C, P and T

based on S-40

The Lorentz transformation of a Dirac or Majorana field:

$$U(\Lambda)^{-1}\Psi(x)U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x)$$

$$D(1+\delta\omega) = 1 + \tfrac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu}$$

$$S^{\mu\nu} = \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]$$

besides infinitesimal LTs we need to consider also parity and time reversal.

Parity:

$$\mathcal{P}^\mu{}_\nu = (\mathcal{P}^{-1})^\mu{}_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

let's define the corresponding unitary operator

$$P \equiv U(\mathcal{P})$$

A Dirac or Majorana field transforms as:

$$P^{-1}\Psi(x)P = \underline{D(\mathcal{P})}\Psi(\mathcal{P}x)$$

?

acting with the parity transformation again:

$$P^{-2}\Psi(x)P^{2} = D(\mathcal{P})^{2}\Psi(x)$$

naively we would require $D(\mathcal{P})^2 = 1$, but we need an even number of fields to construct an observable (a term in the lagrangian) and so we should require:

$$D(\mathcal{P})^2 = \pm 1$$

Since

$$P^{-1}\mathbf{P}P = -\mathbf{P}$$

$$P^{-1}\mathbf{J}P = +\mathbf{J}$$

the parity transformation should reverse the 3-momentum, and leave the spin direction of a particle unchanged; we require:

$$P^{-1}b_s^{\dagger}(\mathbf{p})P = \eta\, b_s^{\dagger}(-\mathbf{p})$$

$$P^{-1}d_s^{\dagger}(\mathbf{p})P = \eta\, d_s^{\dagger}(-\mathbf{p})$$

$$\eta^2 = \pm 1$$

the same for both b and d so that it is compatible with the Majorana condition

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

$$P^{-1} b_s^\dagger(\mathbf{p}) P = \eta \, b_s^\dagger(-\mathbf{p})$$

$$P^{-1} d_s^\dagger(\mathbf{p}) P = \eta \, d_s^\dagger(-\mathbf{p})$$

$$\begin{aligned} P^{-1}\Psi(x)P & \\ &= \sum_{s=\pm} \int \widetilde{dp} \left[\left(P^{-1} b_s(\mathbf{p}) P\right) u_s(\mathbf{p}) e^{ipx} + \left(P^{-1} d_s^\dagger(\mathbf{p}) P\right) v_s(\mathbf{p}) e^{-ipx} \right] \\ &= \sum_{s=\pm} \int \widetilde{dp} \left[\eta^* b_s(-\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + \eta d_s^\dagger(-\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right] \\ &= \sum_{s=\pm} \int \widetilde{dp} \left[\eta^* b_s(\mathbf{p}) u_s(-\mathbf{p}) e^{ip\mathcal{P}x} + \eta d_s^\dagger(\mathbf{p}) v_s(-\mathbf{p}) e^{-ip\mathcal{P}x} \right] . \end{aligned}$$

$\mathbf{p} \rightarrow -\mathbf{p}$

$$u_s(-\mathbf{p}) = +\beta u_s(\mathbf{p})$$
$$v_s(-\mathbf{p}) = -\beta v_s(\mathbf{p})$$

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choosing $\eta = -i$:

$$\begin{aligned} P^{-1}\Psi(x)P &= \sum_{s=\pm} \int \widetilde{dp} \left[i b_s(\mathbf{p}) \beta u_s(\mathbf{p}) e^{ip\mathcal{P}x} + i d_s^\dagger(\mathbf{p}) \beta v_s(\mathbf{p}) e^{-ip\mathcal{P}x} \right] \\ &= i\beta \, \Psi(\mathcal{P}x) . \end{aligned}$$

$$D(\mathcal{P}) = i\beta$$

we have just used:

$$u_s(-\mathbf{p}) = +\beta u_s(\mathbf{p})$$
$$v_s(-\mathbf{p}) = -\beta v_s(\mathbf{p})$$

Proof:

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix},$$

$$v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}, \qquad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

by direct calculation:

$$\beta u_s(\mathbf{0}) = +u_s(\mathbf{0})$$
$$\beta v_s(\mathbf{0}) = -v_s(\mathbf{0})$$

$$\beta = \begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

boosting to any frame we get:

$$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})$$
$$v_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})v_s(\mathbf{0})$$

$$K^j = \tfrac{i}{4}[\gamma^j,\gamma^0] = \tfrac{i}{2}\gamma^j\gamma^0$$
$$\beta K^j = -K^j\beta$$

$$u_s(-\mathbf{p}) = +\beta u_s(\mathbf{p})$$
$$v_s(-\mathbf{p}) = -\beta v_s(\mathbf{p})$$

