Synchrobetatron coupling in a storage ring with transverse electrostatic fields

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**Abstract**

The literature on synchrobetatron coupling is extended to include rings with transverse electrostatic fields. The analytical formulas are compared against numerical tracking results for a smooth focusing all-electric ring. Agreement is obtained for tune shifts caused by the synchrobetatron coupling.

**1. Introduction**

There is interest in storage rings with transverse electrostatic guiding and focusing fields, to search for a permanent electric dipole moment (EDM) of a charged particle using a stored polarized beam. I published a paper giving equations of motion for the orbital and spin motion in a storage ring with static electric and magnetic fields [1]. However, I did not treat synchrobetatron coupling in that paper. Numerical tracking results using a model of an all-electric ring indicate that the observed betatron and synchrotron tunes, etc., do not match the values from the naïve one-turn map equations for the betatron and synchrotron oscillations when there is dispersion at the location of the rf cavity. The synchrobetatron coupling shifts the orbital tunes, for example. I shall derive the equations for synchrobetatron coupling for a ring with transverse electrostatic guiding and focusing fields. I shall compare the analytical results for a smooth focusing all-electric ring against results from numerical tracking. The agreement is satisfactory. Note that the synchrobetatron coupling also affects the dynamic aperture, but it is beyond the scope of this paper to present detailed analytical or tracking studies for the dynamic aperture in the longitudinal phase space. I begin by treating a stationary rf bucket above transition. The general case, for motion above or below transition, including both electric and magnetic guiding and focusing fields and a non-stationary rf bucket, is treated in Section 2.5. Some important technical details are presented in Appendix A.

**2. Synchrobetatron coupling**

**2.1. General remarks**

I follow the notation and formalism in Ref. [1]. I treat a particle of mass $m$ and charge $e$, with velocity $v = \beta c$ and Lorentz factor $\gamma = 1 / \sqrt{1 - \beta^2}$. In most of this paper I shall set $c = 1$. The independent variable is the arc-length $s$ along the reference orbit. In some sections I shall use $\theta = s/R$, where the ring circumference is $2\pi R$. A prime denotes differentiation with respect to the independent variable. I shall treat only motion without transverse betatron coupling. From Ref. [1] and references therein, the one-turn map equations are, for uncoupled betatron motion and a stationary rf bucket above transition and $eV_0 > 0$,

$$\Delta H = -eV_0 \sin(\omega_H \Delta t),$$

$$\Delta \phi = \omega_H \Delta t = \frac{2\pi h}{\beta_0} \left( \alpha - \frac{1}{\gamma_0} \right) \frac{\Delta H}{H_0}$$

(2.1a)

(2.1b)

Here $\Delta t = t - t_0$ where $t_0$ is the time of arrival of the reference particle, $\Delta H/H_0$ is the relative energy offset, $\omega_H$ is the rf angular frequency, $h$ is the harmonic number and $\alpha$ is the momentum compaction factor. The above equations implicitly assume that the dispersion is zero at the rf cavity, hence there is no synchrobetatron coupling.
coupling. This is not satisfactory for many lattice designs of storage rings for EDM searches.

2.2. Synchrobetatron theory

There are many papers on synchrobetatron coupling and it is impossible to cite them all. The following papers employ a Hamiltonian formalism [2–6]. There is also early unpublished work by Chao and Morton [7] and a matrix formalism by Huang [8]. I shall employ the Hamiltonian formalism below. Naturally, all of the above references treat only rings with magnetic guiding and focusing fields. A notable paper on synchrobetatron coupling theory was published by Suzuki [21]. (Suzuki himself mentions an earlier paper by Corsten and Hagedoorn [3] and unpublished work by Morton and Chao, which Suzuki cites as a private communication.) I revised the theory, which I shall quote from below, was published by Baartman [4]. Baartman quotes Suzuki for the Hamiltonian, for an all-magnetic ring

\[ H = \frac{R_0}{2} \left[ Kx_0^2 + \frac{p_{x0}^2}{\beta_{x0}^2} \right] - \frac{1}{2} \frac{\hbar^2}{\rho} \frac{1}{\gamma^2} W^2 - \frac{\hbar}{\rho \gamma} \sum_{\ell} \psi \langle \delta p (\theta - \eta) \rangle (\cos \psi + \psi \sin \phi_x). \]  

