The path integral for photons

based on S-57

We will discuss the path integral for photons and the photon propagator more carefully using the Lorentz gauge:

$$Z_0(J) = \int \mathcal{D}A \, e^{iS_0} \,,$$

$$\mathcal{L} = +\tfrac{1}{2} A_\mu (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) A_\nu + J^\mu A_\mu$$

as in the case of scalar field we Fourier-transform to the momentum space:

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{\varphi}(k)(k^2+m^2)\widetilde{\varphi}(-k) + \widetilde{J}(k)\widetilde{\varphi}(-k) + \widetilde{J}(-k)\widetilde{\varphi}(k)\Big]$$

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{A}_\mu(k)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big)\widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k)\widetilde{A}_\mu(k)\Big]$$

we shift integration variables so that mixed terms disappear...

$$\widetilde{\chi}(k) = \widetilde{\varphi}(k) - \frac{\widetilde{J}(k)}{k^2+m^2}$$

Problem: the matrix has zero eigenvalue and cannot be inverted.

To see this, note:

$$k^2 g^{\mu\nu} - k^\mu k^\nu = k^2 P^{\mu\nu}(k)$$

where

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2$$

is a projection matrix

$$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k)$$

and so the only allowed eigenvalues are 0 and +1

Since

$$P^{\mu\nu}(k) k_\nu = 0$$

$$g_{\mu\nu} P^{\mu\nu}(k) = 3$$

it has one 0 and three +1 eigenvalues.

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{A}_\mu(k)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big)\widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k)\widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k)\widetilde{A}_\mu(k)\Big]$$

We can decompose the gauge field $\widetilde{A}_\mu(k)$ into components aligned along a set of linearly independent four-vectors, one of which is k_μ and then this component does not contribute to the quadratic term because

$$P^{\mu\nu}(k)k_\nu = 0$$

and it doesn't even contribute to the linear term because

$$\partial^\mu J_\mu(x) = 0 \longrightarrow k^\mu \widetilde{J}_\mu(k) = 0$$

and so there is no reason to integrate over it; we define the path integral as integral over the remaining three basis vector; these are given by

$$k^\mu \widetilde{A}_\mu(k) = 0$$

which is equivalent to

$$\partial^\mu A_\mu(x) = 0$$

Lorentz gauge

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2$$

Within the subspace orthogonal to k_μ the projection matrix is simply the identity matrix and the inverse is straightforward; thus we get:

$$Z_0(J) = \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde{J}_\mu(k)\,\frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}\,\tilde{J}_\nu(-k)\right]$$

$$= \exp\left[\frac{i}{2}\int d^4x\,d^4y\,J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right]$$

going back to the position space

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\,\frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}$$

propagator in the Lorentz gauge (Landau gauge)

we can again neglect the term with momenta because the current is conserved and we obtain the propagator in the Feynman gauge:

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}\,\tilde{\Delta}^{\mu\nu}(k) \qquad \tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\epsilon}$$

The path integral for nonabelian gauge theory

based on S-71

Now we want to evaluate the path integral for nonabelian gauge theory:

$$Z(J) \propto \int \mathcal{D}A \, e^{iS_{\text{YM}}(A,J)} \, ,$$

$$S_{\text{YM}}(A,J) = \int d^4x \left[-\tfrac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + J^{a\mu} A^a_\mu \right]$$

for U(1) gauge theory, the component of the gauge field parallel to the four-momentum k^μ did not appear in the action and so it should not be integrated over; since the U(1) gauge transformation is of the form $A_\mu(x) \to A_\mu(x) - \partial_\mu \Gamma(x)$, excluding the components parallel to k^μ removes the gauge redundancy in the path integral.

nonabelian gauge transformation is nonlinear:

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x)$$

$$A_\mu(x) = A^a_\mu(x) T^a$$

for an infinitesimal transformation:

$$
\begin{aligned}
U(x) &= I - ig\theta(x) + O(\theta^2) \\
&= I - ig\theta^a(x)T^a + O(\theta^2)
\end{aligned}
$$

we have:

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

$$A_\mu(x) \to A_\mu(x) + ig[A_\mu(x), \theta(x)] - \partial_\mu\theta(x)$$

or, in components:

$$A_\mu(x) = A_\mu^a(x)T^a$$

$$
\begin{aligned}
A_\mu^a(x) &\to A_\mu^a(x) - gf^{abc}A_\mu^b(x)\theta^c(x) - \partial_\mu\theta^a(x) \\
&= A_\mu^a(x) - [\delta^{ac}\partial_\mu + gf^{abc}A_\mu^b(x)]\theta^c(x) \\
&= A_\mu^a(x) - [\delta^{ac}\partial_\mu - igA_\mu^b(-if^{bac})]\theta^c(x) \\
&= A_\mu^a(x) - [\delta^{ac}\partial_\mu - igA_\mu^b(T_{\mathrm{A}}^b)^{ac}]\theta^c(x) \\
&= A_\mu^a(x) - D_\mu^{ac}\theta^c(x) \; ,
\end{aligned}
$$

↑ the covariant derivative in the adjoint representation
(instead of ∂_μ that we have for the U(1) transformation)

we have to remove the gauge redundancy in a different way!

Consider an ordinary integral of the form:

$$Z \propto \int dx\, dy\, e^{iS(x)}$$

the integral over y is redundant
we can simply drop it and define:

$$Z \equiv \int dx\, e^{iS(x)}$$

this is how we dealt with gauge redundancy in the abelian case

or we can get the same result by inserting a delta function:

$$Z = \int dx\, dy\, \delta(y)\, e^{iS(x)}$$

this is what we are going to do for the nonabelian case

the argument of the delta function can be shifted by an arbitrary function of x

$$Z = \int dx\, dy\, \delta(y - f(x))\, e^{iS(x)}$$

$$Z = \int dx\, dy\, \delta(y - f(x))\, e^{iS(x)}$$

if $y = f(x)$ is a unique solution of $G(x, y) = 0$ for fixed x, we can write:

$$\delta(G(x, y)) = \frac{\delta(y - f(x))}{|\partial G / \partial y|}$$

then we have:

$$Z = \int dx\, dy\, \frac{\partial G}{\partial y}\, \delta(G)\, e^{iS}$$

we dropped the abs. value

generalizing the result to an integral over n variables:

$$Z = \int d^n x\, d^n y\, \det\left(\frac{\partial G_i}{\partial y_j}\right) \prod_i \delta(G_i)\, e^{iS}$$

Now we translate this result to path integral over nonabelian gauge fields:

$$Z = \int d^n x \, d^n y \, \det\left(\frac{\partial G_i}{\partial y_j}\right) \prod_i \delta(G_i) \, e^{iS}$$

i index now represents x and a

x and y ⟶ $A^a_\mu(x)$

y ⟶ $\theta^a(x)$

G becomes the gauge fixing function:

for R_ξ gauge we use:

$$G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x)$$

fixed, arbitrarily chosen function of x

$$Z(J) \propto \int \mathcal{D}A \, \det\left(\frac{\delta G}{\delta \theta}\right) \prod_{x,a} \delta(G) \, e^{iS_{\rm YM}}$$

$$S_{\rm YM}(A,J) = \int d^4x \left[-\tfrac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + J^{a\mu} A^a_\mu\right]$$

$$Z(J) \propto \int \mathcal{D}A \det\left(\frac{\delta G}{\delta\theta}\right) \prod_{x,a} \delta(G)\, e^{iS_{\rm YM}}$$

let's evaluate the functional derivative:

$$G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x)$$
$$A^a_\mu(x) \to A^a_\mu(x) - D^{ac}_\mu \theta^c(x)$$

$$\longrightarrow \quad G^a(x) \to G^a(x) - \partial^\mu D^{ab}_\mu \theta^b(x)$$

and we find:

$$\frac{\delta\varphi_b(y)}{\delta\varphi_a(x)} = \delta_{ba}\delta^4(y-x)$$

$$\frac{\delta G^a(x)}{\delta\theta^b(y)} = -\partial^\mu D^{ab}_\mu \delta^4(x-y)$$

