Formation of a self-magnetic cusp in a highly bunched relativistic annular electron beam

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The formation of a self-magnetic cusp that develops in a highly bunched, relativistic, annular electron beam as the beam current approaches a critical current limit is demonstrated. The self-magnetic cusp is calculated within the framework of a fluid equilibrium model for a beam of periodic, azimuthally symmetric, relativistic bunched annular disks propagating in a perfectly conducting cylindrical pipe with a uniform magnetic focusing field. Magnetic cusp formation may play an important role in determining the operational current limit of high-current bunched relativistic annular beam experiments. © 2006 American Institute of Physics.

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I. INTRODUCTION

It is well known that two fundamental limits exist for high-current unbunched relativistic electron beams propagating in an external magnetic focusing field, i.e., the space-charge-limiting current and the limiting current for general laminar flow equilibria. The space-charge-limiting current arises from the energy depression that an unbunched electron beam undergoes in transitioning from a free space region (e.g., a cathode) to one bounded by a grounded, perfectly conducting cylinder in the presence of an infinite (or strong) external magnetic field. The general laminar flow current limit is a steady-state equilibrium limit for unbunched beams in uniform external magnetic focusing, which includes the effects of self-electric and self-magnetic fields, as well as the initial kinetic energy and canonical angular momentum of the beam at the cathode.

In this paper, we show the existence of an entirely new current limit, namely, the self-magnetic cusp limit, for a highly bunched, relativistic annular electron beam that is immersed in a finite external magnetic focusing field and propagates in a perfectly conducting cylindrical pipe. In particular, this new current limit is applicable to highly bunched, relativistic annular beams whose annuli are close to the perfectly conducting cylinder, which are often utilized in high-power microwave devices such as relativistic klystrons and relativistic backward wave oscillators. For such beams, there are three key parameters that distinguish various regimes in the parameter space. They are \( \mu = \omega a / c \), \( \alpha = 2 \pi \text{ar} / L_{\text{rest}} \), and \( \lambda = 2 N_e e^2 / m_e c \omega a^2 \), where \( \omega_e = eB/m_e c \) is the electron cyclotron frequency for an external uniform magnetic focusing field \( B_{\text{ext}} = B_{z} \hat{z} \), \( \alpha \) is the radius of the conducting pipe, \( L_{\text{rest}} \) is the spacing of the bunches in the longitudinal rest frame, and \( N_e \) is the number of electrons per unit length. To be more specific, \( \mu \) is proportional to the applied magnetic field and \( \lambda \) is proportional to the beam current. The general laminar flow equilibria model for unbunched beams is recovered by taking the limit \( \alpha \to \infty \). In the previous work, \( \lambda \to \infty \) the canonical angular momentum is specified as opposed to specifying the density distribution in the present model.

In the space-charge-limiting current model, \( \mu \to \infty \) and \( \alpha \to \infty \), since the magnetic field is assumed to be infinite and the beam is unbunched. We note, however, that the space-charge-limiting current model is fundamentally different than the present model. The space-charge-limiting current is due to the reduction of longitudinal beam energy when the beam transitions free space to a finite conductor boundary. The present model and corresponding current limit is based entirely on transverse beam confinement.

For a bunched beam with a finite \( \alpha \), the transverse rotation can be comparable to the speed of light. This is because the annular beam radius is comparable to \( a \) and beam rotation frequency may be of the order of \( \omega_e / 2 \). While the present model can be used to investigate the entire regime with \( 0 < \mu < \infty \) and \( 0 < \alpha < \infty \), we only focus on the experimental regime of highly bunched beams with \( \alpha \sim 1.0 \) in the remainder of this paper.

Two possible current limits may arise for bunched annular beams within the framework of this model. One limit is due to the intersection of the fast and slow equilibrium rotation solutions, which occurs for \( \mu \ll 1 \). This limit is similar to a limit that we reported in an earlier paper on nonrelativistic bunched annular beam equilibria where the self-magnetic field was neglected. The second current limit is due to the formation of a diamagnetic self-magnetic cusp within the beam profile, which is strong enough to completely negate the external magnetic focusing field at a point inside of the beam, leading to a loss of beam confinement. The self-magnetic cusp will only occur when the beam is bunched and moreover will only be present when the beam is rotating relativistically (\( \mu \gg 1 \)).

The paper is organized as follows. In Sec. II, we formulate the relativistic fluid equilibrium model for bunched annular beams propagating in a perfectly conducting pipe. In
Sec. III, we show the numerical methods for which the fluid momentum equation is solved, as well as the numerical results that demonstrate both the self-magnetic cusp and the fast-slower merger. In Secs. IV and V, we provide a discussion and summary of the results in this paper.

II. FORMULATION OF THE EQUILIBRIUM MODEL

The present model is a relativistic extension of the bunched bunch model presented in Ref. 11. As in Ref. 11, we assume that the beam is composed of a periodic bunch train with spacing \( L \), where each bunch is an azimuthally symmetric charged disk with zero longitudinal thickness. The beam is propagating within a perfectly conducting pipe of radius \( a \) with a speed \( V_z \). An external uniform magnetic field \( B = B_0 \hat{e}_z \) provides confinement in the transverse direction. Each bunch is assumed to be in steady-state transverse equilibrium, i.e., the external magnetic field provides confinement against the repulsive self-electric field, as well as the self-magnetic field. Figure 1 illustrates the model.

The density and velocity distribution functions for the beam are given by

\[
n(r,t) = N_b \sigma(r) \sum_{k=-\infty}^{\infty} \delta(z - V_z t - kL),
\]

\[
V(r,t) = V_z \hat{e}_z + V_0(r) \hat{e}_\theta + V_\theta \hat{e}_\phi,
\]

where \( N_b \) is the number of particles in a bunch, \( \sigma \) contains the radial dependence in the bunch density, and \( \delta \) is the Dirac delta function. Equation (1) yields the normalization condition \( 2\pi \int d\sigma \sigma = 1 \).