(2.2)

The independent variable is \( \theta = s/R \). Also \( p_0 \) is the reference momentum, \( K \) is the (normalized) horizontal focusing gradient, \( D \) is the (horizontal) dispersion function, \( 1/\rho \) is the curvature of the reference orbit (and is zero in non-bends) and \( \phi_x \) is the synchronous phase. The definition of \( \psi \), etc. will be given below. I modified Baartman’s expression to make the \( \delta \)-function periodic

\[ \delta_p (\theta - \eta) = \sum_{k = -\infty}^{\infty} \delta (\theta - \eta) - 2k \pi. \]  

(2.3)

Also, \( x \) and \( p_x \) are canonical variables with respect to the dispersed closed orbit. The total horizontal motion is

\[ x_{tot} = x_0 + D \Delta E / E_0 = \frac{p_{x0} \beta_{x0}}{\rho_0} + D' \Delta E / E_0. \]  

(2.4)

Here \( \Delta E \) is the energy offset and \( W = \Delta E / \omega_{\text{ce}} \) and

\[ \psi = \frac{h}{R} \left( \frac{p_x}{p_0} - D' x \right). \]  

(2.5)

This is the key: the presence of the phase \( \psi \) in the Hamiltonian in Eq. (2.2) yields the kicks to \( x \) and \( p_x \) at the rf cavity, hence the synchrobetron coupling.

For comparison with later work in this paper, let me rewrite the effective Hamiltonian using the notation (or conventions) I employed in Ref. [1]. Also, I shall treat an all-electric ring. I use \( s \) as the independent variable, so my notation is probably closer to that employed by Lee [5] (who cites both Suzuki [2] and Baartman [4]). I treat a model with only one rf bucket, above transition, so \( \phi_x = \pi \). I employ \( H \) and \( \rho \) directly, instead of \( \psi \) and \( E / \omega_{\text{ce}} \). There is a sign difference between my definition of \( \phi \) and the above, basically \( t - t_x \) instead of \( t - t_x \). I define \( \omega = \omega_{\text{ce}}(t - t_x) \) and I redefine

\[ \psi = \omega_{\text{ce}} \left[ t - t_x - \frac{p_0}{\beta_{x0} H_0} \left( D_x \frac{p_x}{p_0} - D' x \right) \right] = \omega_{\text{ce}}(t - t_x) - \frac{h}{R} \left( D_x \frac{p_x}{p_0} - D' x \right). \]  

(2.6)

I change notation from \( H \) to \( K_{ab} \) and put \( c = 1 \). For an all-electric ring, it was shown in Ref. [1] that the term in \( D_x \) (i.e. the momentum compaction) is multiplied by a factor of \( 1 + 1/\gamma^2 \). Hence the linearized Hamiltonian is

\[ K_{ab} = \frac{p_0}{2} \left[ Kx_0^2 + \frac{p_{x0}^2}{\beta_{x0}^2} \right] - \frac{1}{2p_0^2} \left[ \left( 1 + \frac{1}{\gamma^2} \right) D_x \frac{p_x}{p_0} - \frac{1}{\gamma^2} \left( \Delta H \right)^2 / H_0 \right] + \frac{eV_0}{\omega_{\text{ce}}} \cos \psi \delta_p (s - \sigma_t). \]  

(2.7)

Then the equations of motion in the longitudinal plane are

\[ \frac{dH}{ds} = -eV_0 \sin \psi \delta_p (s - \sigma_t). \]  

