{align*}
\]

(C.6)

These equations can be extended to describe the RMS beam dynamics by considering that, for a uniform density elliptical beam, the envelope radii are related to the second moments of the distribution, \( \langle x^2 \rangle \) and \( \langle y^2 \rangle \), by

\[
\begin{align*}
    r_x &= 2\sqrt{\langle x^2 \rangle} = 2r_{x,rms} \equiv 2\sigma_x, \\
    r_y &= 2\sqrt{\langle y^2 \rangle} = 2r_{y,rms} \equiv 2\sigma_y,
\end{align*}
\]

(C.7)

so that

\[
    \epsilon_x = 4\epsilon_{x,rms} \quad \text{and} \quad \epsilon_y = 4\epsilon_{y,rms}.
\]

(C.8)

We obtain the RMS K-V envelope equations

\[
\begin{align*}
    \sigma''_x + k_x(z)\sigma_x - \frac{K_{sc}}{2(\sigma_x + \sigma_y)} - \frac{\epsilon_{x,rms}^2}{\sigma_x^3} &= 0, \\
    \sigma''_y + k_y(z)\sigma_y - \frac{K_{sc}}{2(\sigma_x + \sigma_y)} - \frac{\epsilon_{y,rms}^2}{\sigma_y^3} &= 0.
\end{align*}
\]

(C.9)
These equations were first derived by Kapchinskiy and Vladimirskiy [26] for a stationary uniform beam in a quadrupole focusing transport lattice. Solving the K-V envelope equations is equivalent to finding the betatron amplitude function in the presence of the space-charge force. Although for beams with nonuniform charge density the beam emittance does not preserve due to the presence of the nonlinear self-field forces [27], results discovered by Lapostolle [28] and Sacherer [29] suggest that the RMS K-V envelope equations Eq. (C.9) are valid for all density distributions with elliptical symmetry.
C. The Kapchinskiy-Vladimirskiy Envelope Equation
Appendix D

Beam Tranverse Emittance

D.1 Phase Space and The Courant-Snyder Invariant

Beam emittance is an important quantifier of beam quality. It is a measure to the coherence property of the beam: the degree to which the beam particles have nearly the same coordinates as the reference particle [16]. Consider a particle beam. Each of the particles in a beam is described by three pairs of position-momentum coordinates. At every instance, each particle is represented by a single point in the six dimensional phase space volume. A beam can then be expressed as a collection of points in the three projection of the phase space volume, namely, $x - p_x$, $y - p_y$ and $z - p_z$, where $x$, $y$ and $z$ are the position coordinates, $p_x$, $p_y$ and $p_z$ are the momentum coordinates. In practice, it is more convenient to measure the divergence of a beam. So the unnormalized phase space projection, $x - x'$, $y - y'$ and $\phi - \Delta W$ are used instead. Here $x' = dx/dz$ and $y' = dy/dz$ are the divergence angles, $\phi$ is the phase and $\Delta W$ is the energy variable.

In most accelerators, linear focusing forces dominate. With linear focusing, ellipti-
cal distributions in phase space remain elliptical [16]. The trajectory of each particle in the phase space projection traces out an ellipse, called trajectory ellipse, defined by
\[
\epsilon_x = \gamma x^2 + 2\alpha xx' + \beta x'^2. \tag{D.1}
\]
This is sometimes called the Courant-Snyder invariant. Here \(\gamma, \alpha\) and \(\beta\) are the Twiss parameters with the normalization \(\gamma \beta - \alpha^2 = 1\). The phase space area enclosed by the ellipse is equal to \(\pi \epsilon_x\).

For a matched beam, the phase space isodensity contours of the beam are concentric and geometrically similar to the trajectory ellipses of the particles [16]. By choosing a particular density contour that encloses a certain portion of the total number of particles, one can define the beam emittance to be the Courant-Snyder invariant for the chosen ellipse, as defined in Eq. (D.1) (see Fig. D.1). The emittance of a beam is conserved when Liouville's theorem is satisfied in the six dimensional phase space and when the forces in the three orthogonal directions are uncoupled [16]. A detailed discussion on the topic is given in Lawson [30].

**D.2 RMS Emittance**

The elliptical shape of the phase space contours can be distorted when, for example, nonlinear forces are present. The calculation of the emittance defined by these contours becomes ambiguous. The phase space area occupied by the beam does not reflect the degrade of the beam quality due to dilution of phase space density. The concept of RMS emittance, based on statistical moments of any particle distribution, is a quantity that can effectively reflect the degradation of the beam quality under the presence of nonlinear forces.