The distribution functions in Eqs. (1) and (2) satisfy the continuity equation

\[
\frac{\partial n(r,t)}{\partial t} + \nabla \cdot [n(r,t)V(r,t)] = 0,
\]

which yields \( \sigma(r \partial \sigma V_z) / \partial r \equiv 0 \), i.e., \( r \partial \sigma V_z \) is a constant. Since the beam density at the wall is zero, \( \partial(n \rho^2 V_z) / \partial r \equiv 0 \), it follows that the radial fluid velocity is also zero, i.e.,

\[
V_r = 0.
\]

In order to simplify the analysis, we perform a Lorentz transformation into the longitudinal comoving frame, i.e., the beam rest frame. The only nonzero bunch fluid velocity is in the azimuthal direction, which is given by

\[
V^\text{rest}(r) = \frac{V^\text{rest}_0 \hat{e}_\theta}{r} = \gamma V^\text{rest}_0 \hat{e}_\theta,
\]

where \( \gamma = (1 - V_z^2 / c^2)^{-1} \). We note that in this frame the bunch spacing is \( L^\text{rest} = \gamma L \).

The equilibrium momentum equation within the plane of each bunch is given by

\[
- \frac{m_e (V^\text{rest}_0)^2 \hat{e}_\theta}{r \sqrt{1 - (V^\text{rest}_0/c)^2}} = -e \left[ E^\text{self} + \frac{V^\text{rest}}{c} \times (B_0 \hat{e}_z + B^\text{self}) \right],
\]

where the left-hand side of Eq. (3) represents the relativistic centripetal force and \( E^\text{self} \) and \( B^\text{self} \) are the self-generated electric and magnetic fields in the longitudinal rest frame of the beam.

As shown in Ref. 11, \( E^\text{self} \) may be computed by expanding \( \sigma(r) \) in terms of a Fourier-Bessel series. In particular, if \( \sigma \) is a sufficiently well-behaved function, i.e., piecewise continuous in the region \( 0 \leq r \leq a \), then it may be written as

\[
\sigma(r) = \sum_{m=1}^{\infty} \sigma_m J_0(j_m r / a),
\]

where \( J_m(x) \) is the \( m \)th order Bessel function of the first kind, \( J_m(l) \) is the \( n \)th positive zero of \( J_m(x) \), and \( \{\sigma_m\} \) is the set of expansion coefficients. By utilizing a Green’s function technique that ensures the boundary condition \( E^\text{self}_{\text{pipe}} = 0 \), the self-electric field in the plane of the bunch was found to be \( 11 \)

\[
E^\text{self} = -2\pi N_b e \sum_{m=1}^{\infty} \sigma_m J_1(j_m r / a) \text{coth}(j_m \pi a / \alpha) \hat{e}_\theta,
\]

where \( \alpha = 2\pi a / L^\text{rest} \).

In the present analysis, the self-magnetic field is generated from the relativistic transverse bunch currents

\[
J^\text{rest} = -N_b e \kappa(r) \sum_{k=-\infty}^{\infty} \delta(z - kL) e_{\phi},
\]

where

\[
\kappa(r) = \sigma(r) V^\text{rest}_0(r).
\]

Surface currents on the conductor wall provide the magnetic boundary condition \( B^\text{self}_{\text{wall}} = 0 \). The presence of the conductor also provides a second effect, which is to impose a constancy constraint on the total self-magnetic flux through any transverse slice, i.e.,

\[
2\pi \int B^\text{self}_z(r,z) r dr = \text{const} = 0.
\]

In the case of a high-power microwave source, the particle beam will not be rotating at the cathode and hence will not be generating any magnetic field in the \( z \) direction. Moreover, since the beam pulse is short (\( \sim 1 \mu s \)) compared to the transverse magnetic diffusion time of the pipe, which is typically copper, \( \tau_B \approx 4\pi \mu_a^2 c^2 / 1.0 - 10.0 \text{ ms} \), the mag-
netic flux should be conserved as the beam evolves from the unbunched state to a fully bunched state. Therefore, the constant in Eq. (10) is precisely zero.

The self-magnetic field can be calculated by expanding $\kappa(r)$ in a Dini series, namely,

$$k(r) = \sum_{m=1}^{\infty} \kappa_m J_0(j_{0,m}r/a). \quad (11)$$

Note that by symmetry of the system, $B_{self}$ will only have a longitudinal component in the plane of the bunch. It will be shown in the Appendix that the expression for the magnetic self-field in the plane of the bunch is given by

$$B_{self} = \left\{ -2\pi N_e e c \sum_{m=1}^{\infty} \kappa_m J_0(j_{0,m}r) \coth(\pi j_{0,m}/\alpha) \\
+ 4\alpha^2 N_e e c \sum_{m=1}^{\infty} \kappa_m n J_1(j_{0,m}r) I_0(n\alpha r) \right\} \hat{e}_z, \quad (12)$$

where $\alpha = r/a$ and $I_m(x)$ is the $m$th modified Bessel function of the first kind.