(2.8a)

\[ \frac{dt}{ds} = \frac{\partial K_{ab}}{\partial E} = -\frac{1}{p_0^2} \left[ \left( 1 + \frac{1}{\gamma^2} \right) D_x \frac{p_x}{p_0} - \frac{1}{\gamma^2} \Delta H \right]. \]  

(2.8b)

The equations of motion in the transverse plane are

\[ \frac{dx}{ds} = -\frac{\partial K_{ab}}{\partial p_x} = -p_x \frac{eV_0 D_x}{\beta_{x0} H_0} \sin \psi \delta_p (s - \sigma_t). \]  

(2.9a)

\[ \frac{dp_x}{ds} = -\frac{\partial K_{ab}}{\partial x} = -p_x \frac{eV_0 p_0 D_x}{\beta_{x0} H_0} \sin \psi \delta_p (s - \sigma_t). \]  

(2.9b)

The equations for the longitudinal motion agree with Eq. (2.1), with additional terms in the phase \( \psi \). The \( x \) and \( p_x \) transverse motion couples to the longitudinal motion via the phase \( \psi \). The above equations of motion are for an all-electric ring. They are essentially the same as for an all-magnetic ring, with just the correct substitution for the momentum compaction factor.

2.3. Canonical transformation

Note that I did not really derive the effective Hamiltonian in Eq. (2.7); instead I obtained it by analogy with the Hamiltonian in Eq. (2.2). Let me derive the effective synchrobetron Hamiltonian from first principles. Suzuki [6] (in a 2002 paper) has given a nice summary of the canonical transformations. To develop the notation and formalism, I treat an all-magnetic ring in this section, so I set \( \Phi = 0 \) below. I shall treat an all-electric ring later. The original Hamiltonian is

\[ K = -\left( 1 + \frac{x}{\rho_0} \right) \left[ h^2 - m^2 - p_x^2 - p_z^2 \right]^{1/2} - eA_z. \]  

(2.10)

We wish to express \( x \) and \( p_x \) as a sum of betatron and dispersion terms \( \Delta H = H - H_0 \) for brevity)

\[ x = x_0 + x_d \]  

\( p_x = p_{x0} + p_{x0} \) \( x_d = D_x H / \beta_{x0} H_0 \) \( p_{x0} = p_{x0} D_x H / \beta_{x0} H_0 \) \( \Delta H / \beta_{x0} H_0 \) \( \Delta H / \beta_{x0} H_0 \) \( \Delta H / \beta_{x0} H_0 \) \( \Delta H / \beta_{x0} H_0 \) \( \Delta H / \beta_{x0} H_0 \) \( \Delta H / \beta_{x0} H_0 \)

(2.11)

The derivative \( D_x \) is taken with respect to the arc-length \( s \). We also define offsets

\[ \varepsilon = H - H_0, \qquad \tau = t - \frac{s}{\rho_0} \]  

(2.12)

To perform the required canonical transformation, we introduce a generating function from \( (x, p_x) \) to \( (x_d, p_{x0}) \) and \( (z, p_z) \) to \( (z_d, p_{z0}) \) and \( (H,t) \) to \( (\varepsilon, \tau) \). Suzuki [6] has published the answer:

\[ G = \Delta H \left( \frac{s}{\rho_0} + 2p_{x0} x_d + p_z \right) + 2p_{z0} z_d \]  

(2.13)

Note that this generating function does not depend on the vector or scalar potentials, hence it will be the same for an all-electric ring, which I shall treat in the next section. Note that there is no change to \( z \) and \( p_z \), because \( z = z_d \) and \( p_z = p_{z0} \). Then

\[ x_d = \frac{\partial G}{\partial p_{x0}} = x - \frac{D_x H - H_0}{H_0}. \]  

(2.14a)
\[ p_x = \frac{\partial G}{\partial x} = p_{0\beta} + p_{0\beta} D_s H - H_0, \]  

(2.14b)

\[ \varepsilon = \frac{\partial G}{\partial \varepsilon} = H - H_0, \]  

