Maxwell's equations

based on S-54

Our next task is to find a quantum field theory description of spin-1 particles, e.g. photons.

Classical electrodynamics is governed by Maxwell's equations:

electric field

charge density

$$\nabla\cdot\mathbf{E} = \rho$$

$$\nabla\times\mathbf{B} - \dot{\mathbf{E}} = \mathbf{J}$$

current density

$$\nabla\times\mathbf{E} + \dot{\mathbf{B}} = 0$$

$$\nabla\cdot\mathbf{B} = 0$$

magnetic field

can be solved by writing fields in terms of a scalar potential and a vector potential

$$\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}\,,$$

$$\mathbf{B} = \nabla\times\mathbf{A}\,.$$

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$$\mathbf{B} = \nabla \times \mathbf{A} \,.$$

The potentials uniquely determine the fields, but the fields do not uniquely determine the potentials, e.g.

arbitrary function of spacetime

$$\varphi' = \varphi + \dot{\Gamma} \,,$$

$$\mathbf{A}' = \mathbf{A} - \nabla\Gamma \,,$$

result in the same electric and magnetic fields.

gauge transformation
(a change of potentials that does not change the fields)

More elegant relativistic notation:

four-vector potential, or gauge field

$$A^\mu \equiv (\varphi, \mathbf{A})$$

the field strength

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F^{\mu\nu} = -F^{\nu\mu}$$

in components:

$$F^{0i} = E^i \,,$$

$$F^{ij} = \varepsilon^{ijk} B_k \,.$$

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$$F^{ij} = \varepsilon^{ijk} B_k \ .$$

$$\nabla \cdot \mathbf{E} = \rho$$

$$\nabla \times \mathbf{B} - \dot{\mathbf{E}} = \mathbf{J}$$

$$\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

The first two Maxwell's equations can be written as:

$$\partial_\nu F^{\mu\nu} = J^\mu$$

charge-current density 4-vector

$$J^\mu \equiv (\rho, \mathbf{J})$$

taking the four-divergence:

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu J^\mu$$

$$F^{\mu\nu} = -F^{\nu\mu}$$

we find that the electromagnetic current is conserved:

$$\partial_\mu J^\mu = 0$$

$$\dot{\rho} + \nabla \cdot \mathbf{J} = 0$$

The last two Maxwell's equations can be written as:

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\rho F^{\mu\nu} = 0$$

automatically satisfied!

The gauge transformation in four-vector notation:

$\varphi' = \varphi + \dot{\Gamma}\,,$

$A^\mu \equiv (\varphi, \mathbf{A})$

$\mathbf{A}' = \mathbf{A} - \nabla\Gamma\,,$

$$A'^\mu = A^\mu - \partial^\mu \Gamma$$

The field strength transforms as:

$F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$

$$F'^{\mu\nu} = F^{\mu\nu} - (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\Gamma$$

= 0 (derivatives commute)

$$\boxed{F'^{\mu\nu} = F^{\mu\nu}}$$

the field strength is gauge invariant!

Next we want to find an action that results in Maxwell's equations as the equations of motion; it should be Lorentz invariant, gauge invariant, parity and time-reversal invariant and no more than second order in derivatives; the only candidate is:

$S = \int d^4x\, \mathcal{L}$

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu$$

we will treat the current as an external source

$$S = \int d^4x\, \mathcal{L}$$

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu$$

obviously gauge invariant

$$J^\mu(A'_\mu - A_\mu) = -J^\mu \partial_\mu \Gamma$$
$$= -(\partial_\mu J^\mu)\Gamma - \partial_\mu(J^\mu \Gamma)$$

$$\partial_\mu J^\mu = 0$$

total divergence

In terms of the gauge field:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu + J^\mu A_\mu$$
$$= +\tfrac{1}{2}A_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + J^\mu A_\mu - \partial^\mu K_\mu$$

total divergence

$$K_\mu = \tfrac{1}{2}A^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

equations of motion:

$$(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + J^\mu = 0$$

equivalent to the first two Maxwell's equations!

$$\partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)A_\nu$$

Electrodynamics in Coulomb gauge

based on S-55

Next step is to construct the hamiltonian and quantize the electromagnetic field ...