Substituting Eqs. (8) and (12) into Eq. (6) yields the equilibrium force balance equation in the dimensionless form

$$\frac{\langle \dot{\omega}^2 \rangle}{\mu \nu \hat{r}} = \ddot{\omega} - \frac{\lambda}{\nu} \sum_{m=1}^{\infty} \tilde{\sigma}_m J_1(j_{0,m}r) \coth(\pi j_{0,m}/\alpha) \\
- \lambda \alpha \dot{\omega} - 2\alpha \sum_{m=1}^{\infty} \tilde{\kappa}_m n J_1(j_{0,m}r) I_0(n\alpha r) \\
- 2\alpha^2 \sum_{m=1}^{\infty} \tilde{\kappa}_m n J_1(j_{0,m}r) I_0(n\alpha r) \hat{e}_z. \quad (13)$$

In general, for a given $\sigma(r)$ and a set of values for $\alpha$, $\mu$, and $\lambda$, Eq. (13) yields two solutions for the normalized beam rotation $\hat{\sigma}(r)$, namely, slow and fast rotation solutions. However, since $B_{self}$ in Eq. (12) implicitly depends on $\kappa(r) = r\sigma(r)\omega(r)$ or equivalently, $\omega(r)$ because $\sigma(r)$ is fixed, then typically Eq. (13) can only be solved using an iterative numerical scheme. The numerical technique that we employ to solve Eq. (13) will be discussed explicitly in the next section.

We finish this section by mentioning two important limits for Eq. (13): the unbunched beam limit ($L_{rest} \rightarrow 0$, $\alpha \rightarrow \infty$, $N_j/L_{rest} = \text{const}$) and the single-bunch limit (rest $\rightarrow \infty$, $\alpha \rightarrow 0$).

In the limit of an unbunched beam, $\coth(j_{0,m}\pi/\alpha) \rightarrow \alpha/j_{0,m}\pi$ and the factor $I_0(n\alpha r)/I_1(n\alpha) \rightarrow e^{-n\alpha(1-r)/\hat{r}}$ goes to zero exponentially fast. Hence, Eq. (13) in the unbunched beam limit is

$$\frac{\sigma^2}{\mu \nu \hat{r} \dot{\sigma}} = \ddot{\sigma} - \frac{\lambda}{\nu} \sum_{m=1}^{\infty} \tilde{\sigma}_m J_1(j_{0,m}r) I_0(n\alpha r) \hat{e}_r - \lambda \alpha \dot{\sigma} - 2\alpha \sum_{m=1}^{\infty} \tilde{\kappa}_m n J_1(j_{0,m}r) I_0(n\alpha r) \hat{e}_r. \quad (14)$$

The self-electric field $E_{self}$ in Eq. (8) becomes

$$E_{self} = -\frac{4\pi N_e e a}{c L_{rest} r} \sum_{m=1}^{\infty} \sigma_m J_1(j_{0,m}r) I_0(n\alpha r) \hat{e}_r, \quad (15)$$

where in line 2 of Eq. (15) we have utilized the Bessel function identity\(^{13}\)

$$\int_0^1 dxx J_0(y)x = J_1(y)/y. \quad (16)$$

Notice that the expression in line 3 of Eq. (15) is exactly the same as one obtains from Gauss’s law for a beam of charge per unit length, $N_j/L_{rest}$. Likewise, $B_{self}$ in Eq. (12) becomes

$$B_{self} = \left\{ -\frac{4\pi N_e e a}{c L_{rest} r} \sum_{m=1}^{\infty} \kappa_m J_0(j_{0,m}r/a) I_0(n\alpha r) \\
+ \frac{8\pi N_e e a}{c L_{rest} r} \sum_{m=1}^{\infty} \kappa_m J_1(j_{0,m}r) n I_0(n\alpha r) \right\} \hat{e}_z. \quad (17)$$

$$= -\frac{4\pi N_e e a}{c L_{rest} r} \int_r^a dr' \sum_{m=1}^{\infty} \kappa_m J_0(j_{0,m}r'/a) I_0(n\alpha r') + \frac{8\pi N_e e a}{c L_{rest} r} \sum_{m=1}^{\infty} \kappa_m J_1(j_{0,m}r) n I_0(n\alpha r) \hat{e}_z, \quad (18)$$

$$= -\frac{4\pi N_e e a}{c L_{rest} r} \int_r^a dr' \kappa(r') + \frac{8\pi N_e e a}{c L_{rest} r} \int_0^a dr \int_r^a dr' \kappa(r') \hat{e}_z. \quad (19)$$
where in line 2 of Eq. (17) we have used Eq. (16) and the Bessel function identity:

\[ \int_0^1 dx J_1(yx) = 1/y - J_0(y)/y. \]  \tag{18} 

The first integral in line 3 of Eq. (17) is precisely the expression for the magnetic field of an unbunched beam that is derivable from Ampere’s law. The double integral in line 3 is the additional magnetic field generated by the surface currents on the pipe to ensure zero total magnetic flux within the entire pipe.

In the limit of a single-bunch beam \((\alpha \to 0)\), \(\coth(j_0m/\alpha) \approx 1\). Hence, \(\mathbf{E}_{\text{self}}\) is

\[ \mathbf{E}_{\text{self}} = -2\pi N_{\sigma} \sum_{m=1}^{\infty} \sigma_m J_1(j_0m\bar{r}) \hat{\mathbf{e}}_r. \]  \tag{19} 

In order to calculate \(\mathbf{B}_{\text{self}}\) from Eq. (12), we need to take the asymptotic limit of the series over \(n\). By performing the substitutions,

\[ \lim_{\alpha \to 0} \sum_{n=1}^{\infty} \rightarrow \int_0^\infty dx \]  \tag{20a} 

and

\[ n\alpha \rightarrow x, \]  \tag{20b} 

we find

\[ \mathbf{B}_{\text{self}} = \frac{2N_{\sigma}}{c} \sum_{m=1}^{\infty} \kappa_m \left( \begin{array}{c} j_0(j_0m) \\ \coth(j_0m) \\ \end{array} \right) \hat{\mathbf{e}}_r. \]  \tag{21} 

Hence, Eq. (13) in the single-bunch limit is

\[ \frac{\bar{\omega}^2}{\mu_1 - \bar{r}^2 \bar{\omega}^2} = \bar{\omega}^2 - \frac{\lambda}{\bar{r}} \sum_{m=1}^{\infty} \kappa_m J_0(j_0m) \]  

\[ - \bar{\omega} \sum_{m=1}^{\infty} \kappa_m J_0(j_0m) \]  

\[ + \frac{2}{\pi} J_1(j_0) \int_0^\infty dx I_0(\bar{x}\bar{r}) \left( \frac{1}{j_0m + x^2} I_1(x) \right). \]  \tag{22} 