(2.14c)

\[ t = \frac{\partial G}{\partial t} = \tau + \frac{s}{p_0} D_s H - H_0 + p_{0\beta} \frac{\partial G}{\partial p_0} \left( D_s H - H_0 \right). \]  

(2.14d)

Also write \( \tau = s/p_0 = \tau + \tau_\beta \), where

\[ \tau_\beta = \frac{p_{0\beta} D_s H - p_{0\beta} D_s H}{p_0 H_0}. \]  

(2.15)

Collect old variables on the left hand side and new variables on the right hand side to obtain

\[ x = \tau + p_{0\beta} \varepsilon \]  

(2.16)

Then the transformed Hamiltonian is

\[ K_{ab} = K + \frac{\partial G}{\partial \varepsilon} - \left( 1 + \frac{x}{p_0} \right) \left( H_0 + \varepsilon \right) - m^2 - (p_{0\beta} + p_{0\beta}^2 + p_{0\beta}^3)^{1/2} - \varepsilon H_0 \]  

(2.17)

The term in \( p_{0\beta} \) in the square root is neglected by Suzuki and others. Basically, the one-turn average of \( p_{0\beta} \) vanishes, because \( \langle p_{0\beta} \rangle \propto \frac{1}{\beta} \), and for the longitudinal motion a one-turn average is a good approximation. We need to quantize the vector potential. It includes the guiding and focusing fields and the rf cavity. We can neglect higher multipoles, hence for our purposes the answer is

\[ -\varepsilon A_\tau \approx \frac{p_{0x} x + p_{0x}^2 + p_{0x}^2}{2p_0} K_1 \left( x^2 - x^2 \right) + \frac{e_0}{\alpha_\tau} \cos \left( \alpha_\tau \left( \frac{t - s}{p_0} \right) \right) \delta_p (s - s_\tau). \]  

(2.18)

Then, neglecting many cross-terms in \( p_{0\beta}^2 \) and \( p_{0\beta}^2 \), etc., we obtain

\[ K_{ab} \approx -\left( 1 + \frac{x}{p_0} \right) \left[ \left( 2H_0 + \varepsilon \right) - m^2 - (p_{0\beta} + p_{0\beta}^2 + p_{0\beta}^3)^{1/2} - \varepsilon H_0 \right] \]  

(2.19)

Note that terms in \( x p_{0\beta} \), etc., have been neglected. Now use Hill's equation

\[ D_x + K_x D_x = \frac{1}{p_0}. \]  

(2.20)

Then

\[ \frac{p_0}{2} K_x^2 = \frac{p_0 e_0^2}{2p_0^2 H_0} K_x D_x = \frac{e_0^2}{2p_0^2 H_0} \left( D_x - D_0 D_s^* \right) \]  

(2.21)

The term in \( D_0 \) is neglected in Ref. [6], probably because its one-turn average also vanishes. Then, dropping the constant term in the Hamiltonian, we obtain

\[ K_{ab} \approx \frac{p_0^2 + p_0^2}{2p_0^2} - \frac{p_0}{2} \left( K_x^2 + K_x^2 \right) - \frac{1}{2p_0^2} \left( \frac{D_x}{p_0^2} - \frac{1}{p_0^2} \right) \]  

(2.22)

This agrees with the previously stated expression in Eq. (2.22).

2.4. Canonical transformation: all-electric ring

Now I shall treat an all-electric ring. The Hamiltonian is

\[ K = -\left( 1 + \frac{x}{p_0} \right) \left( H_0 - \Phi^2 - m^2 \right) \]  

(2.23)

The vector potential is nonzero only in the rf cavity. Note by definition that the electrostatic potential is quadratic and higher (i.e. non-bending) if \( 1/p_0 = 0 \). We say that \( 1/p_0 = 1/r_0 \) in all bends (isoelectric model). Expand the potential in a Taylor series, with \( \Phi(0) = 0 \). Because the ring has midplane symmetry, the Taylor series contains all powers of \( x \) but only even powers of \( z \). Using a general-relativistic notation for the partial derivatives,

\[ \Phi = \Phi_x x + \Phi_x z \frac{x^2}{2} + \Phi_z \frac{z^2}{2} \]  