An interesting consequence of $\eta = -i$:

Consider a state of an electron and a positron with zero center of mass momentum:

$$|\phi\rangle = \int \widetilde{dp}\ \phi(\mathbf{p}) b_s^\dagger(\mathbf{p}) d_{s'}^\dagger(-\mathbf{p})|0\rangle$$

momentum space wave function
with definite parity: $\phi(-\mathbf{p}) = (-)^\ell \phi(\mathbf{p})$

Then we find:

$$P|0\rangle = P^{-1}|0\rangle = |0\rangle$$

$$\begin{aligned} P^{-1}|\phi\rangle &= \int \widetilde{dp}\ \phi(\mathbf{p}) \Big(P^{-1} b_s^\dagger(\mathbf{p}) P\Big)\Big(P^{-1} d_{s'}^\dagger(-\mathbf{p}) P\Big) P^{-1}|0\rangle \\ &= (-i)^2 \int \widetilde{dp}\ \phi(\mathbf{p}) b_s^\dagger(-\mathbf{p}) d_{s'}^\dagger(\mathbf{p})|0\rangle \\ &= (-i)^2 \int \widetilde{dp}\ \phi(-\mathbf{p}) b_s^\dagger(\mathbf{p}) d_{s'}^\dagger(-\mathbf{p})|0\rangle \\ &= -(-)^\ell |\phi\rangle\ . \end{aligned}$$

an electron-positron pair has an intrinsic parity of -1

For the two Weyl components of a Dirac field we get:

$$P^{-1}\Psi(x)P = \sum_{s=\pm}\int \widetilde{dp}\left[ib_s(\mathbf{p})\beta u_s(\mathbf{p})e^{ip\mathcal{P}x} + id_s^\dagger(\mathbf{p})\beta v_s(\mathbf{p})e^{-ip\mathcal{P}x}\right]$$
$$= i\beta\,\Psi(\mathcal{P}x)\ .$$

$$\Psi = \begin{pmatrix}\chi_a \\ \xi^{\dagger\dot{a}}\end{pmatrix}$$

$$P^{-1}\chi_a(x)P = i\xi^{\dagger\dot{a}}(\mathcal{P}x)$$
$$P^{-1}\xi^{\dagger\dot{a}}(x)P = i\chi_a(\mathcal{P}x)$$

$$\beta = \begin{pmatrix}0 & I \\ I & 0\end{pmatrix}$$

a parity transformation exchanges a left-handed field for a right-handed one!

Taking the hermitian conjugate:

$$P^{-1}\chi^{\dagger\dot{a}}(x)P = i\xi_a(\mathcal{P}x)\ ,$$
$$P^{-1}\xi_a(x)P = i\chi^{\dagger\dot{a}}(\mathcal{P}x)\ .$$

we see that these transformations are compatible with the Majorana condition:

$$\chi_a(x) = \xi_a(x)$$

Time reversal:

$$\mathcal{T}^{\mu}{}_{\nu} = (\mathcal{T}^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

let's define the corresponding anti-unitary operator

$$T \equiv U(\mathcal{T})$$

A Dirac or Majorana field transforms as:

$$T^{-1}\Psi(x)T = D(\mathcal{T})\Psi(\mathcal{T}x)$$

?

$$D(\mathcal{T})^2 = \pm 1$$

Since

$$T^{-1}\mathbf{P}T = -\mathbf{P}$$

$$T^{-1}\mathbf{J}T = -\mathbf{J}$$

we also require:

$$T^{-1}b_s^{\dagger}(\mathbf{p})T = \zeta_s\, b_{-s}^{\dagger}(-\mathbf{p})$$

$$T^{-1}d_s^{\dagger}(\mathbf{p})T = \zeta_s\, d_{-s}^{\dagger}(-\mathbf{p})$$

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left[b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right]$$

$$T^{-1} b_s^\dagger(\mathbf{p}) T = \zeta_s \, b_{-s}^\dagger(-\mathbf{p})$$

$$T^{-1} d_s^\dagger(\mathbf{p}) T = \zeta_s \, d_{-s}^\dagger(-\mathbf{p})$$

$$T^{-1}\Psi(x)T$$

$$= \sum_{s=\pm} \int \widetilde{dp} \left[\left(T^{-1} b_s(\mathbf{p}) T \right) u_s^*(\mathbf{p}) e^{-ipx} + \left(T^{-1} d_s^\dagger(\mathbf{p}) T \right) v_s^*(\mathbf{p}) e^{ipx} \right]$$

