Beta functions in quantum electrodynamics

Let's calculate the beta function in QED:

\[ \mathcal{L}_0 = i \overline{\Psi} \partial \Psi - m \overline{\Psi} \Psi - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \]

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \]

\[ \mathcal{L}_1 = Z_1 e \overline{\Psi} A \Psi + \mathcal{L}_{ct} \]

\[ \mathcal{L}_{ct} = i(Z_2 - 1) \overline{\Psi} \partial \Psi - (Z_m - 1)m \overline{\Psi} \Psi - \frac{1}{4}(Z_3 - 1) F^{\mu \nu} F_{\mu \nu} \]

the dictionary:

\[ e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \bar{\mu}^{\varepsilon/2} e \]

\[ \alpha = \frac{e^2}{4\pi} \]

\[ \alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^{2} \bar{\mu}^{\varepsilon} \alpha \]

\[ Z_1 = 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \]

\[ Z_2 = 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \]

\[ Z_3 = 1 - \frac{2\alpha}{3\pi} \frac{1}{\varepsilon} + O(\alpha^2) \]

Note \( Z_1 = Z_2 \)!
following the usual procedure:

\[ \alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \tilde{\mu}^\varepsilon \alpha \]

\[ \ln \alpha_0 = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu} \]

\[ \ln \left( Z_3^{-1} Z_2^{-2} Z_1^2 \right) = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} \]

\[ Z_1 = 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \]

\[ Z_2 = 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \]

\[ Z_3 = 1 - \frac{2\alpha}{3\pi} \frac{1}{\varepsilon} + O(\alpha^2) \]

we find:

\[ E_1(\alpha) = \frac{2\alpha}{3\pi} + O(\alpha^2) \]

\[ \beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3) \]

\[ \beta(\alpha) = \alpha^2 E'_1(\alpha) \]

\[ \beta(\alpha) = \alpha^2 E'_1(\alpha) \]
or equivalently:

\[ \beta(e) = \frac{e^3}{12\pi^2} + O(e^5) \]

For a theory with \( N \) Dirac fields with charges \( Q_i e \):

\[
\begin{align*}
Z_1 &= 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \\
Z_2 &= 1 - \frac{\alpha}{2\pi} \frac{1}{\varepsilon} + O(\alpha^2) \\
Z_3 &= 1 - \frac{2\alpha}{3\pi} \frac{1}{\varepsilon} + O(\alpha^2)
\end{align*}
\]

we find:

\[
\frac{Z_{1i}}{Z_{2i}} = 1
\]

\[
\beta(e) = \frac{\sum_{i=1}^N Q_i^2}{12\pi^2} e^3 + O(e^5)
\]

\[
\alpha = \frac{e^2}{4\pi}
\]

\[
\dot{\alpha} = \frac{e\dot{e}}{2\pi}
\]
For completeness, let's calculate the beta functions in scalar ED:

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 ,
\]

\[
\mathcal{L}_0 = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} ,
\]

\[
\mathcal{L}_1 = i Z_1 e [\varphi^\dagger \partial^\mu \varphi - (\partial^\mu \varphi^\dagger) \varphi] A_\mu - Z_4 e^2 \varphi^\dagger \varphi A^\mu A_\mu
\]

\[
- \frac{1}{4} Z_\lambda \lambda (\varphi^\dagger \varphi)^2 + \mathcal{L}_{ct} ,
\]

\[
\mathcal{L}_{ct} = -(Z_2-1) \partial^\mu \varphi^\dagger \partial_\mu \varphi - (Z_{m1}) m^2 \varphi^\dagger \varphi - \frac{1}{4} (Z_3-1) F^{\mu\nu} F_{\mu\nu}
\]

the dictionary:

\[
e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \tilde{\mu} \varepsilon/2 e .
\]

\[
e_0^2 = Z_3^{-2} Z_2^{-1} Z_4 \tilde{\mu} \varepsilon e^2 .
\]

\[
\lambda_0 = Z_2^{-2} Z_\lambda \tilde{\mu} \varepsilon \lambda .
\]

\[
Z_1 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \ldots ,
\]

\[
Z_2 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \ldots ,
\]

\[
Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \ldots ,
\]

\[
Z_4 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \ldots ,
\]

\[
Z_\lambda = 1 + \left( \frac{3e^4}{2\pi^2 \lambda} + \frac{5\lambda}{16\pi^2} \right) \frac{1}{\varepsilon} + \ldots ,
\]

