

# Method of Perturbative-PIC Simulation for CSR Effect

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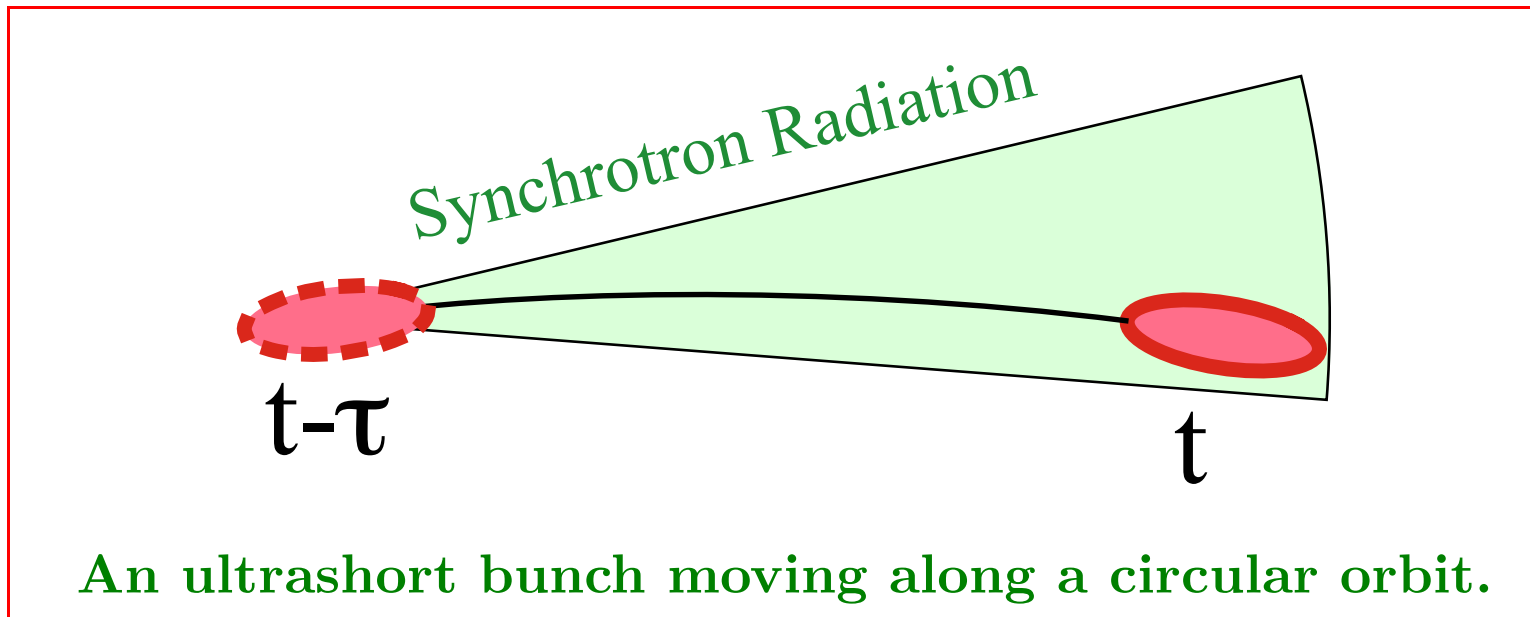
## OUTLINE

- Why Do We Want to Do This?
  - Perturbation Expansion of the Retardation of CSR
  - PIC Method for the Evaluation of the Retarded Field
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## The Problem

How to evaluate numerically the interaction between a short bunch and its synchrotron radiation?



- The interaction is collective in nature — bunch particle distribution is involved.
- The interaction involved a retarded bunch particle distribution.