### III. NUMERICAL SOLUTION OF THE MOMENTUM EQUATION

In this section, we present the numerical solutions to the normalized momentum equation, Eq. (13). We first describe a general numerical scheme, which involves an iterative high-dimensional Newton’s method, for solving Eq. (13) for a general beam cross section, \(\bar{\sigma}(\bar{r})\). Specifically, we obtain \(\bar{\omega}(\bar{r})\) and hence, \(\mathbf{B}_{\text{self}}^{\text{rel}}(\bar{r})\), which satisfy Eq. (13), self-consistently. We then demonstrate this procedure on the special case of an annular beam with a quadratic density profile.

The numerical method begins by dividing the cross section of the beam into \(I\) equal components of length \(\Delta \bar{r}\). Typically, \(I=1000\) is chosen, which yields accurate results within a relatively modest computational time. Our goal is to solve for the correct \(\bar{\omega}(\bar{r})\) and \(\mathbf{B}_{\text{self}}^{\text{rel}}(\bar{r})\) that satisfy Eq. (13) for all \(1 \leq i \leq I\). The discretized version of Eq. (13), for which we solve numerically is given by

\[ \frac{\bar{\omega}_{i}^2}{\mu_1 - \bar{r}^2 \bar{\omega}_{i}^2} = \bar{\omega}_{i}^2 + \frac{\lambda f_i \bar{\omega}_{i}}{\sum_{j=1}^{I} \bar{\omega}_{j} g_{ij}} = 0, \]  \tag{23a} 

where

\[ f_i = \frac{1}{r_i} \sum_{m=1}^{M} \left( \frac{2\bar{\sigma}_i \bar{r} \bar{J}_1(\bar{r}_i)}{\bar{J}_1(\bar{r}_i)^2} \right) \times \left( \frac{\bar{J}_1(j_0m\bar{r}_i)}{\bar{J}_1(\bar{r}_i)^2} - \frac{\bar{\omega}_i}{\bar{r}_i} \frac{\bar{J}_1(j_0m\bar{r}_i)}{\bar{J}_1(\bar{r}_i)^2} \right) \]  \tag{23b} 

\[ g_{ij} = \frac{2\bar{\omega}_i \bar{r}_i \bar{J}_1(\bar{r}_i)}{\bar{J}_1(\bar{r}_i)^2} \times \left( \frac{\bar{J}_1(j_0m\bar{r}_i)}{\bar{J}_1(\bar{r}_i)^2} - \frac{\bar{\omega}_i}{\bar{r}_i} \frac{\bar{J}_1(j_0m\bar{r}_i)}{\bar{J}_1(\bar{r}_i)^2} \right) \]  \tag{23c} 

\(\bar{\omega}_i = \bar{\omega}(r_i)\), \(\sigma_i = \sigma(r_i)\), \(M\) is the number of Bessel expansion modes used, and \(N\) is the number of bunches in the bunch train considered. We note the following approximate relations of \(f_i\) and \(g_{ij}\) with the self-electric and self-magnetic fields

\[ \mathbf{E}_{\text{self}}(\bar{r}_i) = -\bar{\lambda}_i \mathbf{B}_0 f_i, \]  \tag{24a} 

and

\[ \mathbf{B}_{\text{self}}^{\text{rel}}(\bar{r}_i) = -\bar{\lambda} \mathbf{B}_0 \sum_{j=1}^{I} \bar{\omega}_j g_{ij}. \]  \tag{24b} 

The total number of Bessel modes needed to accurately calculate \(\bar{\omega}(\bar{r})\) and \(\mathbf{B}_{\text{self}}^{\text{rel}}(\bar{r})\) can depend on \(\bar{\sigma}(\bar{r})\). For example, an annular beam with a radial thickness much smaller than \(a\), \(M \sim 1000\) may be necessary. The total number of bunches necessary in Eq. (23c) depends on the parameter \(\alpha\). Qualitatively, small values of \(\alpha\) require higher values of \(N\). For \(\alpha=1.0\), choosing \(N=50\) yields accurate answers.

By specifying \(\bar{\sigma}(\bar{r})\) at the beginning of the scheme, we can compute the vector and matrix elements, \(f_i\) and \(g_{ij}\), which remain fixed for the rest of the numerical scheme. The iteration then begins by specifying an initial guess for \(\bar{\omega}(r_i)\). If the value of \(\lambda\) is sufficiently small, then the appropriate initialization for \(\bar{\omega}(r_i)\) is the exact solution of Eq. (13) for \(\lambda=0\). When \(\lambda=0\) the self-electric and self-magnetic fields vanish, and the exact slow and fast rotation solutions to Eq. (13) are given by

\[ \bar{\omega}(r_i) = 0 \] (slow root)  \tag{25a} 

and

\[ \bar{\omega}(r_i) = \frac{\mu}{\bar{r}^2 + \mu^2 r^2} \] (fast root).  \tag{25b} 

With the initial \(\bar{\omega}(r_i)\) specified, a standard nonlinear equation solver method, such as Newton’s method, is capable of solving Eq. (23). There typically exists a \(\lambda^* > 0\) in which the slow and fast solutions exist, despite the fact that using either
Eq. (25a) or Eq. (25b) does not yield a solution. In this case, a solution can be found by using the numerical solution for \( \bar{\omega}(r) \) at \( \lambda = \lambda^* - \Delta \lambda^* \), where \( \Delta \lambda^* \) is sufficiently small.