Note \( Z_1 = Z_2 = Z_4 \) !
following the usual procedure we find:

\[ \beta_e(e, \lambda) = \frac{e^3}{48\pi^2} + \ldots , \]

\[ \beta_\lambda(e, \lambda) = \frac{1}{16\pi^2} \left( 5\lambda^2 - 6\lambda e^2 + 24e^4 \right) + \ldots . \]

Generalizing to the case of arbitrary number of complex scalar and Dirac fields:

\[ \beta_e(e, \lambda) = \frac{1}{12\pi^2} \left( \sum_{\Psi} Q_{\Psi}^2 + \frac{1}{4} \sum_{\varphi} Q_{\varphi}^2 \right) e^3 + \ldots \]
Schwinger-Dyson equations

The path integral

\[ Z(J) = \int \mathcal{D}\varphi \, e^{i[S + \int d^4y \, J_a \varphi_a]} \]

doesn't change if we change variables \( \varphi_a(x) \to \varphi_a(x) + \delta \varphi_a(x) \)
assuming the measure \( \mathcal{D}\varphi \) is invariant under the change of variables
thus we have:

\[ 0 = \delta Z(J) \]

\[ = i \int \mathcal{D}\varphi \, e^{i[S + \int d^4y \, J_b \varphi_b]} \int d^4x \left( \frac{\delta S}{\delta \varphi_a(x)} + J_a(x) \right) \delta \varphi_a(x) \]

taking n functional derivatives with respect to \( J_{a_j}(x_j) \) and setting \( J = 0 \) we get:

\[ 0 = \int \mathcal{D}\varphi \, e^{iS} \int d^4x \left[ i \frac{\delta S}{\delta \varphi_a(x)} \varphi_{a_1}(x_1) \ldots \varphi_{a_n}(x_n) \right. \]

\[ + \sum_{j=1}^n \varphi_{a_1}(x_1) \ldots \delta_{a a_j} \delta^4(x - x_j) \ldots \varphi_{a_n}(x_n) \left. \right] \delta \varphi_a(x) \]
A comment on functional derivative:

\[
\frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} = \delta_{ba} \delta^4(y-x)
\]

\[
\delta \varphi_b(y) = \delta_{ba} \delta^4(y-x) \delta \varphi_a(x)
\]

Similarly:

\[
\frac{\delta F(\varphi_a(x))}{\delta \varphi_b(y)} = \frac{\partial F}{\partial \varphi_a(x)} \delta \varphi_a(x) = \frac{\partial F}{\partial \varphi_a(x)} \delta_{ba} \delta^4(y-x)
\]

\[
\delta F(\varphi_a(x)) = \sum_b \int dy^4 \frac{\partial F}{\partial \varphi_a(x)} \delta_{ba} \delta^4(y-x) \delta \varphi_b(y) = \frac{\partial F}{\partial \varphi_a(x)} \delta \varphi_a(x)
\]

or, for the action:

\[
\delta S = \int dx^4 \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]
since \( \delta \varphi_a(x) \) is arbitrary, we can drop it together with the integral over \( d^4x \).

Since the path integral computes vacuum expectation values of T-ordered products, we have:

\[
0 = i\langle 0| T \frac{\delta S}{\delta \varphi_a(x)} \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) |0\rangle \\
+ \sum_{j=1}^{n} \langle 0| T \varphi_{a_1}(x_1) \cdots \delta_{aa_j} \delta^4(x-x_j) \cdots \varphi_{a_n}(x_n) |0\rangle
\]

\textbf{Schwinger-Dyson equations}

For a free field theory of one scalar field we have:

\[
\frac{\delta S}{\delta \varphi(x)} = (\partial_x^2 - m^2) \varphi(x)
\]