# Radiation Field of a Moving Charge in Free Space

The field radiated from a moving charge at  $(\vec{X}_1, t_1)$  and observed at  $(\vec{X}, t)$  in the lab frame is

$$\vec{E}(\vec{X}, t, \vec{X}_1, t_1) = \frac{e(\vec{n} - \vec{\beta}_1)}{\gamma_1^2 r^2 (1 - \vec{n} \cdot \vec{\beta}_1)^3} \Big|_{\text{ret}} + \frac{e}{c} \frac{\vec{n} \times [(\vec{n} - \vec{\beta}_1) \times \dot{\vec{\beta}}_1]}{r (1 - \vec{n} \cdot \vec{\beta}_1)^3} \Big|_{\text{ret}}$$

$$\vec{B}(\vec{X}, t, \vec{X}_1, t_1) = \vec{n} \times \vec{E}$$

where  $\vec{r} = \vec{X} - \vec{X}_1$ ,  $\vec{n} = \vec{r}/r$ ,  $\vec{\beta}_1 = \vec{v}_1(t_1)/c$ , and the retardation condition is  $r = c(t - t_1) = c\tau$ .

For an electron in an ultra-relativistic beam,  $\vec{v}_1$  and  $\dot{\vec{v}}_1$  are dominated by the bunch centroid motion and, therefore, can be well approximated by the velocity and acceleration of the bunch centroid at  $t_1$ .

## Lorenz Force due to Synchrotron Radiation

- Let  $\vec{X}(t) = \vec{X}_c(t) + \vec{x}(t)$  be a global coordinate of a bunch particle in the lab frame, where  $\vec{X}_c(t)$  is the global coordinate of the bunch centroid and  $\vec{x}$  is the coordinate of the particle respect to its bunch centroid.
- The Lorenz force on a particle at  $\vec{X}$  due to the radiation from its bunch is

$$\vec{F}_{\text{SR}}(\vec{X}, t) = \int_{-\infty}^{\infty} \vec{G}(\vec{X} - \vec{X}_1, t, \tau) f(\vec{X}_1, \vec{P}_1, t - \tau) d\vec{X}_1 d\vec{P}_1$$

$$\vec{G}(\vec{X} - \vec{X}_1, t, \tau) = e \left[ \vec{E}(\vec{X}, t, \vec{X}_1, t_1) + \vec{\beta} \times \vec{B}(\vec{X}, t, \vec{X}_1, t_1) \right]$$

where  $\tau = |\vec{X} - \vec{X}_1|/c$  is the retardation of the radiation field and  $f(\vec{X}_1, \vec{P}_1, t - \tau)$  is the retarded bunch particle distribution.

## Traditional PIC Method Cannot Be Easily Used:

Chief of difficulties in a self-consistent simulation of the CSR problem is how to construct the retarded particle distribution  $f(\vec{X}_1, \vec{P}_1, t - \tau)$  numerically. It is very difficult, if not impossible, to use the traditional PIC method for this problem.

## Current CSR Codes:

The most codes developed for the CSR problem so far have thus adapted the approaches of (a) non-self-consistent; (b) tracking of a limited number of macroparticles and the particle-radiation interaction being calculated with particle-to-particle individually; or (c) solving the Vlasov-Maxwell equation in phase space.

## Our Approach:

- The retardation of  $f(\vec{X}_1, \vec{P}_1, t - \tau)$  is treated perturbatively.
- The expansion of the retarded distribution can be reconstructed effectively with the PIC method during the simulation.
- This perturbative-PIC calculation of the retarded distribution can be applied to the calculation of either the radiation field or the Liénard-Wiechert potential of the radiation field.