In this paper, we use a modified Powell algorithm, which is a variation of Newton’s method found in the FORTRAN 90 IMSL library. Convergence to a solution will generally occur within 50 iterations of the algorithm, if a solution does exists and the initial \( \bar{\omega}(r_i) \) is appropriately chosen.

We now demonstrate the numerical procedure for the special case of an annular beam with a quadratic cross section given by

\[
\sigma(r) = \begin{cases} 
0, & r \leq r_i, \\
3(r_o - r)(r - r_i) \pi \bar{r}^2, & r_i \leq r \leq r_o, \\
0, & r \leq r_o,
\end{cases}
\]

where \( \bar{r} = (r_i + r_o)/2 \) is an average beam radius and \( \delta = r_o - r_i \) is the beam width. For this example, we have chosen \( \bar{r}/a = 0.8 \) and \( \delta \alpha = 0.2 \), which corresponds to \( r/a = 0.7 \) and \( r_o/a = 0.9 \). For this example, we have also set \( \alpha = 1.0, I = 1000, M = 1000, \) and \( N = 50 \). Figure 2 shows a plot of \( \bar{\sigma}(\bar{r}) \) vs \( \bar{r} \). By fixing \( \bar{\sigma}(\bar{r}) \), we immediately specify the radial electric field, \( E^{\text{self}}(\bar{r}) \), from Eq. (19). Figure 3 shows a plot of the normalized radial electric field \( E^{\text{self}}(\bar{r}) a^2/2N_b e \) vs \( \bar{r} \). The following plots, Figs. 4(a), 4(b), 5(a), and 5(b), show the solutions to \( \bar{\sigma}(\bar{r}) \) and \( B^{\text{self}}(r) \) in Eq. (13) for specific values of \( \mu \).

In Fig. 4(a), we show plots of \( \bar{\sigma}(\bar{r}) \) for \( \mu = 0.1 \) with \( \lambda = 0.003 \) and \( \lambda = 0.005 \). The top curves correspond to the fast rotation solutions and the bottom curves correspond to the slow rotation solutions. Note that as \( \lambda \) increases the fast and slow rotation solutions become closer. At a critical value \( \lambda_{\text{crit}} \), the two solutions merge at one point, and for any \( \lambda > \lambda_{\text{crit}} \) no rotation solutions exist. We refer to this parameter limit as a fast-slow merger. In this example, the fast-slow merger occurs approximately at \( \lambda_{\text{crit}} \approx 0.005 \). To complete the numerical example for \( \mu = 0.1 \), we show the plots of \( B^{\text{self}}(\bar{r})/B_0 \) for the slow rotation solutions in Fig. 4(b). We have only plotted \( B^{\text{self}}(r) \) for the slow rotation solutions since the fast solutions are typically not accessible in high-power microwave sources. In general, we find that there regions within the beam cross section where the self-magnetic field is paramagnetic \( [B^{\text{self}}(\bar{r})/B_0] > 0 \), and other regions that are diamagnetic \( [B^{\text{self}}(\bar{r})/B_0] < 0 \).

In Fig. 5(a), we show plots of the slow rotation roots of \( \bar{\sigma}(\bar{r}) \) for the case of \( \mu = 10.7 \) for the cases of \( \lambda = 0.05 \) and \( \lambda = 0.1115 \). The fast rotation roots, which are not shown, are well separated from the slow roots and are given approxi-
mately by Eq. (25b). Figure 5(b) shows plots of $B^{\text{self}}(\tilde{r})/B_0$ for these two cases. Notice that as $\lambda$ increases to a critical value $\lambda_{\text{cusp}} \approx 0.1116$, $B^{\text{self}}(\tilde{r})/B_0$ forms a cusp shape at a specific point $r^* = 0.7881$, with $B^{\text{self}}(\tilde{r})/B_0 = -1$. Near the cusp, we find numerically that the magnetic field scales approximately as $B^{\text{self}}(r) + B_0 \propto (r^* - r)^{\alpha_1}$ with $\alpha_1 \approx 0.5$ for $r < r^*$, and $B^{\text{self}}(r) + B_0 \propto (r - r^*)^{\alpha_2}$ with $\alpha_2 \approx 0.73$ for $r > r^*$. Beyond the cusp with $\lambda > \lambda_{\text{cusp}}$, we are unable to find numerical solutions to the equilibrium momentum equation. We will further discuss this point in the next section.

For the geometric parameters and the applied magnetic field in Fig. 5, the fast root of $\bar{\omega}(r)$ exists for $\lambda > \lambda_{\text{cusp}}$. However, we have not fully examined the physics of the fast root for $\lambda > \lambda_{\text{cusp}}$.

IV. DISCUSSION

In this section, we further discuss the results presented in Sec. III, and point out the potential significance of these solutions for the equilibrium flow in high-power microwave sources.

First, we discuss the physics of two current limits. To understand how the fast-slow merger and self-magnetic cusp arise within the theory it is helpful to first analyze the properties of Eq. (13) in the limit of small $\lambda$. From Eqs. (25a) and (25b), we know the solutions to $\bar{\omega}(r)$ when $\lambda = 0$. For small $\lambda$, the lowest order corrections to the slow and fast roots will be $O(\lambda)$, that is

$$\bar{\omega}_{\text{slow}}(\tilde{r}) = \delta \bar{\omega}_{\text{slow}}(\tilde{r})$$

and

$$\bar{\omega}_{\text{fast}}(\tilde{r}) = \bar{\omega}_{\text{fast},0}(\tilde{r}) + \delta \bar{\omega}_{\text{fast}}(\tilde{r})$$

where $\bar{\omega}_{\text{fast},0}(\tilde{r}) = \mu/(1 + \mu^2 r^2)^{1/2}$.