SD eq. for \( n=1 \):

\[
(-\partial_x^2 + m^2) i\langle 0| T \varphi(x) \varphi(x_1) |0\rangle = \delta^4(x-x_1)
\]

\( \Delta(x-x_1) = i\langle 0| T \varphi(x) \varphi(x_1) |0\rangle \) is a Green's function for the Klein-Gordon operator as we already know
$0 = i\langle 0| T \frac{\delta S}{\delta \varphi_a(x)} \varphi_{a_1}(x_1) \ldots \varphi_{a_n}(x_n)|0\rangle$

$$+ \sum_{j=1}^{n} \langle 0| T \varphi_{a_1}(x_1) \ldots \delta_{a_j} \delta^4(x-x_j) \ldots \varphi_{a_n}(x_n)|0\rangle$$

in general Schwinger-Dyson equations imply

$$\langle 0| T \frac{\delta S}{\delta \varphi_a(x)} \varphi_{a_1}(x_1) \ldots \varphi_{a_n}(x_n)|0\rangle = 0 \quad \text{for} \quad x \neq x_1, \ldots, n$$

thus the classical equation of motion is satisfied by a quantum field inside a correlation function, as far as its spacetime argument differs from those of all other fields.

if this is not the case we get extra contact terms
Ward-Takahashi identity:

For a theory with a continuous symmetry we can consider transformations that result in \( \delta \mathcal{L} = 0 \):

\[
\partial_\mu j^\mu(x) = \delta \mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]

\[
0 = \int \mathcal{D}\varphi e^{iS} \int d^4x \left[ i \frac{\delta S}{\delta \varphi_a(x)} \varphi_a(x_1) \ldots \varphi_a(x_n) + \sum_{j=1}^n \varphi_a(x_1) \ldots \delta a_j \delta^4(x-x_j) \ldots \varphi_a(x_n) \right] \delta \varphi_a(x)
\]

summing over \(a\) and dropping the integral over \(d^4x\)

\[
0 = \partial_\mu \langle 0 | T j^\mu(x) \varphi_a(x_1) \ldots \varphi_a(x_n) | 0 \rangle + i \sum_{j=1}^n \langle 0 | T \varphi_a(x_1) \ldots \delta \varphi_a(x) \delta^4(x-x_j) \ldots \varphi_a(x_n) | 0 \rangle
\]

Ward-Takahashi identity

thus, conservation of the Noether current holds in the quantum theory, with the current inside a correlation function, up to contact terms (that depend on the infinitesimal transformation).
Ward identities in QED

When discussing QED we used the result that a scattering amplitude for a process that includes an external photon with momentum $k^\mu$ should satisfy (as a result of Ward identity):

$$k^\mu M_\mu = 0$$

we used it, for example, to obtain a simple formula for the photon polarization sum that we needed for calculations of cross sections:

$$\sum_{\lambda=\pm} \varepsilon_\lambda^\mu(k)\varepsilon_\lambda^{\rho\ast}(k) \rightarrow g^{\mu\rho}$$

now we are going to prove it!
To simplify the discussion let’s treat all particles in the LSZ formula as outgoing (incoming particles have $k_i^0 < 0$):

$$
\langle f|i \rangle = i \int d^4 x_1 e^{-i k_1 x_1} (-\partial_1^2 + m^2) \cdots \langle 0|T \varphi(x_1) \cdots |0\rangle
$$

it can be also written as:

$$
\langle f|i \rangle = \lim_{k_i^2 \rightarrow -m^2} (k_1^2 + m^2) \cdots \langle 0|T \tilde{\varphi}(k_1) \cdots |0\rangle
$$

$$
\tilde{\varphi}(k) = i \int d^4 x e^{-i k x} \varphi(x)
$$

we do not fix $k_i^2 = -m^2$

must include an overall energy-momentum delta function

$$
\langle 0|T \tilde{\varphi}(k_1) \cdots |0\rangle = (2\pi)^4 \delta^4(\sum_i k_i) \mathcal{F}(k_1^2, k_i \cdot k_j)
$$
\[ \langle f | i \rangle = \lim_{k_i^2 \to -m^2} (k_1^2 + m^2) \ldots \langle 0 | T \bar{\varphi}(k_1) \ldots | 0 \rangle \]