# Expansion of the Retardation of $f(\vec{X}_1, \vec{P}_1, t - \tau)$

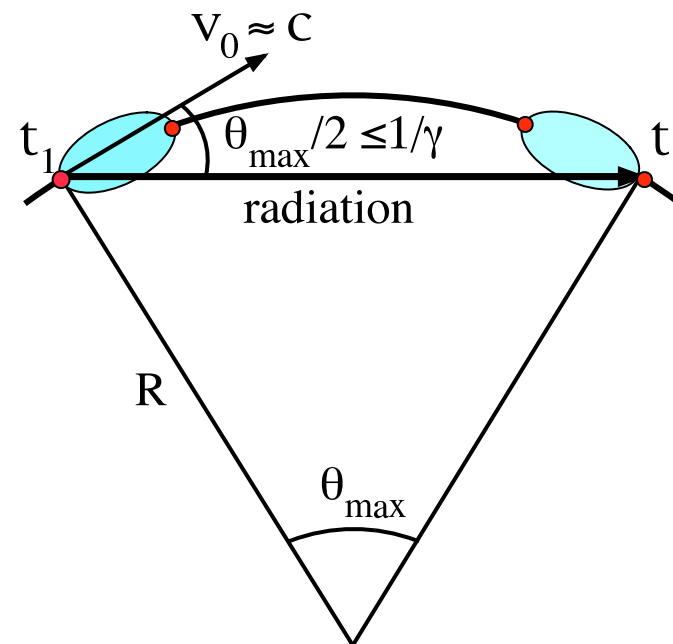
## Maximal Retardation of CSR

The maximal retardation of the radiation-particle interaction is between particles in the tail and particles in the head of a bunch. Including those particles in the tail of the distribution ( $\sim 4\sigma_z$ ),

$$\theta_{\max} \sim 2 \left( \frac{24\sigma_z}{R} \right)^{1/3}$$

For an example, if  $R \sim 10$  m,  
 $\gamma \sim 1000$ ,  $\sigma_z \sim 1$  mm,

$$\theta_{\max} < 0.3 < 16^\circ$$



**retardation  $\ll$  time scale of slow emittance growth !**

## Expansion of the Retardation of $f(\vec{X}_1, \vec{P}_1, t - \tau)$

**Fact:** The time dependence of a bunch particle distribution has two significantly different time scales: A fast time scale is of the linear dynamics of a bunch as its centroid moves along the lattice and its beam sizes vary with lattice functions. A slow time scale is of the slow beam-size growth due to nonlinear perturbations.

**Retardation on the Fast Time Scale:** Since the linear dynamics is known, the retardation on the fast time scale can be eliminated in the distribution analytically by using the normalized variables.

**Retardation on the Slow Time Scale:** Since the retardation in the distribution is usually much shorter than the slow time scale, the retardation on the slow time scale can be approximated by an expansion in terms of the retardation.

## Elimination of Fast Time Dependence of $f(\vec{X}, \vec{P}, t)$

The fast time dependence of  $f(\vec{X}, \vec{P}, t - \tau)$  can be eliminated by using the normalized variables  $(\vec{q}, \vec{p})$ . For an ultra-relativistic beam, the transformation of  $(\vec{X}, \vec{P})$  to  $(\vec{q}, \vec{p})$  is symplectic. The force integral of the radiation simply becomes

$$\vec{F}_{\text{SR}}(\vec{q}, t) = \int_{-\infty}^{\infty} \vec{G}_1(\vec{q}, \vec{q}', \tau) \rho(\vec{q}', t - \tau) d\vec{q}'$$

$$\rho(\vec{q}, t) = \int_{-\infty}^{\infty} f(\vec{q}, \vec{p}, t) d\vec{p}$$

$$\vec{x} = ( \beta_x^{1/2}(\theta)q_1, \quad \beta_y^{1/2}(\theta)q_2, \quad q_3 ), \quad \theta = v_0 t / R$$

$$\vec{x}_1 = ( \beta_x^{1/2}(\theta - \delta\theta)q'_1, \quad \beta_y^{1/2}(\theta - \delta\theta)q'_2, \quad q'_3 ), \quad \delta\theta = v_0 \tau / R$$

$$\vec{X} = \vec{X}_c(t) + \vec{x}, \quad \vec{X}_1 = \vec{X}_c(t - \tau) + \vec{x}_1$$

**Both  $\tau$  and  $\vec{G}_1$  are only functions of  $(\vec{q}, \vec{q}')$ , not  $(\vec{p}, \vec{p}')$ .**

## Retardation in $\rho(\vec{q}, t - \tau)$

- Without any nonlinear perturbation and if a bunch matches with linear ring,  $f(\vec{q}, \vec{p}, t)$  is a constant of motion and  $\rho(\vec{q}, t)$  does not depend on  $t$  explicitly.
- With the perturbation of CSR,  $\rho(\vec{q}, t)$  depends on  $t$  weakly. If the emittance growth due to CSR is insignificant in the time scale of the retardation, i.e.

$$\left\langle \frac{\tau_{\max}}{\rho(\vec{q}, t)} \frac{\partial \rho(\vec{q}, t)}{\partial t} \right\rangle \sim \frac{\tau_{\max}}{\sigma} \frac{d\sigma}{dt} \ll 1$$

the retardation of the weakly  $t$ -dependence of  $\rho(\vec{q}, t - \tau)$  can be treated perturbatively.