Recall, the functional determinant can be written as a path integral over complex Grassmann variables:

$$\int d^n\bar\psi\, d^n\psi\, \exp\left(-i\bar\psi_i M_{ij}\psi_j\right) \propto \det M$$

$$\det \frac{\delta G^a(x)}{\delta\theta^b(y)} \propto \int \mathcal{D}c\,\mathcal{D}\bar c\, e^{iS_{\rm gh}}$$

$$S_{\rm gh} = \int d^4x\, \mathcal{L}_{\rm gh}$$

where:

$$\mathcal{L}_{\rm gh} = \bar c^a \partial^\mu D^{ab}_\mu c^b$$

Faddeev-Popov ghosts

the ghost lagrangian can be further written as:

$$\begin{aligned}\mathcal{L}_{\text{gh}} &= \bar{c}^a \partial^\mu D_\mu^{ab} c^b \\ &= -\partial^\mu \bar{c}^a D_\mu^{ab} c^b \\ &= -\partial^\mu \bar{c}^a \partial_\mu c^a + ig \partial^\mu \bar{c}^a A_\mu^c (T_{\text{A}}^c)^{ab} c^b \\ &= \underline{-\partial^\mu \bar{c}^a \partial_\mu c^a + g f^{abc} A_\mu^c \partial^\mu \bar{c}^a c^b} \,.\end{aligned}$$

we drop the total divergence

$$D_\mu^{ac} \;=\; \delta^{ac}\partial_\mu - igA_\mu^b (T_{\text{A}}^b)^{ac}$$

Comments:

- ghost fields interact with the gauge field; however ghosts do not exist and we will see later (when we discuss the BRST symmetry) that the amplitude to produce them in any scattering process is zero. The only place they appear is in loops! Since they are Grassmann fields, a closed loop of ghost lines in a Feynman diagram comes with a minus sign!

- For abelian gauge theory $f^{abc} = 0$ and thus there is no interaction term for ghost fields; we can absorb its path integral into overall normalization.

At this point we have:

$$Z(J) \propto \int \mathcal{D}A \, \det\left(\frac{\delta G}{\delta\theta}\right) \prod_{x,a} \delta(G) \, e^{iS_{\rm YM}}$$

$$\det \frac{\delta G^a(x)}{\delta\theta^b(y)} \propto \int \mathcal{D}c \, \mathcal{D}\bar{c} \, e^{iS_{\rm gh}}$$

$$G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x)$$

fixed, arbitrarily chosen function of x

The path integral is independent of $\omega^a(x)$! Thus we can multiply it by arbitrary functional of ω and perform a path integral over ω ; the result changes only the overall normalization of $Z(J)$.

we can multiply $Z(J)$ by:

$$\exp\left[-\frac{i}{2\xi}\int d^4x \, \omega^a\omega^a\right]$$

integral over ω is trivial

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu$$

gauge fixing term

$$S_{\rm gf} = \int d^4x \, \mathcal{L}_{\rm gf}$$

our final result is:

$$Z(J) \propto \int \mathcal{D}A \, \mathcal{D}\bar{c} \, \mathcal{D}c \, \exp\Big(iS_{\rm YM} + iS_{\rm gh} + iS_{\rm gf}\Big)$$

next time we will derive Feynman rules from this action...