The solutions to the lowest order perturbations are straightforward to find and are given by

$$\delta \bar{\omega}_{\text{slow}}(\tilde{r}) = -\lambda f(\tilde{r})$$

and

$$\delta \bar{\omega}_{\text{fast}}(\tilde{r}) = -\frac{\lambda f(\tilde{r}) + \bar{\omega}_{\text{fast},0}(\tilde{r}) g_0(\tilde{r})}{1 + \mu^2 r^2}$$

where we have defined

$$f(\tilde{r}) = \frac{1}{2} \sum_{m=1}^{\infty} \bar{\alpha}_r J_2(\tilde{r} J_{0m}/\alpha) \coth(\pi J_{0m}/\alpha)$$

and

$$g_0(\tilde{r}) = \sum_{m=1}^{\infty} \bar{\alpha}_g J_2(\tilde{r} J_{0m}/\alpha) \coth(\pi J_{0m}/\alpha) - \frac{2 \alpha}{\pi} \sum_{m=1}^{\infty} \bar{\alpha}_r J_2(\tilde{r} J_{0m}/\alpha)$$

Moreover, in Eq. (29b), we are denoting $\bar{\alpha}(\tilde{r}) = \bar{\alpha}(\tilde{r}) \bar{\omega}_{\text{fast},0}(\tilde{r})$. With Eqs. (28) and (29), we can understand the parameter regimes in which the fast-slow merger and the self-magnetic cusp occur.

The fast-slow merger is a limit which was investigated for nonrelativistically rotating bunched annular beams. This fast-slow merger is illustrated in Fig. 4. In general, fast-slow mergers will occur when the fast and slow roots are “marginally” separated at $\lambda = 0$. The increase of $\lambda$ will then cause the two roots to intersect at one point. From Eqs. (28) and (29), it is obvious that this marginal separability will occur when $\mu \ll 1$. In this case, the difference between the two roots is given by

$$\bar{\omega}_{\text{fast}}(\tilde{r}) - \bar{\omega}_{\text{slow}}(\tilde{r}) = \lambda f(\tilde{r}) \mu^2 r^2 - \frac{\lambda \bar{\alpha}(\tilde{r}) \bar{\omega}_{\text{fast},0}(\tilde{r}) g_0(\tilde{r})}{1 + \mu^2 r^2} \sim O(\mu, \lambda \mu^2).$$

We note that since $\bar{\omega}_{\text{fast},0}(\tilde{r}) \sim O(\mu)$ and $g_0(\tilde{r}) \sim O(\mu)$, the last term in Eq. (30) is $O(\lambda \mu^3)$. It is obvious that the regime $\mu \ll 1$ corresponds with nonrelativistic rotation, since $V_b(\tilde{r})/c = \bar{\omega}(\tilde{r}) \tilde{r} \ll 1$.

The generation of the self-magnetic cusp, which is shown in Fig. 5, is the other type of limit that can occur in bunched annular beams. In this case, the self-magnetic field generated by the slow root is non-negligible compared to the external magnetic field and the self-electric field. This fact implies that the slow rotation root must be sufficiently large in magnitude, which typically implies that the rotation is relativistic. Therefore the separation between the fast and
slow root must be sufficiently large to prevent the onset of the fast-slow merger prior to the slow root becoming relativistic. This large separation occurs when \( \mu \gg 1 \). Notice that in this regime, \( \delta \phi_{\text{fast}}(\tilde{r}) \approx 1/2 \tilde{r} \sim O(1) \) and \( g_0(\tilde{r}) \sim O(1) \). For \( \mu \gg 1 \), the fast root’s deviation from \( \tilde{\omega}_{\text{fast}}(\tilde{r}) \) is small since \( \delta \phi_{\text{fast}}(\tilde{r}) \approx O(\lambda/\mu^2) \ll 1 \) from Eq. (28b). The slow root is therefore capable of reaching relativistic rotation for sufficiently large \( \lambda \), since by Eq. (28a), \( \delta \phi_{\text{slow}}(\tilde{r}) \sim O(\lambda) \).

Second, we now discuss the general properties of the self-magnetic cusp. As we mentioned previously, the self-magnetic cusp occurs at \( \tilde{r} \approx 0.7881 \) inside of the beam cross section for a critical value \( \lambda_{\text{cusp}} \approx 0.1116 \). For all values \( 0 \leq \lambda < \lambda_{\text{cusp}} \), \( \delta \phi_{\text{slow}}(\tilde{r}) \) has only one zero, which is precisely the zero point of \( E_{\text{self}}(\tilde{r}) \). We label the zero point of the electric field as \( \tilde{r}^* \). Near \( \tilde{r}^* \), the electric field can be expanded linearly, i.e., \( E_{\text{self}}(\tilde{r}) \approx E_{\text{self}}(\tilde{r}^*) (\tilde{r} - \tilde{r}^*) \), where the prime represents the first radial derivative. Hence for \( 0 \leq \lambda < \lambda_{\text{cusp}} \), Eq. (13) also implies that \( \tilde{\omega}_{\text{slow}}(\tilde{r}) \) has a linear dependence, i.e., \( \tilde{\omega}_{\text{slow}}(\tilde{r}) \approx \tilde{\omega}_{\text{slow}}(\tilde{r}^*)(\tilde{r} - \tilde{r}^*) \). However, at \( \lambda = \lambda_{\text{cusp}} \), the self-magnetic field forms a cusp shape at \( \tilde{r}^* \) that is equal in magnitude and opposite to the external magnetic field, i.e., \( B_{\text{self}}(\tilde{r}) = -B_0 \). In this critical case, the spatial dependency of the slow root near \( \tilde{r}^* \) also changes to \( \tilde{\omega}_{\text{slow}}(\tilde{r}) \approx \tilde{\omega}_{\text{slow}}(\tilde{r}^*) (\tilde{r} - \tilde{r}^*) \). Near \( \tilde{r}^* \), \( \tilde{\omega}_{\text{slow}}(\tilde{r}) \approx \tilde{\omega}_{\text{slow}}(\tilde{r}^*) (\tilde{r} - \tilde{r}^*) \). Likewise, the dependency of \( B_{\text{self}}(\tilde{r}) \) will also become nondifferentiable near \( \tilde{r}^* \), i.e., \( B_{\text{self}}(\tilde{r}) \approx \tilde{B}_0 \cdot \tilde{B}_e / \tilde{r} \tilde{r}^* \). In general, the constants \( \tilde{\omega}_{\text{slow}}(\tilde{r}) \approx \tilde{\omega}_{\text{slow}}(\tilde{r}^*) (\tilde{r} - \tilde{r}^*) \), and \( \tilde{B}_{\text{self}}(\tilde{r}) \approx \tilde{B}_{\text{self}}(\tilde{r}^*) (\tilde{r} - \tilde{r}^*) \).