- For a bunch that is slightly mismatched with linear ring, as long as the oscillation of the bunch is not significant in the time scale of the retardation, the perturbative treatment of the retardation in  $\rho(\vec{q}, t - \tau)$  should still be valid.

## Expansion of Weak $\tau$ -Dependence of $\rho(\vec{q}, t - \tau)$

$$\rho(\vec{q}, s - \tau_s) = \sum_{n=0} \frac{1}{n!} \frac{\partial^n \rho(\vec{q}, s)}{\partial s^n} (-\tau_s)^n$$

where  $s = v_0 t$  and  $\tau_s = v_0 \tau$ . Note that this expansion preserves the normalization condition of the distribution.

The force integral of SCR is then

$$\vec{F}_{\text{SR}}(\vec{q}, s) = \sum_{n=0} \vec{F}_n(\vec{q}, s)$$

$$\vec{F}_n(\vec{q}, s) = \int_{-\infty}^{\infty} \vec{G}_1(\vec{q}, \vec{q}', \tau_s) \frac{(-\tau_s)^n}{n!} \frac{\partial^n \rho(\vec{q}', s)}{\partial s^n} d\vec{q}'$$

If the time step in the simulation is small enough,

$$\vec{F}_{\text{SR}}(\vec{q}, s) \simeq \vec{F}_0(\vec{q}, s) + \vec{F}_1(\vec{q}, s) + O\left(\tau_{\text{max}}^2 \frac{\partial^2 \rho(\vec{q}, s)}{\partial s^2}\right)$$

where  $\vec{F}_0$  can be evaluated with the standard PIC method.

## How To Evaluate $\vec{F}_1$ , which needs $\partial\rho/\partial s$ ?

✗  $\partial\rho/\partial s$  can be obtained directly with numerical differentiation during the simulation — but it could be numerically unstable!

✓ Since  $\vec{G}_1(\vec{q}, \vec{q}', \tau_s)$  does not depend on the momenta,  $\partial\rho/\partial s$  can be obtained without the numerical differentiation with  $s$ .

To manipulate  $\partial\rho/\partial s$ , we start with the Liouville's equation,

$$\frac{\partial f}{\partial s} = -\frac{d\vec{p}}{ds} \cdot \frac{\partial f}{\partial \vec{p}} - \frac{d\vec{q}}{ds} \cdot \frac{\partial f}{\partial \vec{q}}$$

and the equation of motion with the kick approximation for CSR,

$$\begin{cases} \frac{d\vec{q}}{ds} = \frac{\partial H_0}{\partial \vec{p}} \\ \frac{d\vec{p}}{ds} = -\frac{\partial H_0}{\partial \vec{q}} + \sum_k \Delta s_k \vec{F}_{\text{SR}}(\vec{q}, s) \delta(s - s_k) \end{cases}$$

where  $H_0$  is the Hamiltonian without SCR,  $s_k$  is the location of the kick, and  $\Delta s_k$  is the duration between two successive kicks.

Substituting the equation of motion into the Liouville's equation yields

$$\begin{aligned}
\frac{\partial \rho}{\partial s} &= \int_{-\infty}^{\infty} \frac{\partial H_0}{\partial \vec{q}} \cdot \frac{\partial f}{\partial \vec{p}} d\vec{p} - \int_{-\infty}^{\infty} \frac{\partial H_0}{\partial \vec{p}} \cdot \frac{\partial f}{\partial \vec{q}} d\vec{p} \\
&\quad - \sum_k \Delta s_k \delta(s - s_k) \int_{-\infty}^{\infty} \vec{F}_{\text{SR}}(\vec{q}, s) \cdot \frac{\partial f}{\partial \vec{p}} d\vec{p} \\
&= -\frac{\partial}{\partial \vec{q}} \cdot \left\langle \frac{\partial H_0}{\partial \vec{p}} \right\rangle_{\vec{p}}
\end{aligned}$$

For an ultra-relativistic beam,

$$\left\langle \frac{\partial H_0}{\partial p_i} \right\rangle_{\vec{p}} = \frac{1}{\beta_i(s)} \int_{-\infty}^{\infty} p_i f(\vec{q}, \vec{p}, s) d\vec{p} = \frac{1}{\beta_i(s)} \langle p_i \rangle_{\vec{p}}, \quad i = 1 \text{ or } 2$$

$$\left\langle \frac{\partial H_0}{\partial p_3} \right\rangle_{\vec{p}} = \left[ 1 + \frac{\beta_x^{1/2}(s) q_1}{R} \right] \rho(\vec{q}, s)$$

where  $\beta_1 = \beta_x$  and  $\beta_2 = \beta_y$ .