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu$$
$$= -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu + J^\mu A_\mu$$

Which A_μ should we quantize?

too much freedom due to gauge invariance

There is no time derivative of A^0 and so this field has no conjugate momentum (and no dynamics).

To eliminate the gauge freedom we choose a gauge, e.g.

$$\nabla \cdot \mathbf{A}(x) = 0$$

Coulomb gauge

an example of a manifestly relativistic gauge is Lorentz gauge:

$$\partial^\mu A_\mu = 0$$

$$\nabla \cdot \mathbf{A}(x) = 0$$

We can impose the Coulomb gauge by acting with a projection operator:

$$A_i(x) \to \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2}\right) A_j(x)$$

in the momentum space it corresponds to multiplying $\widetilde{A}_i(k)$ by the matrix $\delta_{ij} - k_i k_j / \mathbf{k}^2$, that projects out the longitudinal component.

(also known as transverse gauge)

the lagrangian in terms of scalar and vector potentials:

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \\ &= -\tfrac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2} \partial^\mu A^\nu \partial_\nu A_\mu + J^\mu A_\mu \end{aligned}$$

$A^\mu \equiv (\varphi, \mathbf{A})$

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2} \dot{A}_i \dot{A}_i - \tfrac{1}{2} \nabla_j A_i \nabla_j A_i + J_i A_i \\ & + \tfrac{1}{2} \nabla_i A_j \nabla_j A_i + \dot{A}_i \nabla_i \varphi \\ & + \tfrac{1}{2} \nabla_i \varphi \nabla_i \varphi - \rho \varphi \, . \end{aligned}$$

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i$$
$$+\, \tfrac{1}{2}\nabla_i A_j \nabla_j A_i + \dot{A}_i \nabla_i \varphi$$
$$+\, \tfrac{1}{2}\nabla_i\varphi\nabla_i\varphi - \rho\varphi\,.$$

integration by parts

$$0 \leftarrow \nabla_j(\nabla_i A_i) \qquad \nabla_i A_i = 0$$

integration by parts

$$\nabla_i \dot{A}_i \rightarrow 0 \qquad \nabla_i A_i = 0$$

equation of motion

integration by parts

$$-\nabla^2\varphi = \rho$$

$$-\varphi\nabla^2\varphi = \varphi\rho$$

Poisson's equation

unique solution:

$$\varphi(\mathbf{x},t) = \int d^3y\, \frac{\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

we get the lagrangian:

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i + \mathcal{L}_{\rm coul}$$

$$\mathcal{L}_{\rm coul} = -\frac{1}{2}\int d^3y\, \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i + \mathcal{L}_{\text{coul}}$$

the equation of motion for a free field ($J_i = 0$):

$$-\partial^2 A_i(x) = 0$$

massless Klein-Gordon equation

the general solution:

$$\mathbf{A}(x) = \sum_{\lambda=\pm} \int \widetilde{dk} \left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-ikx} \right]$$

$$k^0 = \omega = |\mathbf{k}|$$

$$\widetilde{dk} = d^3k/(2\pi)^3 2\omega$$

polarization vectors (orthogonal to k)

we can choose the polarization vectors to correspond to right- and left-handed circular polarizations:

$$\boldsymbol{\varepsilon}_+(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1, -i, 0)$$

$$\boldsymbol{\varepsilon}_-(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1, +i, 0)$$

$$\mathbf{k} = (0, 0, k)$$

in general:

$$\mathbf{k}\cdot\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0 \,,$$

$$\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k})\cdot\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) = \delta_{\lambda'\lambda} \,,$$

$$\sum_{\lambda=\pm} \varepsilon^*_{i\lambda}(\mathbf{k})\varepsilon_{j\lambda}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \,.$$

$$\mathbf{A}(x) = \sum_{\lambda=\pm} \int \widetilde{dk} \left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-ikx} \right]$$

following the procedure used for a scalar field we can express the operators in terms of fields:

$$a_\lambda(\mathbf{k}) = +i\, \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot \int d^3x\; e^{-ikx}\, \overleftrightarrow{\partial_0}\, \mathbf{A}(x)$$

$$a^\dagger_\lambda(\mathbf{k}) = -i\, \boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) \cdot \int d^3x\; e^{+ikx}\, \overleftrightarrow{\partial_0}\, \mathbf{A}(x)$$

$$f \overleftrightarrow{\partial_\mu} g = f(\partial_\mu g) - (\partial_\mu f) g$$

to find the hamiltonian we start with the conjugate momenta:

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i \dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i + \mathcal{L}_{\rm coul}$$

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i$$

$$\nabla_i A_i = 0 \longrightarrow \nabla_i \Pi_i = 0$$

the hamiltonian density is then

$$\begin{aligned} \mathcal{H} &= \Pi_i \dot{A}_i - \mathcal{L} \\ &= \tfrac{1}{2}\Pi_i \Pi_i + \tfrac{1}{2}\nabla_j A_i \nabla_j A_i - J_i A_i + \mathcal{H}_{\rm coul} \end{aligned}$$

$$\mathcal{H}_{\rm coul} = -\mathcal{L}_{\rm coul}$$

we impose the canonical commutation relations:

$$A_i(x) \to \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2}\right) A_j(x)$$

with the projection operator

$$[A_i(\mathbf{x},t), \Pi_j(\mathbf{y},t)] = i\left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{y})$$

$$= i \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)$$

$$[A_i, A_j] = [\Pi_i, \Pi_j] = 0$$

these correspond to the canonical commutation relations for creation and annihilation operators:

(the same procedure as for the scalar field)

$$[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')] = 0\,,$$

$$[a^\dagger_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = 0\,,$$

$$[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = (2\pi)^3 2\omega\, \delta^3(\mathbf{k}'-\mathbf{k})\delta_{\lambda\lambda'}\,.$$

creation and annihilation operators for photons with helicity +1 (right-circular polarization) and -1 (left-circular polarization) now we can write the hamiltonian in terms of creation and annihilation operators:

$$\mathcal{H} = \Pi_i \dot{A}_i - \mathcal{L}$$
$$= \tfrac{1}{2}\Pi_i\Pi_i + \tfrac{1}{2}\nabla_j A_i \nabla_j A_i - J_i A_i + \mathcal{H}_{\text{coul}}$$

(the same procedure as for the scalar field)

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\ \omega\ a^\dagger_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) + 2\mathcal{E}_0 V - \int d^3x\ \mathbf{J}(x)\cdot\mathbf{A}(x) + H_{\text{coul}}$$

$$H_{\text{coul}} = \frac{1}{2}\int d^3x\, d^3y\ \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

2-times the zero-point energy of a scalar field

$$\mathcal{E}_0 = \tfrac{1}{2}(2\pi)^{-3}\int d^3k\ \omega$$

this form of the hamiltonian of electrodynamics is used in calculations of atomic transition rates, ... in particle physics the hamiltonian doesn't play a special role; we start with the lagrangian with specific interactions, calculate correlation functions, plug them into LSZ to get transition amplitudes ...

LSZ reduction for photons

based on S-56

Next step is to get the LSZ formula for the photon. The derivation closely follows the scalar field case; the only difference is due to the presence of polarization vectors:

For a scalar field we found that in order to obtain a transition amplitude we simply replace the creation and annihilation operators in the transition amplitude by:

$$a^\dagger(\mathbf{k})_{\text{in}} \to i\int d^4z_1\, e^{+ikz_1}(-\partial^2+m^2)\varphi(z_1)$$

$$a(\mathbf{k}')_{\text{out}} \to i\int d^4z_2\, e^{-ik'z_2}(-\partial^2+m^2)\varphi(z_2)$$

similarly, for an incoming and outgoing photon we simply replace:

$$\mathbf{A}(x) = \sum_{\lambda=\pm}\int \widetilde{dk}\left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})a_\lambda(\mathbf{k})e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k})a^\dagger_\lambda(\mathbf{k})e^{-ikx}\right]$$