The Feynman rules for nonabelian gauge theory

based on S-72

The lagrangian for nonabelian gauge theory is:

$$\begin{aligned}\mathcal{L}_{\rm YM} &= -\tfrac{1}{4}F^{e\mu\nu}F^{e}_{\mu\nu} \\ &= -\tfrac{1}{4}(\partial^\mu A^{e\nu} - \partial^\nu A^{e\mu} + gf^{abe}A^{a\mu}A^{b\nu})(\partial_\mu A^e_\nu - \partial_\nu A^e_\mu + gf^{cde}A^c_\mu A^d_\nu) \\ &= -\tfrac{1}{2}\partial^\mu A^{e\nu}\partial_\mu A^e_\nu + \tfrac{1}{2}\partial^\mu A^{e\nu}\partial_\nu A^e_\mu \\ &\quad - gf^{abe}A^{a\mu}A^{b\nu}\partial_\mu A^e_\nu - \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu\ .\end{aligned}$$

the gauge fixing term for R_ξ gauge:

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1}\partial^\mu A^e_\mu \partial^\nu A^e_\nu$$

we can write the gauge fixed lagrangian in the form:

$$\begin{aligned}\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} &= \tfrac{1}{2}A^{e\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{e\nu} + \tfrac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu} \\ &\quad - gf^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu\end{aligned}$$

The gluon propagator in the R_ξ gauge:

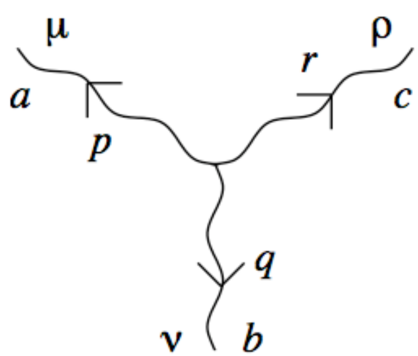
$$\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} = \underline{\tfrac{1}{2} A^{e\mu}(g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu) A^{e\nu} + \tfrac{1}{2}\xi^{-1} A^{e\mu}\partial_\mu\partial_\nu A^{e\nu}}$$
$$- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu - \tfrac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu$$

going to the momentum space and taking the inverse of the quadratic term

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi \frac{k_\mu k_\nu}{k^2}\right)$$

The three-gluon vertex:

$$\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} = \tfrac{1}{2}A^{e\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{e\nu} + \tfrac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu}$$
$$- g f^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu$$



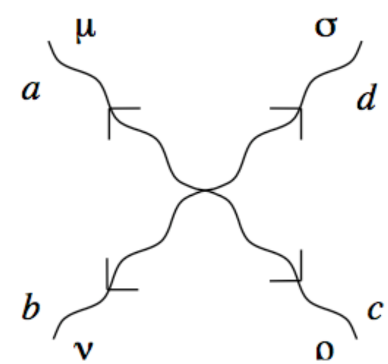
all photons outgoing

the derivative acting on an outgoing particle brings (-i momentum) of the particle

$$i\mathbf{V}^{abc}_{\mu\nu\rho}(p,q,r) = i(-gf^{abc})(-ir_\mu g_{\nu\rho})$$
$$+ [\,5 \text{ permutations of } (a,\mu,p), (b,\nu,q), (c,\rho,r)\,]$$
$$= gf^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}]\ .$$

The four-gluon vertex:

$$\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} = \tfrac{1}{2} A^{e\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{e\nu} + \tfrac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu}$$
$$- g f^{abc} A^{a\mu} A^{b\nu} \partial_\mu A^c_\nu \underline{- \tfrac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^c_\mu A^d_\nu}$$



$$i\mathbf{V}^{abcd}_{\mu\nu\rho\sigma} = -ig^2 f^{abe} f^{cde} g_{\mu\rho} g_{\nu\sigma}$$
$$+ [\, 5 \text{ permutations of } (b,\nu), (c,\rho), (d,\sigma)\,]$$
$$= -ig^2 [\, f^{abe} f^{cde} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$$
$$+ f^{ace} f^{dbe} (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma})$$
$$+ f^{ade} f^{bce} (g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\sigma\nu})\,] \,.$$