## Final Formulas for Computing the Retarded Field

$$\vec{F}_0(\vec{q}, s) = \int_{-\infty}^{\infty} \vec{G}_1(\vec{q}, \vec{q}', \tau_s) \rho(\vec{q}', s) d\vec{q}'$$

$$\vec{F}_1(\vec{q}, s) = - \sum_{i=1}^2 \int_{-\infty}^{\infty} \vec{G}_i(\vec{q}, \vec{q}', s) \langle p_i \rangle_{\vec{p}} d\vec{q}'$$

$$- \int_{-\infty}^{\infty} \vec{G}_3(\vec{q}, \vec{q}', s) \rho(\vec{q}', s) d\vec{q}'$$

$$\vec{G}_i(\vec{q}, \vec{q}', s) = \frac{1}{\beta_i(s)} \frac{\partial(\tau_s \vec{G}_1)}{\partial q'_i}, \quad i = 1, 2$$

$$\vec{G}_3(\vec{q}, \vec{q}', s) = \left[ 1 + \frac{\beta_x^{1/2}(s) q'_1}{R} \right] \frac{\partial(\tau_s \vec{G}_1)}{\partial q'_3}$$

$\langle \vec{p} \rangle_{\vec{p}}$  as a function of  $\vec{q}$  and  $\rho(\vec{q}, s)$  can be constructed on a mesh in PIC simulation by using a similarly computational procedure.

## Retardation $\delta\theta = \tau_s/R$ as a Function of $(\vec{q}, \vec{q}')$

$$\begin{aligned}
 \delta\theta^2 = & \left[1 + \beta_x^{1/2}(\theta)\tilde{q}_1\right]^2 + \left[1 + \beta_x^{1/2}(\theta - \delta\theta)\tilde{q}'_1\right]^2 \\
 & + \left[\beta_y^{1/2}(\theta)\tilde{q}_2 - \beta_y^{1/2}(\theta - \delta\theta)\tilde{q}'_2\right]^2 + \tilde{q}_3^2 + \tilde{q}'_3{}^2 \\
 & - 2 \left\{ \left[1 + \beta_x^{1/2}(\theta)\tilde{q}_1\right] \left[1 + \beta_x^{1/2}(\theta - \delta\theta)\tilde{q}'_1\right] + \tilde{q}_3\tilde{q}'_3 \right\} \cos \delta\theta \\
 & + 2 \left\{ \tilde{q}_3 \left[1 + \beta_x^{1/2}(\theta - \delta\theta)\tilde{q}'_1\right] - \tilde{q}'_3 \left[1 + \beta_x^{1/2}(\theta)\tilde{q}_1\right] \right\} \sin \delta\theta
 \end{aligned}$$

where  $\vec{q} = \vec{q}/R$ . For a given mesh,  $\delta\theta$  can be solved numerically at all the grid points prior to particle tracking. If the change of  $\beta_x$  is small in the time scale of  $\delta\theta$ , one may use the approximation

$$\beta_x(\theta - \delta\theta) \simeq \beta_x(\theta) + 2R\alpha_x(\theta)\delta\theta$$

**If  $R \gg \max(|\vec{x}|, |\vec{x}_1|)$ ,  $\delta\theta \simeq 2 \left[3(\tilde{q}_3 - \tilde{q}'_3)\right]^{1/3}$**

If  $R \sim 10$  m,  $\sigma_z \sim 1$  mm,  $\sigma_x \sim 0.1$  mm, the error in this approximation is less than 1%.