$T^{-1} i T = -i$

$$= \sum_{s=\pm} \int \widetilde{dp} \left[\zeta_s^* b_{-s}(-\mathbf{p}) u_s^*(\mathbf{p}) e^{-ipx} + \zeta_s d_{-s}^\dagger(-\mathbf{p}) v_s^*(\mathbf{p}) e^{ipx} \right]$$

$$= \sum_{s=\pm} \int \widetilde{dp} \left[\zeta_{-s}^* b_s(\mathbf{p}) u_{-s}^*(-\mathbf{p}) e^{ip\mathcal{T}x} + \zeta_{-s} d_s^\dagger(\mathbf{p}) v_{-s}^*(-\mathbf{p}) e^{-ip\mathcal{T}x} \right]$$

$\mathbf{p} \rightarrow -\mathbf{p}$

$s \rightarrow -s$

$$u_{-s}^*(-\mathbf{p}) = -s \, \mathcal{C} \gamma_5 u_s(\mathbf{p})$$

$$v_{-s}^*(-\mathbf{p}) = -s \, \mathcal{C} \gamma_5 v_s(\mathbf{p})$$

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choosing $\zeta_s = s$:

$$T^{-1}\Psi(x)T = \mathcal{C}\gamma_5 \Psi(\mathcal{T}x)$$

$$D(\mathcal{T}) = \mathcal{C}\gamma_5$$

we have just used:

$$u^*_{-s}(-\mathbf{p}) = -s\,\mathcal{C}\gamma_5 u_s(\mathbf{p})$$
$$v^*_{-s}(-\mathbf{p}) = -s\,\mathcal{C}\gamma_5 v_s(\mathbf{p})$$

Proof:

$$\mathcal{C}\overline{u}_s(\mathbf{p})^{\mathrm{T}} = v_s(\mathbf{p})$$
$$\mathcal{C}\overline{v}_s(\mathbf{p})^{\mathrm{T}} = u_s(\mathbf{p})$$

$\overline{u}^{\mathrm{T}*} = \overline{u}^\dagger = \beta u$ → complex conjugate

$$u^*_s(\mathbf{p}) = \mathcal{C}\beta v_s(\mathbf{p})$$
$$v^*_s(\mathbf{p}) = \mathcal{C}\beta u_s(\mathbf{p})$$

by direct calculation:

$$\gamma_5 u_s(\mathbf{0}) = +s\,v_{-s}(\mathbf{0})$$
$$\gamma_5 v_s(\mathbf{0}) = -s\,u_{-s}(\mathbf{0})$$

boosting to any frame we get:

$$\gamma_5 K^j = K^j \gamma_5$$

$$\gamma_5 u_s(\mathbf{p}) = +s\,v_{-s}(\mathbf{p})$$
$$\gamma_5 v_s(\mathbf{p}) = -s\,u_{-s}(\mathbf{p})$$

$$u_s(-\mathbf{p}) = +\beta u_s(\mathbf{p})$$
$$v_s(-\mathbf{p}) = -\beta v_s(\mathbf{p})$$

$$u^*_{-s}(-\mathbf{p}) = -s\,\mathcal{C}\gamma_5 u_s(\mathbf{p})$$
$$v^*_{-s}(-\mathbf{p}) = -s\,\mathcal{C}\gamma_5 v_s(\mathbf{p})$$

For the two Weyl components of a Dirac field we get:

$$\Psi = \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix}$$

$$T^{-1}\Psi(x)T = \mathcal{C}\gamma_5\Psi(\mathcal{T}x)$$

$$\mathcal{C} = \begin{pmatrix} -\varepsilon^{ab} & 0 \\ 0 & -\varepsilon_{\dot{a}\dot{b}} \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}$$

$$T^{-1}\chi_a(x)T = +\chi^a(\mathcal{T}x)$$

$$T^{-1}\xi^{\dagger\dot{a}}(x)T = -\xi^\dagger_{\dot{a}}(\mathcal{T}x)$$

time reversal takes left-handed Weyl fields into left-handed Weyl fields!