The key attributes of the self-magnetic cusp are that it has a finite bunch length, annular electron beam to show the formation of the self-magnetic cusp, the same effect will occur in an annular electron beam with a finite bunch length, which we will now describe. As an example, suppose that a periodic annular electron beam has finite length bunches that are symmetric about a maximum in their charge density, such as

\[
n(r, t) = (2\pi r_{\text{rms}}^2)^{-1/2} N_0 \sigma \sum_{k=-\infty}^{\infty} \exp[-(z - V_{lt} k L)^2/2 r_{\text{rms}}^2],
\]

where \( r_{\text{rms}} \) is the rms bunch length and \( \sigma \) is given by Eq. (26). In the rest frame of the beam, the cold-fluid equilibrium momentum equation, i.e., Eq. (6), may be applied to the beam cross section at the point of maximum density, i.e., where the electric field is maximum. The radial electric field can be computed as a function of \( r \), and its shape will be very similar to the one shown in Fig. 3. The only differences are that the zero point, \( \tilde{r}^* \), of \( E_{\text{self}}(\tilde{r}) \) for the finite bunch length case will be shifted to the left of the zero point in Fig. 3, and that the values of the maxima and minima will also be reduced. However, as we explained in the previous paragraph, the formation of the self-magnetic cusp occurs due to the zero point of \( E_{\text{self}}(\tilde{r}) \). Hence, the self-magnetic cusp limit can form in the middle of a bunch that has finite bunch length, and therefore this limit can be applied to finite length bunches. The difference between the zero bunch length and finite bunch length self-magnetic cusp limits is the value of \( \lambda_{\text{cusp}} \). As the bunch length increases, \( \lambda_{\text{cusp}} \) increases as well in order to offset the reduction in \( E_{\text{self}}(\tilde{r}) \). In the case of a continuous unbunched beam, one can view it as a bunched beam with infinite \( r_{\text{rms}} \). In this case, \( \lambda_{\text{cusp}} \) increases to infinity. Hence, an unbunched beam does not allow for formation of self-magnetic cusps.

The third issue in this discussion pertains to what will really happen to a bunched beam as the current is increased to the point of \( \lambda = \lambda_{\text{cusp}} \) and beyond. This point is difficult to address within the framework of the present analysis that involves the conditions for beam equilibrium assuming a predetermined beam density function. Obviously, if the beam is not in equilibrium then the density function must be affected in such a way as to reduce the magnitude of the electric field. There are two likely possibilities for beam density changes. One possibility is that the radial cross section changes, i.e., either the beam radial thickness increases and/or the maximum splits into two near the self-magnetic cusp. A second possibility is that the bunch length increases. In either scenario the radial electric field will reduce in magnitude thereby removing the restriction on beam equilibrium.

The density change that will occur in a real beam at \( \lambda = \lambda_{\text{cusp}} \) largely depends upon allowable equilibrium energy states subject to energy conservation as well as the beam initial conditions, e.g., the rate of longitudinal compression.

Finally, the formation of the self-magnetic cusp may have important consequences for high-current annular electron beams, such as the ones found in high-power klystrons. Klystrons that operate in the \( \approx 1 \) GHz regime have large beam pipe radii, which are of the order of 10.0 cm. Moreover, the external magnetic fields needed to confine the high beam current, which may be as large as 10 kA, are in the 1 T regime. Hence, these klystrons are found to be operating in the regime where \( \mu > 10^3 \). For example, the Los Alamos National Laboratory 1.3 GHz relativistic klystron amplifier experiment is operating at \( \mu = 3.5 \). The Air Force Research Laboratory relativistic klystron oscillator experiment is operating at an even higher value of \( \mu = 35.9 \). These experimental values of \( \mu \) are well within the regime where a self-magnetic field cusp may form at a critical value of \( \lambda = \lambda_{\text{cusp}} \).

V. SUMMARY

In this paper we have shown the existence of a new phenomenon that can occur in relativistic bunched annular electron beams, namely the formation of a self-magnetic cusp. The self-magnetic cusp calculation was developed within the framework of a bunched annular beam equilibrium fluid model which includes the effects of the beam self-electric and self-magnetic fields as well as the effect of a conductor pipe. The self-magnetic cusp, along with the fast-slow merger reported previously, can be viewed as a type of limit for bunched annular beams since beyond a certain value \( \lambda = \lambda_{\text{cusp}} \), no slow root equilibrium solution can be found.

The key attributes of the self-magnetic cusp are that it occurs at the same point as the zero crossing of the electric field for the slow root solution when \( \lambda = \lambda_{\text{cusp}} \) it is equal and
opposite in magnitude to the external magnetic field, and for values \( \lambda > \lambda_{\text{cusp}} \) no slow root solution has been found. To develop a self-magnetic cusp within a bunched annular beam, and hence, avoid the fast-slow merger, it is necessary to have the parameter \( \mu \) sufficiently large. This implies that \( \mu \gg 1 \), and hence the beam is rotating relativistically. We note that bunched annular beams with \( \mu \gg 1 \) are of special interest to the field of high-power microwave sources, and therefore this theoretical work may have direct experimental capability.