(2.24)

Then put \( x = x_0 + x \) and expand to quadratic order. We need a more detailed expression for \( \Phi_x \) below. We equate the centripetal acceleration on the reference orbit. Then

\[ \frac{\gamma_0 m v_0^2}{r_0} = -E_x = \Phi_x. \]  

(2.25)

Hence

\[ \Phi_x = \frac{p_0}{r_0}. \]  

(2.26)

We shall need this below, and as noted above, the bending field is zero if \( 1/p_0 = 0 \). As noted above, the generating function is the same as in Eq. (2.13). Hence the transformation of variables is the same as in Eq. (2.14). We obtain the transformed Hamiltonian

\[ K_{ab} = K + \frac{\partial G}{\partial \varepsilon} - \left( 1 + \frac{x}{p_0} \right) \left( H_0 + \varepsilon \right) - m^2 - (p_{0\beta} + p_{0\beta}^2 + p_{0\beta}^3)^{1/2} \]  

(2.27)

We again neglect \( p_{0\beta} \) in the square root. Then

\[ K_{ab} \approx \left( 1 + \frac{x}{p_0} \right) \left[ \left( 2H_0 + \varepsilon \right) - m^2 - (p_{0\beta} + p_{0\beta}^2 + p_{0\beta}^3)^{1/2} - \varepsilon H_0 \right] \]  

(2.28)

The linear term in \( \varepsilon \) cancels

\[ -\left( 1 + \frac{x}{p_0} \right) \left[ \left( H_0 - \Phi^2 - m^2 \right) - \frac{\varepsilon}{p_0} \right] \]  

(2.29)

The linear term in \( x \) also cancels. In non-bends such a term does not exist. In bends we set \( p_0 = r_0 \) and use Eq. (2.26):

\[ -\left( 1 + \frac{x}{p_0} \right) \left[ \left( H_0 - \Phi^2 - m^2 \right) - \frac{\varepsilon}{p_0} \right] \]  

(2.30)

Then, continuing with the expansion, we obtain

\[ K_{ab} \approx \left( 1 + \frac{x}{p_0} \right) \left[ \left( 2H_0 + \varepsilon \right) - m^2 - (p_{0\beta} + p_{0\beta}^2 + p_{0\beta}^3)^{1/2} - \varepsilon H_0 \right] \]  

(2.31)
We now apply Hill’s equation twice. First we recall that for an
constant term in the Hamiltonian

\[ K_{xx} \propto \phi_1 + \phi_2 K_{xz} + \phi_3 z^2 - \frac{1}{2\rho_0} \left( 1 + \frac{1}{\gamma_0^2} \right) \frac{\varepsilon(x_0 + x_0^2)}{2p_0} \]

Then the terms in \( \xi_0 \) cancel

\[ p_0 K_{xx} \xi_1 - \left( 1 + \frac{1}{\gamma_0^2} \right) \frac{\varepsilon x_0}{2p_0} - \frac{\varepsilon x_0}{2p_0} \left[ D_x^* + \left( 1 + \frac{1}{\gamma_0^2} \right) \frac{1}{\rho_0} \right] \]

The term in \( D_x^* \) is again neglected. Next the terms in \( x_0 \varepsilon \) combine

\[ \frac{p_0}{2} K_{xx} \xi_2 + \left( 1 + \frac{1}{\gamma_0^2} \right) \frac{\varepsilon x_0}{2p_0} = \frac{x_0 \varepsilon}{2p_0} \left[ D_x^* + \left( 1 + \frac{1}{\gamma_0^2} \right) \frac{1}{\rho_0} \right] \]

As before, the term in \( D_x^* \) is neglected. Then, dropping the constant term in the Hamiltonian

\[ K_{ab} \propto \frac{p_0^2}{2p_0} + \frac{p_0^2}{2p_0} q(K_{xx} + K_{xz})^2 \]

This agrees with the expression in Eq. (2.7), which justifies the analogy used to obtain Eq. (2.7).