Taking the hermitian conjugate:

$$T^{-1}\chi^{\dagger\dot{a}}(x)T = -\chi^\dagger_{\dot{a}}(\mathcal{T}x)$$

$$T^{-1}\xi_a(x)T = +\xi^a(\mathcal{T}x)$$

we see that these transformations are compatible with the Majorana condition:

$$\chi_a(x) = \xi_a(x)$$

We will also need transformation properties of fermion bilinears:

$$\overline{\Psi} A \Psi$$

some product of gamma matrices, such that $\overline{A} = A$ so that $\overline{\Psi} A \Psi$ is hermitian.

$$\overline{A} \equiv \beta A^{\dagger} \beta$$

$$\overline{\Psi} = \Psi^{\dagger} \beta$$

Parity:

$$P^{-1} \Psi(x) P = \ i\beta\, \Psi(\mathcal{P}x)$$

$$P^{-1} \overline{\Psi}(x) P = -i \overline{\Psi}(\mathcal{P}x) \beta$$

$$P^{-1} \left(\overline{\Psi} A \Psi \right) P = \overline{\Psi} \left(\beta A \beta \right) \Psi$$

we easily find:

$$\beta 1 \beta = +1 \ ,$$

$$\beta i\gamma_5 \beta = -i\gamma_5 \ ,$$

$$\beta \gamma^0 \beta = +\gamma^0 \ ,$$

$$\beta \gamma^i \beta = -\gamma^i \ ,$$

$$\beta \gamma^0 \gamma_5 \beta = -\gamma^0 \gamma_5 \ ,$$

$$\beta \gamma^i \gamma_5 \beta = +\gamma^i \gamma_5 \ .$$

$$P^{-1}\left(\overline{\Psi}A\Psi\right)P = \overline{\Psi}\left(\beta A\beta\right)\Psi$$

$$\begin{aligned}
\beta 1 \beta &= +1\,, \\
\beta i\gamma_5 \beta &= -i\gamma_5\,, \\
\beta \gamma^0 \beta &= +\gamma^0\,, \\
\beta \gamma^i \beta &= -\gamma^i\,, \\
\beta \gamma^0\gamma_5 \beta &= -\gamma^0\gamma_5\,, \\
\beta \gamma^i\gamma_5 \beta &= +\gamma^i\gamma_5\,.
\end{aligned}$$

And so the corresponding bilinears transform as:

$$\begin{aligned}
P^{-1}\left(\overline{\Psi}\Psi\right)P &= +\,\overline{\Psi}\Psi\,, \\
P^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)P &= -\,\overline{\Psi}i\gamma_5\Psi\,, \\
P^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)P &= +\,\mathcal{P}^\mu{}_\nu\overline{\Psi}\gamma^\nu\Psi\,, \\
P^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)P &= -\,\mathcal{P}^\mu{}_\nu\overline{\Psi}\gamma^\nu\gamma_5\Psi\,,
\end{aligned}$$

scalar

pseudoscalar

vector

axial vector
(pseudovector)

even under a parity transformation

odd

Time reversal:

$$T^{-1}\Psi(x)T = \mathcal{C}\gamma_5\Psi(\mathcal{T}x)$$

$$T^{-1}\overline{\Psi}(x)T = \overline{\Psi}(\mathcal{T}x)\gamma_5\mathcal{C}^{-1}$$

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -(\gamma^\mu)^{\mathrm{T}}$$

$$\mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5$$

$$T^{-1}\left(\overline{\Psi}A\Psi\right)T = \overline{\Psi}\left(\gamma_5\mathcal{C}^{-1}A^*\mathcal{C}\gamma_5\right)\Psi$$

$$T^{-1}AT = A^*$$

we easily find:

$$\begin{aligned}
\gamma_5\mathcal{C}^{-1}1^*\mathcal{C}\gamma_5 &= +1\,, \\
\gamma_5\mathcal{C}^{-1}(i\gamma_5)^*\mathcal{C}\gamma_5 &= -i\gamma_5\,, \\
\gamma_5\mathcal{C}^{-1}(\gamma^0)^*\mathcal{C}\gamma_5 &= +\gamma^0\,, \\
\gamma_5\mathcal{C}^{-1}(\gamma^i)^*\mathcal{C}\gamma_5 &= -\gamma^i\,, \\
\gamma_5\mathcal{C}^{-1}(\gamma^0\gamma_5)^*\mathcal{C}\gamma_5 &= +\gamma^0\gamma_5\,, \\
\gamma_5\mathcal{C}^{-1}(\gamma^i\gamma_5)^*\mathcal{C}\gamma_5 &= -\gamma^i\gamma_5\,.
\end{aligned}$$