\[ \langle 0 | T \bar{\varphi}(k_1) \ldots | 0 \rangle = (2\pi)^4 \delta^4(\sum_i k_i) \mathcal{F}(k_i^2, k_i \cdot k_j) \]

\[ \langle f | i \rangle = i(2\pi)^4 \delta^4(\sum_i k_i) i\mathcal{T} \]

near \( k_i^2 = -m^2 \) we can write:

\[ \mathcal{F}(k_i^2, k_i \cdot k_j) = \frac{i\mathcal{T}}{(k_1^2 + m^2) \ldots (k_n^2 + m^2)} + \text{nonsingular} \]

\text{residue of the pole}

\text{multivariable pole}

\text{contributions that do not have this multivariable pole do not contribute to } i\mathcal{T}! \]

(for simplicity we work with scalar fields, but the same applies to fields of any spin)
near \(k_i^2 = -m^2\) we can write:

\[
F(k_i^2, k_i \cdot k_j) = \frac{i\mathcal{T}}{(k_1^2 + m^2) \cdots (k_n^2 + m^2)} + \text{nonsingular}
\]

this means that in Schwinger-Dyson equations:

\[
\langle 0|T \frac{\delta S}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n)|0\rangle
= i \sum_{j=1}^{n} \langle 0|T\phi_{a_1}(x_1) \cdots \delta_{a_{a_j}} \delta^4(x-x_j) \cdots \phi_{a_n}(x_n)|0\rangle
\]

in the momentum space a contact term is a function of \(k_1 + k_2\); it doesn’t have the right pole structure to contribute!

contact terms in a correlation function \(F\) do not contribute to the scattering amplitude!
Let's consider a scattering process in QED with an external photon: with momentum \( k^\mu \)

the LSZ formula:

\[
\langle f| i \rangle = i \varepsilon^\mu \int d^4x \, e^{-ikx} (-\partial^2) \ldots \langle 0| T A_\mu(x) \ldots |0 \rangle
\]

the classical equation of motion in the Lorentz gauge:

\[
-Z_3 \partial^2 A_\mu = \frac{\partial L}{\partial A_\mu}
\]

\( (g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) A_\nu + J^\mu = 0 \)

thus we find:

\[
\langle f| i \rangle = i Z_3^{-1} Z_1 \varepsilon^\mu \int d^4x \, e^{-ikx} \ldots \left[ \langle 0| T j_\mu(x) \ldots |0 \rangle + \text{contact terms} \right]
\]

contact terms cannot generate the proper singularities and thus these do not contribute to the scattering amplitude!
\[ \langle f | i \rangle = i Z_3^{-1} Z_1 \varepsilon^\mu \int d^4 x \, e^{-i k x} \ldots \left[ \langle 0 | T j_\mu(x) \ldots | 0 \rangle + \text{contact terms} \right] \]

Replace by \( k^\mu \)

\[ i k^\mu \rightarrow -\partial^\mu e^{-i k x} \]

Integrate by parts

\[ \partial^\mu \langle 0 | T j_\mu(x) \ldots | 0 \rangle \]

Now we use Ward-Takahashi identity:

\[ \partial^\mu \langle 0 | T j_\mu(x) \ldots | 0 \rangle = \text{contact terms} \]

Contact terms do not contribute to the scattering amplitude!

And thus we find:

\[ k^\mu M_\mu = 0 \]
Finally, we will derive another consequence of Ward identity that we used:

\[ Z_1 = Z_2 \]

consider the correlation function:

\[
C_{\alpha\beta}^{\mu}(k, p', p) \equiv i Z_1 \int d^4x \, d^4y \, d^4z \, e^{ikx - ip'y + ipz} \langle 0 | T j^{\mu}(x) \Psi_\alpha(y) \bar{\Psi}_\beta(z) | 0 \rangle
\]

\[ j^{\mu} = e \bar{\Psi} \gamma^{\mu} \Psi \]

in the momentum space:

\[
C_{\alpha\beta}^{\mu}(k, p', p) = (2\pi)^4 \delta^4(k + p - p') \left[ \frac{1}{i} \tilde{S}(p') i \tilde{V}^{\mu}(p', p) \frac{1}{i} \tilde{S}(p) \right]_{\alpha\beta}
\]

later we will use:

\[
k_{\mu} C_{\alpha\beta}^{\mu}(k, p', p) = -i (2\pi)^4 \delta^4(k + p - p') \left[ \tilde{S}(p') k_{\mu} \tilde{V}^{\mu}(p', p) \tilde{S}(p) \right]_{\alpha\beta}
\]
\[
C_{\alpha\beta}^\mu(k, p', p) \equiv i Z_1 \int d^4x \, d^4y \, d^4z \, e^{ikx - ip'y + ipz} \langle 0 | T j_\mu(x) \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle 
\]

multiply by \( k^\mu \)

integrate by parts

\[
k^\mu C_{\alpha\beta}^\mu(k, p', p) = -Z_1 \int d^4x \, d^4y \, d^4z \, e^{ikx - ip'y + ipz} \partial_\mu \langle 0 | T j_\mu(x) \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle 
\]

The Ward identity:

\[
-\partial_\mu \langle 0 | T J_\mu^\mu(x) \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n) | 0 \rangle = i \sum_{j=1}^n \langle 0 | T \phi_{a_1}(x_1) \ldots \delta \phi_{a_j}(x) \delta^4(x-x_j) \ldots \phi_{a_n}(x_n) | 0 \rangle 
\]

\[
\mathcal{L} = i \, Z_2 \overline{\Psi} \phi \Psi + \ldots 
J_\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a 
\]

\[
\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} = i Z_2 \overline{\Psi} \gamma^\mu 
\frac{\partial \mathcal{L}}{\partial (\partial_\mu \overline{\Psi})} = 0 
\delta \Psi(x) = -ie \Psi(x) 
\delta \overline{\Psi}(x) = +ie \overline{\Psi}(x) 
\]

becomes:

\[
J_\mu = Z_2 e \overline{\Psi} \gamma^\mu \Psi = Z_2 j_\mu^\mu 
\]

\[
-Z_2 \partial_\mu \langle 0 | T j_\mu^\mu(x) \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle = +e \delta^4(x-y) \langle 0 | T \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle 
\]

\[
-e \delta^4(x-z) \langle 0 | T \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle 
\]
\[
k_\mu C_{\alpha\beta}^{\mu}(k, p', p) = -Z \int d^4x \, d^4y \, d^4z \, e^{ikx-ip'y+ipz} \partial_\mu \langle 0 | Tj^\mu(x) \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle
\]
\[
-Z_2 \partial_\mu \langle 0 | Tj^\mu(x) \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle = +e \delta^4(x-y) \langle 0 | T\Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle
\]
\[
-e \delta^4(x-z) \langle 0 | T\Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle
\]
\[
\langle 0 | T\Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle = \frac{1}{i} \int \frac{d^4q}{(2\pi)^4} e^{iq(y-z)} \tilde{S}(q)_{\alpha\beta}
\]
\[
k_\mu C_{\alpha\beta}^{\mu}(k, p', p) = -iZ_2^{-1} Z_1 (2\pi)^4 \delta^4(k+p-p') \left[ e\tilde{S}(p) - e\tilde{S}(p') \right]_{\alpha\beta}
\]
before we found:
\[
k_\mu C_{\alpha\beta}^{\mu}(k, p', p) = -i(2\pi)^4 \delta^4(k+p-p') \left[ \tilde{S}(p')k_\mu V^\mu(p', p)\tilde{S}(p) \right]_{\alpha\beta}
\]
thus we get:
\[
(p'-p)_\mu \tilde{S}(p')V^\mu(p', p)\tilde{S}(p) = Z_2^{-1} Z_1 e \left[ \tilde{S}(p) - \tilde{S}(p') \right]
\]
or:
\[
(p'-p)_\mu V^\mu(p', p) = Z_2^{-1} Z_1 e \left[ \tilde{S}(p')^{-1} - \tilde{S}(p)^{-1} \right]
\]
In the OS scheme, near \((p' - p)^2 = 0, \ p^2 = p'^2 = -m^2\), we have:

\[
V^\mu(p', p) = e\gamma^\mu
\]

\[
\tilde{S}(p)^{-1} = p' + m
\]

and so we get \(Z_1 = Z_2\)!