2.5. General case

It is obvious by now that the general case of a ring with both electric and magnetic guiding and focusing fields is given simply by substituting the appropriate expression for the momentum compaction factor. From Ref. [1], if the guide fields are \( E_0 \) and \( B_0 \) for the electric and magnetic fields (radial and vertical, respectively), then the momentum compaction factor is

\[ \alpha = \left[ 1 + \frac{1}{\gamma_0^2} \frac{E_0/\rho_0}{B_0 + E_0/\rho_0} \right] \left( D_x^* / \rho_0 \right) \]

Hence I shall skip the unnecessary algebra and quote the answer for the effective Hamiltonian. It is straightforward to include motion below transition. For completeness of the exposition, I also generalize to a non-stationary rf bucket. The effective Hamiltonian is

\[ K_{ab} = \left[ \frac{p_0^2}{2p_0} + \frac{p_0^2}{2p_0} q(K_{xx} + K_{xz})^2 \right] - \frac{1}{2\rho_0} \left[ 1 + \frac{1}{\gamma_0^2} \frac{E_0/\rho_0}{B_0 + E_0/\rho_0} \right] \left( D_x^* / \rho_0 \right) \]

This Hamiltonian is valid both above and below transition. For a stationary rf bucket, \( \phi_2 = 0 \) below transition and \( \phi_2 = \pi \) above transition. This generalizes Eq. (2) in Ref. [5], and agrees with it in the case of an all-magnetic ring, if one recognizes that \( \varepsilon(1/\rho_0^2) = \Delta \rho_0 / \rho_0 \) in an all-magnetic ring. Note that Lee [5] treats multiple rf cavities, but it is straightforward to generalize Eq. (2.39) to include multiple rf cavities.

2.6. Comparison with tracking

In this section, I shall compare the results from the above formalism with the output from numerical tracking simulations. I shall treat an all-electric smooth focusing model, viz. a pure radial electric field (Such a model has no vertical focusing, but I shall track only motion in the horizontal plane.) The electrostatic potential is \( \Phi = E_0 \rho_0 \ln(1+x_0/\rho_0) \). From Ref. [1], the dispersion for this model is a constant, viz. \( D_x = \rho_0 \), \( D_y = 0 \). For ease of calculation, I replace the \( \delta \)-function in the rf cavity by its one-turn average, i.e. \( \delta_0 \) below transition. Then from Eq. (2.37) the linear dynamical terms are

\[ K_{\text{smooth}} = \left( 1 - \frac{eV_{N\theta h}}{2\pi H_0 \rho_0} \right) \left( \frac{p_0^2}{2p_0} + \frac{p_0^2}{2p_0} q(K_{xx} + K_{xz})^2 \right) - \frac{1}{2\rho_0} \left[ 1 + \frac{1}{\gamma_0^2} \frac{E_0/\rho_0}{B_0 + E_0/\rho_0} \right] \left( D_x^* / \rho_0 \right) \]

Fig. 1 displays the phase space plot for the above model, of an orbit in the longitudinal phase space. One particle was tracked, with an initial value of \( x = p_x = 0 \) and \( \Delta H / H_0 = 10^{-6} \). The synchrobetatron coupling causes the motion not to be a perfect ellipse. The Fourier transform (power spectrum) of the motion is plotted in Fig. 2. The power spectrum indicates peaks at both the synchrotron tune and betatron tunes. It is well known that for this model, the (square of the) horizontal betatron tune is given by (see Ref. [1] and references therein)

\[ \nu^2_x = 1 + \frac{1}{\gamma_0^2} \]

From Eq. (2.40), the synchrobetatron coupling shifts the (squared) betatron tune to

\[ \nu^2_x \rightarrow \left( 1 - \frac{eV_{N\theta h}}{2\pi H_0 \rho_0} \right) \left( 1 + \frac{1}{\gamma_0^2} \right) \]