## Green's Function in $(\vec{q}, \vec{q}')$ -Coordinate

$$\vec{G}_1(\vec{q}, \vec{q}', \tau) = \frac{e^2 r}{b_3^3} \left[ b_1 \vec{n} + b_2 (\vec{n} \cdot \vec{\beta} - 1) \vec{\beta}_1 + b_3 (\vec{n} \cdot \vec{\beta} - 1) \frac{1}{c} \dot{\vec{\beta}}_1 \right]$$

where

$$\begin{aligned} \vec{r} = & [(R + x) - (R + x_1) \cos \delta\theta - z_1 \sin \delta\theta] \vec{e}_x \\ & + (y - y') \vec{e}_y + [(R + x_1) \sin \delta\theta + z - z_1 \cos \delta\theta] \vec{e}_z \end{aligned}$$

$$\vec{\beta}_1 = \vec{e}_x \sin \delta\theta + \vec{e}_z \cos \delta\theta$$

$$\frac{1}{c} \dot{\vec{\beta}}_1 = \frac{1}{R} (-\vec{e}_x \cos \delta\theta + \vec{e}_z \sin \delta\theta)$$

$$b_1 = b_2 (1 - \cos \delta\theta) - \frac{b_3}{R} \sin \delta\theta$$

$$b_2 = \frac{1}{\gamma^2} + r \left( \frac{1}{c} \dot{\vec{\beta}}_1 \cdot \vec{n} \right)$$

$$b_3 = r (1 - \vec{n} \cdot \vec{\beta}_1)$$

# PIC Method for Evaluating Quadratures of $\vec{F}_0$ and $\vec{F}_1$

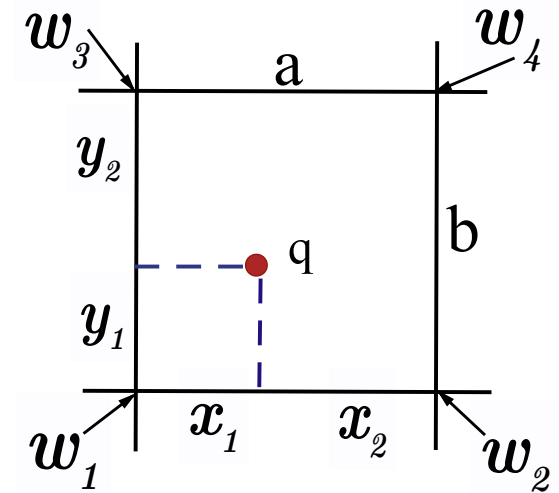
(Construct  $\rho(\vec{q}, s)$  and  $\langle \vec{p} \rangle_{\vec{p}}$  on grids with “Cloud-in-Cell”)

- Weight of  $q$  in horizontal,

$$\text{“at the left”} = \frac{x_2}{a}, \quad \text{“at the right”} = \frac{x_1}{a}$$

- Weight of  $q$  in vertical,

$$\text{“at the bottom”} = \frac{y_2}{b}, \quad \text{“at the top”} = \frac{y_1}{b}$$



⇒ Weight of  $q$  at four corners:

$$w_1 = \frac{x_2}{a} \frac{y_2}{b}, \quad w_2 = \frac{x_1}{a} \frac{y_2}{b}, \quad w_3 = \frac{x_2}{a} \frac{y_1}{b}, \quad w_4 = \frac{x_1}{a} \frac{y_1}{b}$$

Contribution to  $\rho(\vec{q}, s)$ :  $\rho(j) \leftarrow \rho(j) + w_j$ ,  $j = 1, \dots, 4$

Contribution to  $\langle \vec{p} \rangle_{\vec{p}}$ :  $\langle \vec{p} \rangle_{\vec{p}}(j) \leftarrow \langle \vec{p} \rangle_{\vec{p}}(j) + \vec{p} w_j$

## SUMMARY

- A 1st-order perturbation expansion of the retarded bunch particle distribution is developed for a study of the CSR effect numerically without a need of memorizing the history of the distribution.
- The use of the PIC method provides a smooth particle distribution constructed on a mesh in configuration space so that the expansion of the retarded distribution can be calculated on the mesh with a desired accuracy.
- The current approach of the perturbative-PIC method neglects the effect of beam pipe on the radiation field and, therefore, can only be applied to the unshielded CSR problem. Methods of including the shielding effect need to be explored.