Topics of future research include further exploration of the fast root equilibrium solution in the regime \( \lambda = \lambda_{\text{cusp}} \) and studies of the effects of finite bunch length.

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**APPENDIX: DERIVATION OF THE SELF-MAGNETIC FIELD**

We derive the expression for the self-magnetic field in Eq. (12) as follows. The total self-generated magnetic field which incorporates the effect of the conducting boundary has the following properties:

\[
\nabla \cdot \mathbf{B}^{\text{self}} = 0, \quad (A1a)
\]

\[
\nabla \times \mathbf{B}^{\text{self}} = 4\pi \mathbf{J}^{\text{self}}, \quad (A1b)
\]

\[B_{r}^{\text{self}} \big|_{r=a} = 0, \quad (A1c)
\]

\[2\pi \int B_{z}^{\text{self}}(r,z) rdr = \text{const} = 0, \quad (A1d)
\]

where

\[
\mathbf{J}^{\text{self}} = -N_{e} e\sigma(r)V_{\phi}(r)\sum_{k=-\infty}^{\infty} \delta(z-kL)\hat{\mathbf{e}}_{\phi}. \quad (A2)
\]

It proves to be useful to break \( \mathbf{B}^{\text{self}} \) into two parts: one part due to the source (bunch current) and a second part due to the wall correction; that is

\[
\mathbf{B}^{\text{self}} = \mathbf{B}^{\text{source}} + \mathbf{B}^{\text{wall}}, \quad (A3)
\]

where the magnetic field components satisfy

\[
\nabla \cdot \mathbf{B}^{\text{source}} = 0, \quad (A4a)
\]

\[
\nabla \times \mathbf{B}^{\text{source}} = \frac{4\pi}{c} \mathbf{J}^{\text{self}}, \quad (A4b)
\]

\[
\nabla \cdot \mathbf{B}^{\text{wall}} = 0, \quad (A4c)
\]

\[
\nabla \times \mathbf{B}^{\text{wall}} = 0. \quad (A4d)
\]

The source magnetic field can be calculated from a vector potential

\[
\mathbf{B}^{\text{source}} = \nabla \times \mathbf{A}, \quad (A5)
\]

where

\[
\mathbf{A} = \frac{1}{c} \int d^{3}r' \mathbf{J}^{\text{self}}(r') \frac{1}{|r-r'|}. \quad (A6)
\]

Substituting Eq. (A2) and the Bessel function identity,

\[
\frac{1}{|r-r'|} = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{e^{i(n\theta - \theta')}}{r^{n+1}} \int_{-\infty}^{\infty} d\lambda e^{i(\lambda z - \lambda')} I_{n}(\lambda r_{\perp}) K_{n}(\lambda r_{\parallel}) \quad (A7)
\]

into Eq. (A6), we obtain

\[
\mathbf{A} = A_{\phi} \hat{\mathbf{e}}_{\phi}, \quad (A8a)
\]

\[
A_{\phi} = -\frac{2\pi N_{e} e}{cL} \int_{0}^{a} dr' r' \kappa(r') \left( \frac{r_{\leq}}{r_{\geq}} \right), \quad (A8b)
\]

where

\[
\kappa(r) = \sigma(r)V_{\phi}^{\text{est}}(r) \quad (A9)
\]

is the current sheet density within each bunch. Note that the first term on the right-hand side of Eq. (A8b) represents the vector potential for an unbunched beam while the second term is the corrections due to beam bunching.

From Eqs. (A4), it follows that \( \mathbf{B}^{\text{wall}} \) can be expressed in terms of a scalar potential, namely

\[
\mathbf{B}^{\text{wall}} = B_{0}^{\text{wall}} \hat{\mathbf{e}}_{z} - \nabla \phi_{m}, \quad (A10a)
\]

\[
\nabla^{2} \phi_{m} = 0, \quad (A10b)
\]

and where \( B_{0}^{\text{wall}} \) = const is chosen to satisfy the normalization condition in Eq. (A1d). Equation (A10b) implies that \( \phi_{m} \) can be written as

\[
\phi_{m}(r) = \sum_{n=-\infty}^{\infty} \left[ a_{n} e^{i(n\phi - \phi_{0})} + b_{n} e^{-i(n\phi - \phi_{0})} \right], \quad (A11)
\]

where \( \{a_{n}\} \) and \( \{b_{n}\} \) are sets of constants. However, since \( \phi_{m} \) must be finite at the center of the pipe (\( r=0 \)), we must require that \( \{b_{n}\} \) be zero. The magnetic field boundary condition in Eq. (A1c) immediately yields

\[
\phi = -\frac{8\pi N_{e} e}{cL} \int_{0}^{a} dr' r' \kappa(r') \left( \frac{r_{\leq}}{r_{\geq}} \right) \sum_{n=1}^{\infty} \sin[n(\phi - \phi_{0})] I_{n}(n r_{\perp}) K_{n}(n r_{\parallel}) \quad (A12)
\]

By using the expansion for \( \kappa(r) \) in Eq. (11), the Bessel function identity in Eq. (15), and the following Bessel function identities,\(^{14}\)
\[
\int_0^1 dx x J_0(yx) I_0(zx) = (y^2 + z^2)^{-1/2} \left[ z J_0(y) I_1(z) + y J_1(y) I_0(z) \right],
\]
\[
\int_w^1 dx x J_0(yx) K_0(zx) = (y^2 + z^2)^{-1/2} \left[ z J_0(y) K_1(z) + z w J_0(yw) K_1(zw) - z w J_1(yw) K_0(zw) \right],
\]

the self-magnetic field result in Eq. (12) is obtainable after some lengthy algebraic manipulations.