Consider a beam described by an arbitrary normalized distribution \(\rho(x, x')\) on the \(x - x'\) projection of the unnormalized phase space. The first moments of the
Figure D.1: The $x - x'$ projection of the phase space of a particle beam with Gaussian Distribution.
D. Beam Tranverse Emittance

The transverse emittance distribution are given by

\[ \langle x \rangle = \int x \rho(x, x') dx dx' \quad \text{and} \quad \langle x' \rangle = \int x' \rho(x, x') dx dx'. \]  
(D.2)

The \( n^{th} \) moments can be calculated using

\[ \langle x^n \rangle = \int (x - \langle x \rangle)^n \rho(x, x') dx dx', \]
\[ \langle x'^n \rangle = \int (x' - \langle x' \rangle)^n \rho(x, x') dx dx', \]  
(D.3)

The mix term, called the covariance, is given by

\[ \langle xx' \rangle = \int (x - \langle x \rangle)(x' - \langle x' \rangle) \rho(x, x') dx dx'. \]  
(D.4)

We define the RMS phase space ellipse

\[ \epsilon_{x,rms} = \gamma x^2 + 2\alpha xx' + \beta x'^2. \]  
(D.5)

and require that

\[ \langle x^2 \rangle = \beta \epsilon_{x,rms} \quad \text{and} \quad \langle x'^2 \rangle = \gamma \epsilon_{x,rms}. \]  
(D.6)

The RMS emittance is defined by the second moments, sometimes called the variance, and the covariance, given by

\[ \epsilon_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}. \]  
(D.7)

D.3 The Sigma Matrix

Sometimes it is convenient to use the sigma matrix to describe the Courant-Snyder invariant and the state of the beam ellipse. The sigma matrix is defined as

\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'^2} \end{pmatrix}, \]  
(D.8)
where $\sigma_x$ and $\sigma_{x'}$ are the beam width and beam divergence, $\sigma_{xx'}$ is the correlation. The emittance is given by

$$\epsilon = \sqrt{\det \sigma} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}. \quad (D.9)$$

The sigma matrix can also be expressed by the Twiss parameters,

$$\sigma = \epsilon \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}. \quad (D.10)$$

This suggests that the beam width and beam divergence are given by

$$\sigma_x = \sqrt{\epsilon \beta} \quad \text{and} \quad \sigma_{x'} = \sqrt{\epsilon \gamma}. \quad (D.11)$$

Representation of the sigma matrix using Twiss parameters also makes it explicit that the evolution of the sigma matrix follows the beam transfer matrix:

$$\sigma(z_2) = M(z_2|z_1)\sigma(z_1)M(z_2|z_1)^\dagger, \quad (D.12)$$

where $M(z_2|z_1)$ is the transfer matrix that takes the beam from $z_1$ to $z_2$. 
Appendix E

Methods for Transverse Emittance Measurement

The determination of emittance is based on profile measurements. Two different methods have been used to determine the emittance of the HINS proton ion source. Here we follow the discussion in Forck [31] and describe the basics of these methods.

E.1 Solenoid Variation

The emittance can be determined by fitting the beam envelope, measured at one location with different focusing conditions. Consider the $\sigma$ matrix

$$
\sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{12} & \sigma_{22}
\end{pmatrix} = \begin{pmatrix}
\sigma_x^2 & \sigma_{xx'} \\
\sigma_{xx'} & \sigma_{x'}^2
\end{pmatrix},
$$

(E.1)

where $\sigma_x$ and $\sigma_{x'}$ are the beam width and beam divergence, $\sigma_{xx'}$ is the correlation. The emittance is defined as

$$
\epsilon = \sqrt{\det \sigma} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}.
$$

(E.2)
In the RMS definition, it is equal to the phase space area (times a factor of $1/\pi$) enclosed by the Courant-Snyder ellipse of the RMS particle of the beam and is invariant if the beamline comprises only linear elements. The beam RMS emittance can thus be measured by measuring or experimentally deriving the parameters $\sigma_{11}$, $\sigma_{12}$ and $\sigma_{22}$. Since the $\sigma$ matrix is a representation of the beam ellipse in the phase space, it evolves along the beam path. Its transformation is given by

$$\sigma(z_2) = M(z_2|z_1)\sigma(z_1)M(z_2|z_1)^\dagger,$$  \hspace{1cm} (E.3)

where $M(z_2|z_1)$ is the transfer matrix that takes the beam from $z_1$ to $z_2$.

To measure the beam emittance at $z_1$, we need a solenoid whose entrance is placed at $z_1$ and a profile monitor, e.g. wire scanner, at $z_2$. In this case, $M(z_2|z_1)$ is composed of the transfer matrices for a drift space and a solenoid. Due to the fact that the beam coming from the ion source is highly axial symmetric, the decoupled transfer matrix in the Larmor rotating frame is a good approximate to the motion of the beam in both the $x$ and the $y$ plane. The transfer matrix is

$$M(z_2|z_1) = M_{\text{drift}}M_{\text{solenoid}}$$

$$= \left( \begin{array}{cc} 1 & L \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \cos \phi & \frac{1}{g} \sin \phi \\ -g \sin \phi & \cos \phi \end{array} \right)$$

$$= \left( \begin{array}{cc} \cos \phi - gL \sin \phi & \frac{1}{g} \sin \phi + L \cos \phi \\ -g \sin \phi & \cos \phi \end{array} \right),$$  \hspace{1cm} (E.4)

where $L$ is the length of the drift space, $\phi = gl$ is the focusing angle, $l$ is the length of the solenoid, and $g = \frac{eB_z(z)}{2p}$ is the solenoid field strength. In the thick lens approximation, $l_{\text{sol}}$ should be replaced by the effective length of the solenoid while $B_z(z)$ should be replaced by the peak field $B_{z0}$. Here the decoupled transfer matrix $M_{\text{solenoid}}$ is used. The analysis is assumed to be in the Larmor rotating frame. It can be shown that for a round beam with $\sigma_x = \sigma_y$ and $\epsilon_x = \epsilon_y$, under solenoidal focusing channel,
E.2 Slit-Wire Scanner Method

beam envelope analysis in the two transverse planes is identical and decoupled, provided that we interpret the results in the Larmor rotating frame. A more detailed discussion on this topic is given in [17].