For protons at the magic gamma \( p = 0.7 \) GeV/c, with an rf cavity peak voltage of 0.5 MV and \( n = 100 \),

\[ 1 + \frac{1}{\gamma_0^2} = 1.6419, \quad \frac{eV_{N\theta h}}{2\pi H_0 \rho_0} \approx 0.01897 \]

Without synchrobetatron coupling, the small amplitude horizontal betatron tune is \( \sqrt{1.6419} \approx 1.2814 \). With synchrobetatron coupling, the betatron tune is shifted to \( \sqrt{1.6419} \approx 1.2692 \). This agrees with the result observed in the tracking simulations in Fig. 2. The tuneshift is \( \Delta \nu_x \approx -0.0122 \). It is also well known that without synchrobetatron coupling, the small amplitude synchrotron tune from Eq. (2.1) is given by

\[ \nu^2 = \frac{eV_{N\theta h}}{2\pi H_0 \rho_0} \]

For the above parameter values, this yields \( \nu^2 \approx 0.01897 \approx 0.1378 \). The tracking result in Fig. 2 is however \( \nu^2 \approx 0.15 \). For stable small
amplitude motion, the sum of the tunes $\nu_s + \nu_z$ is not changed by the synchrobetatron coupling, so $\Delta \nu_z = -\Delta \nu_s$, hence in this case $\Delta \nu_z \approx 0.0122$. If we add the tuneshift we obtain $\nu_z \approx 0.1378 + 0.0122 \approx 0.1499$, which matches the tracking result $\nu_z \approx 0.15$ in Fig. 2.

3. Conclusion

I have extended the formalism for synchrobetatron coupling to treat rings with transverse electrostatic guiding and focusing fields. I showed that the published formulas, which treat rings with magnetic fields, should be modified by using an appropriate expression for the momentum compaction factor. I applied the formalism to compare the analytical formulas with numerical tracking results for an all-electric ring with a radial electric field and obtained good quantitative agreement with the tracking data.

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Appendix A. Dispersion and transition

I begin with a proof of the existence of the dispersion function in a ring with an arbitrary combination of electric and magnetic guiding and focusing fields. We naturally assume that the ring is designed so that the (on-energy) betatron oscillations are stable. To treat off-energy orbits, we invoke the implicit function theorem: if the on-energy motion has a stable fixed point in the phase space of particle orbits, and the focusing and bending are differentiable functions of the energy, then for sufficiently small energy offsets, i.e. sufficiently small $|\Delta H/H_0|$, a (stable) fixed point will also exist for the off-energy orbits. Note that some care is required to apply the implicit function theorem: it is essential that the dispersion be defined in terms of variations in the total energy (i.e. $\Delta H/H_0$), which is a dynamical invariant in the phase space of the particle orbits. The value of $\gamma$ is not a dynamical invariant in a ring with electrostatic guiding and focusing fields, hence a dispersion function defined in terms of variations in $\gamma$ (i.e. $\Delta \gamma/\gamma_0$), is not well defined.

For quantitative work, it is convenient to employ the parameter $\lambda_p$, where $\Delta H/H_0 = \beta_p \lambda_p$ (see Ref. [1]). The dispersion orbit is then written as $x = D_x \lambda_p$. The frequency slip is given by (Eq. (9.5) in Ref. [1])

$$\left< \frac{\Delta f}{f_0} \right> = \left( \frac{1}{f_0^2} - \alpha^2 \right) \lambda_p.$$  

(A.1)

Here $\alpha$ is the momentum compaction factor and the angle brackets denote an average around the ring. Transition occurs when $\gamma_0^2 - \alpha = 0$. In an all-magnetic ring, $\alpha$ does not depend on $\gamma_0$ and transition occurs when $\gamma_0 = 1/\sqrt{\alpha}$. This yields a physical transition energy if $0 < \alpha < 1$. However, in a ring with electrostatic guiding and focusing fields, the betatron tunes, dispersion and momentum compaction factor all depend on the value of $\gamma_0$. The detailed expressions for the focusing gradient, etc. were given in Ref. [1] and they depend on $\gamma_0$. Hence $\alpha$ depends on $\gamma_0$ and we must solve for transition more carefully. It is not guaranteed that a (real) solution for the transition energy exists.