$$
\begin{aligned}
\gamma_5 \mathcal{C}^{-1} 1^* \mathcal{C} \gamma_5 &= +1 \,, \\
\gamma_5 \mathcal{C}^{-1} (i\gamma_5)^* \mathcal{C} \gamma_5 &= -i\gamma_5 \,, \\
\gamma_5 \mathcal{C}^{-1} (\gamma^0)^* \mathcal{C} \gamma_5 &= +\gamma^0 \,, \\
\gamma_5 \mathcal{C}^{-1} (\gamma^i)^* \mathcal{C} \gamma_5 &= -\gamma^i \,, \\
\gamma_5 \mathcal{C}^{-1} (\gamma^0\gamma_5)^* \mathcal{C} \gamma_5 &= +\gamma^0\gamma_5 \,, \\
\gamma_5 \mathcal{C}^{-1} (\gamma^i\gamma_5)^* \mathcal{C} \gamma_5 &= -\gamma^i\gamma_5 \,.
\end{aligned}
$$

$$
T^{-1}\left(\overline{\Psi} A \Psi\right)T = \overline{\Psi}\left(\gamma_5 \mathcal{C}^{-1} A^* \mathcal{C} \gamma_5\right)\Psi
$$

And so the corresponding bilinears transform as:

even under time reversal

$$
\begin{aligned}
T^{-1}\left(\overline{\Psi}\Psi\right)T &= +\overline{\Psi}\Psi \,, \\
T^{-1}\left(\overline{\Psi} i\gamma_5 \Psi\right)T &= -\overline{\Psi} i\gamma_5 \Psi \,, \\
T^{-1}\left(\overline{\Psi}\gamma^\mu \Psi\right)T &= -\mathcal{T}^\mu{}_\nu \overline{\Psi}\gamma^\nu \Psi \,, \\
T^{-1}\left(\overline{\Psi}\gamma^\mu \gamma_5 \Psi\right)T &= -\mathcal{T}^\mu{}_\nu \overline{\Psi}\gamma^\nu \gamma_5 \Psi \,.
\end{aligned}
$$

odd

Charge conjugation:

$$C^{-1}\Psi(x)C = \mathcal{C}\overline{\Psi}^{\mathrm{T}}(x)$$

$$C^{-1}\overline{\Psi}(x)C = \Psi^{\mathrm{T}}(x)\mathcal{C}$$

$$\Psi^{\mathrm{C}} \equiv \mathcal{C}\overline{\Psi}^{\mathrm{T}} = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix}$$

$$\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\dagger} = \mathcal{C}^{-1} = -\mathcal{C}$$

$$C^{-1}\left(\overline{\Psi}A\Psi\right)C = \Psi^{\mathrm{T}}\mathcal{C}A\mathcal{C}\overline{\Psi}^{\mathrm{T}}$$

taking the transpose of the RHS it can be also written as:

$$C^{-1}\left(\overline{\Psi}A\Psi\right)C = \overline{\Psi}\left(\mathcal{C}^{-1}A^{\mathrm{T}}\mathcal{C}\right)\Psi$$

we easily find:

$$\mathcal{C}^{-1}1^{\mathrm{T}}\mathcal{C} = +1\ ,$$
$$\mathcal{C}^{-1}(i\gamma_5)^{\mathrm{T}}\mathcal{C} = +i\gamma_5\ ,$$
$$\mathcal{C}^{-1}(\gamma^{\mu})^{\mathrm{T}}\mathcal{C} = -\gamma^{\mu}\ ,$$
$$\mathcal{C}^{-1}(\gamma^{\mu}\gamma_5)^{\mathrm{T}}\mathcal{C} = +\gamma^{\mu}\gamma_5\ .$$

$$\mathcal{C}^{-1}\gamma^{\mu}\mathcal{C} = -(\gamma^{\mu})^{\mathrm{T}}$$
$$\mathcal{C}^{-1}\gamma_5\mathcal{C} = \gamma_5$$

For a Majorana field we have:

$$C^{-1}\Psi C = \Psi$$
$$C^{-1}\overline{\Psi}C = \overline{\Psi}$$

$$\longrightarrow \quad C^{-1}(\overline{\Psi}A\Psi)C = \overline{\Psi}A\Psi$$

$$\mathcal{C}^{-1} 1^{\mathrm{T}} \mathcal{C} = +1 \,,$$
$$\mathcal{C}^{-1} (i\gamma_5)^{\mathrm{T}} \mathcal{C} = +i\gamma_5 \,,$$
$$\mathcal{C}^{-1} (\gamma^\mu)^{\mathrm{T}} \mathcal{C} = -\gamma^\mu \,,$$
$$\mathcal{C}^{-1} (\gamma^\mu \gamma_5)^{\mathrm{T}} \mathcal{C} = +\gamma^\mu \gamma_5 \,.$$

$$C^{-1}\left(\overline{\Psi} A \Psi\right) C = \overline{\Psi}\left(\mathcal{C}^{-1} A^{\mathrm{T}} \mathcal{C}\right)\Psi$$