The RMS beam width $\sigma_x$ can be measured at $z_2$ using the profile monitor. Using Eq. (E.3) and Eq. (E.4), the measured beam width $\sigma_x(z_2)$ is related to the elements of the $\sigma$ matrix at $z_1$ by

$$\sigma_x^2(z_2) = \sigma_{11}(z_2) = M_{11}^2 \sigma_{11}(z_1) + 2M_{11} M_{22} \sigma_{12}(z_1) + M_{12}^2 \sigma_{22}(z_1).$$

(E.5)

This is a linear equation for the three unknown $\sigma$ matrix elements $\sigma_{ij}(z_1)$ at the entrance of the solenoid.

Since the transfer matrix $M$ depends on the adjustable solenoid field strength $g$, the RMS beam width $\sigma_x(z_2)$ can be measured as a function of $g$:

$$\sigma_{11}(z_2, g_1) = M_{11}^2(g_1) \sigma_{11}(z_1) + 2M_{11}(g_1) M_{22}(g_1) \sigma_{12}(z_1) + M_{12}^2(g_1) \sigma_{22}(z_1)$$

$$\sigma_{11}(z_2, g_2) = M_{11}^2(g_2) \sigma_{11}(z_1) + 2M_{11}(g_2) M_{22}(g_2) \sigma_{12}(z_1) + M_{12}^2(g_2) \sigma_{22}(z_1)$$

... \hspace{8cm} (E.6)

The solution of this system of equations are the values of the $\sigma$ matrix, $\sigma_{11}(z_1)$, $\sigma_{11}(z_1)$ and $\sigma_{11}(z_1)$, at the entrance of the solenoid, $z_1$. The size and orientation of the phase space ellipse are determined (recall that $\sigma_{ij}$ are the Twiss parameters normalized by $\epsilon$). Three measurements guarantee a unique solution. Usually, for error analysis purpose, more than three measurements are required. The beam matrix $\sigma(z_1)$ can be obtained by least square fitting and thus the beam emittance $\epsilon$ can be calculated.

E.2 Slit-Wire Scanner Method

The other method that has been used for the HINS proton ion source emittance measurement is the slit-wire scanner method. A schematic drawing of the setup is
shown in Fig. E.1. The spatial distribution (distribution in $x$) is scanned by the slit while the angular distribution (distribution in $x'$) is scanned by the wire scanner located a distance $L$ downstream in the beamline. At every position $x$, the slit is held fixed to allow a portion of the beam passing through. The particles in the portion (beamlet) reaching to the plane of the wire scanner travel in a straight line [18]. So the angular distribution at every position $x$ of the beam can be measured by the wire scanner, forming a plot on the $x - x'$ projection of the phase space.

For a measurement where the wire scanner scans $n$ steps for every slit position, up to a total of $m$ steps for the slit, the measured data is a $m \times n$ intensity distribution array $\rho(x_i, x'_j)$. The second moments can be calculated as

$$\langle x^2 \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i - \langle x \rangle)^2 \rho_{ij}}{I_{tot}}, \quad (E.7)$$

$$\langle x'^2 \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (x'_j - \langle x' \rangle)^2 \rho_{ij}}{I_{tot}}, \quad (E.8)$$

$$\langle xx' \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i - \langle x \rangle)(x'_j - \langle x' \rangle) \rho_{ij}}{I_{tot}}, \quad (E.9)$$
where

\[ \langle x \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} x_i \rho_{ij}}{I_{tot}}, \quad (E.10) \]

\[ \langle x' \rangle = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} x'_j \rho_{ij}}{I_{tot}}, \quad (E.11) \]

are the arithmetic mean of the density function \( \rho(x_i, x'_j) \). Here \( I_{tot} = \sum_{i=1}^{m} \sum_{j=1}^{n} \rho_{ij} \) is the normalization factor. From the second moments, we can construct the RMS emittance

\[ \epsilon_{x,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}. \quad (E.12) \]

Also, the Twiss parameters can be calculated [19],

\[ \beta = \frac{\langle x^2 \rangle}{\epsilon_{x,rms}}, \quad \alpha = -\frac{\langle xx' \rangle}{\epsilon_{x,rms}}, \]

\[ \gamma = \frac{\langle x'^2 \rangle}{\epsilon_{x,rms}} = 1 + \frac{\alpha^2}{\beta}. \quad (E.13) \]
E. Methods for Transverse Emittance Measurement
Bibliography