$$a^\dagger_\lambda(\mathbf{k}) = -i\,\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})\cdot\int d^3x\, e^{+ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)$$

$$a_\lambda(\mathbf{k}) = +i\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\int d^3x\, e^{-ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)$$

$$\varepsilon^0_\lambda(\mathbf{k}) \equiv 0$$

$$a^\dagger_\lambda(\mathbf{k})_{\text{in}} \to i\,\varepsilon^{\mu*}_\lambda(\mathbf{k})\int d^4x\, e^{+ikx}(-\partial^2)A_\mu(x)$$

$$a_\lambda(\mathbf{k})_{\text{out}} \to i\,\varepsilon^{\mu}_\lambda(\mathbf{k})\int d^4x\, e^{-ikx}(-\partial^2)A_\mu(x)$$

$$\mathbf{A}(x) = \sum_{\lambda=\pm} \int \widetilde{dk} \left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-ikx} \right]$$

the LSZ formula is then valid if the field is normalized according to the free field formulae:

$$\langle 0|A^i(x)|0\rangle = 0 \, ,$$

$$\langle k, \lambda|A^i(x)|0\rangle = \varepsilon^i_\lambda(\mathbf{k})\bar{e}^{ikx}$$

where a single photon state is normalized according to:

$$\langle k', \lambda'|k, \lambda\rangle = (2\pi)^3 2\omega \, \delta^3(\mathbf{k}' - \mathbf{k})\delta_{\lambda\lambda'}$$

and the renormalization of fields results in the Z-factors in the lagrangian:

$$\mathcal{L} = -\tfrac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + Z_1 J^\mu A_\mu$$

we will discuss this next semester...

Now we want to calculate correlation functions (the derivation again closely follows the scalar field case).

$$\mathbf{A}(x) = \sum_{\lambda=\pm} \int \widetilde{dk} \left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-ikx} \right]$$

the propagator for a free field theory:

$$\langle 0|\mathrm{T} A^i(x) A^j(y)|0\rangle = \tfrac{1}{i}\Delta^{ij}(x-y)$$

$$\Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - i\epsilon} \textstyle\sum_{\lambda=\pm} \varepsilon^{i*}_\lambda(\mathbf{k}) \varepsilon^j_\lambda(\mathbf{k})$$

correlation functions of more fields given in terms of propagators...

Next we want to calculate the path integral for the free EM field:

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}A\; e^{i\int d^4x\,[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu]}$$

we will treat the current as an external source

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}A\; e^{i\int d^4x\,[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+J^\mu A_\mu]}$$

In the Coulomb gauge we integrate over those field configurations that satisfy $\nabla\cdot\mathbf{A}(x)=0$; in addition the zero's component is not dynamical we can replace it by the solution of the equation of motion

$$\mathcal{L}_{\rm coul} = -\frac{1}{2}\int d^3y\,\frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

$$S_{\rm coul} = -\frac{1}{2}\int d^4x\, d^4y\,\delta(x^0-y^0)\,\frac{J^0(x)J^0(y)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

and for the rest of the path integral we will guess the result based on the result we got for a scalar field:

$$Z_0(J) = \exp\left[iS_{\rm coul} + \frac{i}{2}\int d^4x\, d^4y\, J_i(x)\Delta^{ij}(x-y)J_j(y)\right]$$

propagator

$$Z_0(J) = \exp\left[iS_{\rm coul} + \frac{i}{2}\int d^4x\, d^4y\, J_i(x)\Delta^{ij}(x-y)J_j(y)\right]$$

$$S_{\rm coul} = -\frac{1}{2}\int d^4x\, d^4y\, \delta(x^0-y^0)\, \frac{J^0(x)J^0(y)}{4\pi|\mathbf{x}-\mathbf{y}|} \qquad \Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{ik(x-y)}}{k^2-i\epsilon} \sum_{\lambda=\pm} \varepsilon^{i*}_\lambda(\mathbf{k})\varepsilon^j_\lambda(\mathbf{k})$$

we can make it look better:

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right]$$

where

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \tilde{\Delta}^{\mu\nu}(k)$$

$$\tilde{\Delta}^{\mu\nu}(k) \equiv -\frac{1}{\mathbf{k}^2}\delta^{\mu 0}\delta^{\nu 0} + \frac{1}{k^2-i\epsilon}\sum_{\lambda=\pm}\varepsilon^{\mu*}_\lambda(\mathbf{k})\varepsilon^\nu_\lambda(\mathbf{k})$$

$$\varepsilon^0_\lambda(\mathbf{k}) \equiv 0$$

and the Coulomb term is reproduced thanks to:

$$\int_{-\infty}^{+\infty}\frac{dk^0}{2\pi}\, e^{-ik^0(x^0-y^0)} = \delta(x^0-y^0)$$

$$\int \frac{d^3k}{(2\pi)^3}\, \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$$

$$\tilde{\Delta}^{\mu\nu}(k) \equiv -\frac{1}{\mathbf{k}^2}\delta^{\mu 0}\delta^{\nu 0} + \frac{1}{k^2 - i\epsilon}\sum_{\lambda=\pm}\varepsilon_\lambda^{\mu *}(\mathbf{k})\varepsilon_\lambda^{\nu}(\mathbf{k})$$

We can simplify the propagator further...

Let's define:

$$\hat{t}^\mu = (1, \mathbf{0})$$

and $\hat{z}^\mu$ as a unit vector in the $\mathbf{k}$ direction:

$$\hat{t}\cdot k = -k^0$$

$$(0, \mathbf{k}) = k^\mu + (\hat{t}\cdot k)\hat{t}^\mu$$

$$\hat{z}^\mu = \frac{k^\mu + (\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}}$$

$$\hat{t}^2 = -1$$

$$\mathbf{k}^2 = k^2 + (\hat{t}\cdot k)^2$$

now we can replace:

$$\sum_{\lambda=\pm}\varepsilon_\lambda^{i*}(\mathbf{k})\varepsilon_\lambda^{j}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \longrightarrow \sum_{\lambda=\pm}\varepsilon_\lambda^{\mu *}(\mathbf{k})\varepsilon_\lambda^{\nu}(\mathbf{k}) = g^{\mu\nu} + \hat{t}^\mu\hat{t}^\nu - \hat{z}^\mu\hat{z}^\nu$$

and thus we get:

$$\tilde{\Delta}^{\mu\nu}(k) = -\frac{\hat{t}^\mu\hat{t}^\nu}{k^2 + (\hat{t}\cdot k)^2} + \frac{g^{\mu\nu} + \hat{t}^\mu\hat{t}^\nu - \hat{z}^\mu\hat{z}^\nu}{k^2 - i\epsilon}$$

$$\tilde{\Delta}^{\mu\nu}(k) = -\frac{\hat{t}^\mu \hat{t}^\nu}{k^2 + (\hat{t}\cdot k)^2} + \frac{g^{\mu\nu} + \hat{t}^\mu \hat{t}^\nu - \hat{z}^\mu \hat{z}^\nu}{k^2 - i\epsilon}$$

$$\hat{t}^\mu = (1, \mathbf{0})$$

$$\hat{z}^\mu = \frac{k^\mu + (\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}}$$

this looks better but we can simplify the propagator further...

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right]$$

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \tilde{\Delta}^{\mu\nu}(k)$$

the momentum can be replaced by the derivative with respect to x^μ acting on the exponential, and then integrate by parts to obtain $\partial^\mu J_\mu(x)$ which vanishes.

$$\hat{z}^\mu \to \frac{(\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}}$$

and we get:

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left[g^{\mu\nu} + \left(-\frac{k^2}{k^2 + (\hat{t}\cdot k)^2} + 1 - \frac{(\hat{t}\cdot k)^2}{k^2 + (\hat{t}\cdot k)^2}\right)\hat{t}^\mu \hat{t}^\nu\right]$$

= 0

We obtained a very simple formula for the photon propagator:

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \tilde{\Delta}^{\mu\nu}(k)$$

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\epsilon}$$

Feynman gauge

(it would still be in the Coulomb gauge if we had kept the terms proportional to momenta)