The Ward identity means that the kinetic term \(iZ_2 \bar{\Psi} D\Psi\) and the interaction term \(Z_1 e \bar{\Psi} A \Psi\) are renormalized in the same way and so they can be combined into covariant derivative term \(iZ_2 \bar{\Psi} D\Psi\) (which we could have guessed from gauge invariance)!

\[
D^\mu = \partial^\mu - ieA^\mu
\]
Formal development of fermionic PI

We want to derive the fermionic path integral formula (that we previously postulated by analogy with the path integral for a scalar field):

$$Z_0(\bar{\eta}, \eta) = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp \left[ i \int d^4x \bar{\Psi}(i\partial - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta \right]$$

$$= \exp \left[ i \int d^4x \, d^4y \, \bar{\eta}(x)S(x - y)\eta(y) \right],$$

Feynman propagator
inverse of the Dirac wave operator

$$(-i\partial_x + m)S(x - y) = \delta^4(x - y)$$
Let's define a set of anticommuting numbers or Grassmann variables:

\[ \{ \psi_i, \psi_j \} = 0 \quad i = 1, \ldots, n \]

for \( n = 1 \) we have just one number \( \psi \) with \( \psi^2 = 0 \).

We define a function of \( \psi \) by a Taylor expansion:

\[ f(\psi) = a + \psi b \]

the order is important!

if \( f \) itself is commuting then \( b \) has to be an anticommuting number:

\[ \{ b, b \} = \{ b, \psi \} = 0 \]

and we have:

\[ f(\psi) = a + \psi b = a - b\psi \]
Let’s define the left derivative of $f(\psi)$ with respect to $\psi$ as:

$$\partial_\psi f(\psi) = +b$$

Similarly, let’s define the right derivative of $f(\psi)$ with respect to $\psi$ as:

$$f(\psi) \tilde{\partial}_\psi = -b$$

We define the definite integral with the same properties as those of an integral over a real variable; namely linearity and invariance under shifts:

$$\int_{-\infty}^{+\infty} dx \ c f(x) = c \int_{-\infty}^{+\infty} dx \ f(x) \quad \int_{-\infty}^{+\infty} dx \ f(x + a) = \int_{-\infty}^{+\infty} dx \ f(x)$$

The only possible nontrivial definition (up to an overall numerical factor) is:

$$\int d\psi \ f(\psi) = b$$
Let’s generalize this to $n > 1$, we have:

$$f(\psi) = a + \psi_i b_i + \frac{1}{2} \psi_{i_1} \psi_{i_2} c_{i_1 i_2} + \ldots + \frac{1}{n!} \psi_{i_1} \ldots \psi_{i_n} d_{i_1 \ldots i_n}$$

all indices summed over completely antisymmetric on exchange of any two indices

Let’s define the left derivative of $f(\psi)$ with respect to $\psi_j$ as:

$$\frac{\partial}{\partial \psi_j} f(\psi) = b_j + \psi_i c_{ji} + \ldots + \frac{1}{(n-1)!} \psi_{i_2} \ldots \psi_{i_n} d_{ji_2 \ldots i_n}$$

and similarly for the right derivative...
To define (linear and shift invariant) integral note that:

\[ f(\psi) = a + \psi_i b_i + \frac{1}{2} \psi_i \psi_j c_{i j} + \ldots + \frac{1}{n!} \psi_1 \ldots \psi_n d_{i_1 \ldots i_n} \]

the only consistent definition of the integral is:

\[ \int d^n \psi \; f(\psi) = d \]

alternatively we could write the differential in terms of individual differentials:

\[ d^n \psi = d \psi_n \ldots d \psi_1 \]

\[ \{d \psi_i, d \psi_j\} = 0 \]