The ghost lagrangian:

we need ghosts for loop calculations

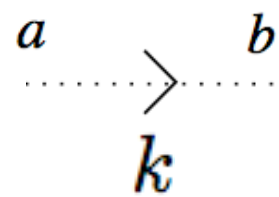
$$\mathcal{L}_{\rm gh} = -\partial^\mu \bar{c}^b D_\mu^{bc} c^c$$

$$= -\partial^\mu \bar{c}^c \partial_\mu c^c + ig\partial^\mu \bar{c}^b A_\mu^a (T_{\rm A}^a)^{bc} c^c$$

$$= \underline{-\partial^\mu \bar{c}^c \partial_\mu c^c} + g f^{abc} A_\mu^a \partial^\mu \bar{c}^b c^c \ .$$

massless complex scalar
they carry charge arrow
(and also a group index)

The ghost propagator:



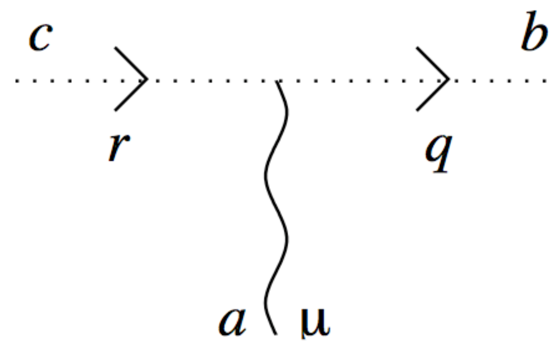
$$\tilde{\Delta}^{ab}(k^2) = \frac{\delta^{ab}}{k^2 - i\epsilon}$$

The ghost-ghost-gluon vertex:

$$\mathcal{L}_{\rm gh} = -\partial^\mu \bar{c}^c \partial_\mu c^c + \underline{g f^{abc} A_\mu^a \partial^\mu \bar{c}^b c^c}$$

the derivative acting on an outgoing particle brings (-i momentum) of the particle

ghosts are complex scalars so their propagator carry a charge arrow



$$i\mathbf{V}_\mu^{abc}(q,r) = i(g f^{abc})(-iq_\mu)$$

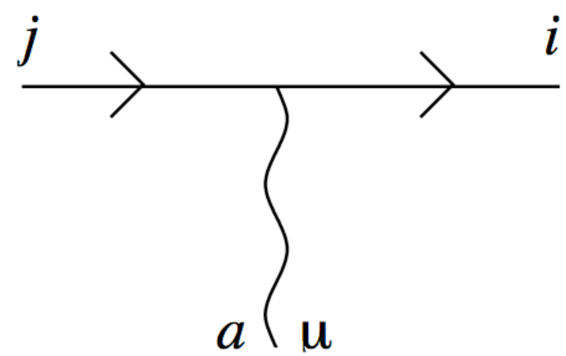
$$= g f^{abc} q_\mu \, .$$

Finally we can include quarks:

$$\mathcal{L}_{\mathrm{q}} = i\overline{\Psi}_i \not{D}_{ij}\Psi_j - m\overline{\Psi}_i\Psi_i$$
$$= i\overline{\Psi}_i\not{\partial}\Psi_i - m\overline{\Psi}_i\Psi_i + \underline{gA^a_\mu\overline{\Psi}_i\gamma^\mu T^a_{ij}\Psi_j}$$

propagator:

$$\tilde{S}_{ij}(p) = \frac{(-\not{p}+m)\delta_{ij}}{p^2+m^2-i\epsilon}$$



vertex:

$$i\mathbf{V}^{\mu a}_{ij} = ig\gamma^\mu T^a_{ij}$$

for fields in different representations we would have $(T^a_{\mathrm{R}})_{ij}$.