And so the corresponding bilinears transform as:

$$C^{-1}\left(\overline{\Psi}\Psi\right)C = +\overline{\Psi}\Psi \,,$$
$$C^{-1}\left(\overline{\Psi} i\gamma_5 \Psi\right)C = +\overline{\Psi} i\gamma_5 \Psi \,,$$
$$C^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)C = -\overline{\Psi}\gamma^\mu\Psi \,,$$
$$C^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)C = +\overline{\Psi}\gamma^\mu\gamma_5\Psi \,.$$

even under charge conjugation

odd

For a Majorana field: $C^{-1}(\overline{\Psi} A \Psi) C = \overline{\Psi} A \Psi$ → $\overline{\Psi}\gamma^\mu\Psi = 0$

Combined C, P and T transformation:

$$P^{-1}\left(\overline{\Psi}\Psi\right)P = +\overline{\Psi}\Psi\ ,$$
$$P^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)P = -\overline{\Psi}i\gamma_5\Psi\ ,$$
$$P^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)P = +\mathcal{P}^\mu{}_\nu\overline{\Psi}\gamma^\nu\Psi\ ,$$
$$P^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)P = -\mathcal{P}^\mu{}_\nu\overline{\Psi}\gamma^\nu\gamma_5\Psi\ ,$$

$$C^{-1}\left(\overline{\Psi}\Psi\right)C = +\overline{\Psi}\Psi\ ,$$
$$C^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)C = +\overline{\Psi}i\gamma_5\Psi\ ,$$
$$C^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)C = -\overline{\Psi}\gamma^\mu\Psi\ ,$$
$$C^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)C = +\overline{\Psi}\gamma^\mu\gamma_5\Psi\ .$$

$$T^{-1}\left(\overline{\Psi}\Psi\right)T = +\overline{\Psi}\Psi\ ,$$
$$T^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)T = -\overline{\Psi}i\gamma_5\Psi\ ,$$
$$T^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)T = -\mathcal{T}^\mu{}_\nu\overline{\Psi}\gamma^\nu\Psi\ ,$$
$$T^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)T = -\mathcal{T}^\mu{}_\nu\overline{\Psi}\gamma^\nu\gamma_5\Psi\ .$$

$$(CPT)^{-1}\left(\overline{\Psi}\Psi\right)CPT = +\overline{\Psi}\Psi\ ,$$
$$(CPT)^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)CPT = +\overline{\Psi}i\gamma_5\Psi\ ,$$
$$(CPT)^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)CPT = -\overline{\Psi}\gamma^\mu\Psi\ ,$$
$$(CPT)^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)CPT = -\overline{\Psi}\gamma^\mu\gamma_5\Psi\ ,$$

$$\mathcal{P}^\mu{}_\nu\mathcal{T}^\nu{}_\rho = -\delta^\mu{}_\rho$$

CPT theorem:

$$(CPT)^{-1}\left(\overline{\Psi}\Psi\right)CPT = +\overline{\Psi}\Psi\,,$$
$$(CPT)^{-1}\left(\overline{\Psi}i\gamma_5\Psi\right)CPT = +\overline{\Psi}i\gamma_5\Psi\,,$$
$$(CPT)^{-1}\left(\overline{\Psi}\gamma^\mu\Psi\right)CPT = -\overline{\Psi}\gamma^\mu\Psi\,,$$
$$(CPT)^{-1}\left(\overline{\Psi}\gamma^\mu\gamma_5\Psi\right)CPT = -\overline{\Psi}\gamma^\mu\gamma_5\Psi\,,$$

even under CPT

odd

General rule: a fermion bilinear with n vector indices is even (odd) under CPT if n is even (odd); this also applies to derivatives ∂_μ .

Thus any hermitian combination of fields and derivatives that is Lorentz invariant (has no uncontracted Lorentz indices) is even under CPT!

Lagrangian is formed from such terms, $\mathcal{L}(x) \to \mathcal{L}(-x)$ under CPT, and so the action $S = \int d^4x\,\mathcal{L}$ is invariant under CPT.

Lorentz invariance ⟷ CPT