\[ \{d \psi_i, \psi_j\} = 0 \]

\[ \int d \psi_i = 0 \]

\[ \int d \psi_i \psi_j = \delta_{ij} \]

to derive the result above.
Consider a linear change of variable: 

$$\psi_i = J_{ij} \psi'_j$$

then we have:

$$f(\psi) = a + \ldots + \frac{1}{n!} (J_{i_1j_1} \psi'_{j_1}) \ldots (J_{i_nj_n} \psi'_{j_n}) \varepsilon_{i_1 \ldots i_n} d$$

integrating over $d^n \psi'$ we get $ (\det J) d$ and thus:

$$\int d^n \psi f(\psi) = (\det J)^{-1} \int d^n \psi' f(\psi)$$

Recall, for integrals over real numbers with $x_i = J_{ij} x'_j$ we have:

$$\int d^n x f(x) = (\det J)^{+1} \int d^n x' f(x)$$
We are interested in gaussian integrals of the form:

$$\int d^n\psi \, \exp\left(\frac{1}{2}\psi^T M \psi\right)$$

for $n = 2$ we have:

$$M = \begin{pmatrix} 0 & +m \\ -m & 0 \end{pmatrix}$$

expanding the exponential:

$$\exp\left(\frac{1}{2}\psi^T M \psi\right) = 1 + m\psi_1\psi_2$$

we find:

$$\int d^2\psi \, \exp\left(\frac{1}{2}\psi^T M \psi\right) = m$$

antisymmetric matrix of (complex) commuting numbers

$$\psi^T M \psi = \psi_i M_{ij} \psi_j$$

$$\psi^T M \psi = 2m\psi_1\psi_2$$
For larger (even) \( n \) we can bring a complex antisymmetric matrix to a block-diagonal form:

\[
U^T M U = \begin{pmatrix}
0 & +m_1 \\
-m_1 & 0 & \\
& & \ddots
\end{pmatrix}
\]

we will later need:

\[(\det U)^2 (\det M) = \prod_{I=1}^{n/2} m_I^2\]

taking:

\[
\psi_i = U_{ij} \psi_j'
\]

we have:

\[
\psi_i = J_{ij} \psi_j'
\]

\[
\int d^n \psi \, f(\psi) = (\det J)^{-1} \int d^n \psi' \, f(\psi)
\]

\[
\int d^n \psi \, \exp\left(\frac{1}{2} \psi^T M \psi\right) = (\det U)^{-1} \prod_{I=1}^{n/2} \int d^2 \psi_I \, \exp\left(\frac{1}{2} \psi_I^T M_I \psi_I\right)
\]

we drop primes

represents 2x2 blocks
\[
\int d^n\psi \exp(\frac{1}{2} \psi^T M \psi) = (\det U)^{-1} \prod_{I=1}^{n/2} \int d^2\psi_I \exp(\frac{1}{2} \psi^T M_I \psi)
\]

using the result for \( n = 2 \) we get:

\[
\int d^n\psi \exp(\frac{1}{2} \psi^T M \psi) = (\det U)^{-1} \prod_{I=1}^{n/2} m_I
\]

we finally get:

\[
\int d^n\psi \exp(\frac{1}{2} \psi^T M \psi) = (\det M)^{1/2}
\]

Recall, for integrals over real numbers we have:

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.
\]

\[
\int d^n x \exp(-\frac{1}{2} x^T M x) = (2\pi)^{n/2} (\det M)^{-1/2}
\]

a complex symmetric matrix
Let’s define complex Grassmann variables:

\[ \chi \equiv \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \]

\[ \bar{\chi} \equiv \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \]

we can invert this to get:

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix}
\]

\[ \psi_i = J_{ij} \psi'_j \]

thus we have:

\[ d^2\psi = d\psi_2 d\psi_1 = (i)^{-1} d\chi d\bar{\chi} \]

also since \( \psi_1 \psi_2 = -i\bar{\chi}\chi \) we have:

\[ \int d\chi \, d\bar{\chi} \, \bar{\chi}\chi = (-i)(-i)^{-1} \int d\psi_2 d\psi_1 \, \psi_1 \psi_2 = 1 \]
A function can be again defined by a Taylor expansion:

\[ f(\chi, \bar{\chi}) = a + \chi b + \bar{\chi} c + \bar{\chi} \chi d \]

the integral is:

\[ \int d\chi d\bar{\chi} \bar{\chi} \chi = (-i)(-i)^{-1} \int d\psi_2 d\psi_1 \psi_1 \psi_2 = 1 \]

\[ \int d\chi d\bar{\chi} f(\chi, \bar{\chi}) = d \]

in particular:

\[ \int d\chi d\bar{\chi} \exp(m\bar{\chi} \chi) = m \]
Let’s consider \( n \) complex Grassmann variables and their conjugates:

Define:

\[
d^n\chi \ d^n\bar{\chi} \equiv d\chi_1 d\bar{\chi}_1 \cdots d\chi_n d\bar{\chi}_n
\]

Under a change of variable:

\[
\chi_i = J_{ij} \chi_j \quad \bar{\chi}_i = K_{ij} \bar{\chi}_j
\]

\[
K_{ij} = J_{ij}^*
\]

(not important)

(The integral doesn’t care whether \( \bar{\chi}_i \) is the complex conjugate of \( \chi_i \))

We have

\[
d^n\chi \ d^n\bar{\chi} = (\text{det } J)^{-1} (\text{det } K)^{-1} \ d^n\chi' \ d^n\bar{\chi}'
\]

We want to evaluate:

\[
\int d^n\chi \ d^n\bar{\chi} \ \exp(\chi^\dagger M \chi)
\]

A general complex matrix can be brought to a diagonal form with all entries positive by a bi-unitary transformation \( V M U \)

\[
\chi = U \chi'
\]

\[
\chi^\dagger = \chi'^\dagger V
\]
under such a change of variable we get:

\[
\int d^n\chi\,d^n\bar{\chi}\,\exp(\chi^\dagger M\chi) = (\det U)^{-1}(\det V)^{-1} \prod_{i=1}^{n} \int d\chi_i d\bar{\chi}_i\,\exp(m_i\bar{\chi}_i\chi_i)
\]

\[
= (\det U)^{-1}(\det V)^{-1} \prod_{i=1}^{n} m_i
\]

\[
\int d\chi\,d\bar{\chi}\,\exp(m\bar{\chi}\chi) = m = \det M.
\]

Analogous integral for commuting complex variable

\[
\int d^n z\,d^n\bar{z}\,\exp(-z^\dagger M z) = (2\pi)^n(\det M)^{-1}
\]

\[
z_i = (x_i + i y_i)/\sqrt{2}
\]

\[
\bar{z} = (x_i - i y_i)/\sqrt{2}
\]

\[
d^n z\,d^n\bar{z} = d^n x\,d^n y
\]
\[
\int d^n\chi \, d^n\bar{\chi} \, \exp(\chi^\dagger M \chi) = \det M
\]

using shift invariance of integrals:
\[
\chi^\dagger \rightarrow \chi^\dagger - \eta^\dagger M^{-1} \quad \chi \rightarrow \chi - M^{-1}\eta
\]

we get:
\[
\int d^n\chi \, d^n\bar{\chi} \, \exp(\chi^\dagger M \chi + \eta^\dagger \chi + \chi^\dagger \eta) = (\det M) \exp(-\eta^\dagger M^{-1}\eta)
\]

generalization for continuous spacetime argument and spin index \( \Psi_\alpha(x) \)

the determinant does not depend on fields or sources and can be absorbed into the overall normalization of the path integral

\[
Z_0(\bar{\eta}, \eta) = \int \mathcal{D}\Psi \, \mathcal{D}\bar{\Psi} \exp\left[ i \int d^4x \, \bar{\Psi}(i\partial - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta \right]
\]
\[
= \exp\left[ i \int d^4x \, d^4y \, \bar{\eta}(x)S(x - y)\eta(y) \right],
\]
\[
(-i\partial_x + m)S(x - y) = \delta^4(x - y)
\]