To illustrate the matter, I display explicit expressions for the dispersion, etc. for a homogenous weak focusing all-electric ring. The field index $n$ is given by $1 + n = -(\nu_s/E_s) (dE_s/dx)$ and is a constant around the ring. It is known to workers in the field that the horizontal and vertical betatron tunes are $\nu_x = \sqrt{2 - \beta_0^2 - n} = \sqrt{1 - n + 1/\gamma_0^2}$ and $\nu_z = \sqrt{n}$, respectively.2 The dispersion for this model is given by (Eq. (7.7) in Ref. [1])

$$D_x = \frac{\gamma_0}{\nu_x^2} \left( 1 + \frac{1}{\gamma_0^2} \right).$$  

(A.2)

I shall restrict $0 < n \leq 1$ because if $n < 0$ then the vertical motion is unstable and if $n > 1$ then the horizontal motion can be unstable, depending on the value of $\gamma_0$.  

\[\text{Fig. 1. Phase space plot of an orbit in the longitudinal phase space for an all-electric ring with a radial electric field. Here $\phi$ is the phase of the longitudinal motion (defined in the text). One particle was tracked, with an initial value of $x = p_x = 0$ and $\Delta H/H_0 = 10^{-4}$.

Fig. 2. Fourier transform (power spectrum) of the motion displayed in Fig. 1. Here $\nu$ is the tune. The power spectrum indicates peaks at both the synchrotron and betatron tunes (and also the mirror tunes).} \]
The momentum compaction factor is (Eq. (9.9) in Ref. [1])

\[ \alpha = \frac{1}{\nu_0^2} \left( 1 + \frac{1}{\nu_0^2} \right)^2. \]  

(A.3)

Note that \( \alpha > 1 \) for \( 0 < n \leq 1 \), which gives us a clue that the motion is always above transition in this model. Then

\[ \frac{1}{\nu_0^2} - \alpha = \frac{1}{\nu_0^2} \left( 1 + \frac{1}{\nu_0^2} \right)^2 - \frac{1 + (1+n)/\nu_0^2}{1 - n + 1/\nu_0^2}. \]  

(A.4)

This is negative for real \( \gamma_0 \) and \( 0 < n \leq 1 \). Notice that the numerator vanishes when \( \nu_0^2 = -(1+n) \), which implies an imaginary transition energy. Hence a homogenous weak focusing all-electric ring always operates above transition, for any real value \( \gamma_0 \geq 1 \). This was the case for the tracking studies reported in Section 2.6.\(^3\) For strong focusing designs of an EDM storage ring, it is possible for the motion to be below transition. The effective Hamiltonian for the general case is given in Section 2.5.

Note that a real transition energy does not need to exist; it is possible for a ring to always operate on only one side of transition.

This is so even for all-magnetic rings. For example, for a homogenous weak focusing all-magnetic ring with field index \( n \) (given by \( n = -(\gamma_0/\nu_0)(\partial B_z/\partial x_0) \)), the horizontal and vertical betatron tunes are \( \nu_x = \sqrt{1-n} \) and \( \nu_z = \sqrt{n} \), respectively, the dispersion is \( D_x = r_0/\nu_x^2 \) and the momentum compaction factor is \( \alpha = 1/\nu_x^2 = 1/(1-n) \). Then \( \alpha > 1 \) for \( 0 < n < 1 \). The motion in a homogenous weak focusing all-magnetic ring is also always above transition.

References


\(^3\) Note that the field index was \( n = 0 \) for the model in Section 2.6. There was no vertical focusing, but I tracked only motion in the horizontal plane.