Continuous symmetries and conserved currents

Consider a set of scalar fields \( \varphi_a(x) \), and a lagrangian density

\[
\mathcal{L}(x) = \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))
\]

let’s make an infinitesimal change: \( \varphi_a(x) \rightarrow \varphi_a(x) + \delta \varphi_a(x) \)

\[
\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \delta \mathcal{L}(x)
\]

\[
\delta \mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial \varphi_a(x)} \delta \varphi_a(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_a(x))} \partial_\mu \delta \varphi_a(x)
\]

variation of the action:

\[
\frac{\delta S}{\delta \varphi_a(x)} = \int d^4y \frac{\delta \mathcal{L}(y)}{\delta \varphi_a(x)}
\]

\[
= \int d^4y \left[ \frac{\partial \mathcal{L}(y)}{\partial \varphi_b(y)} \frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} + \frac{\partial \mathcal{L}(y)}{\partial (\partial_\mu \varphi_b(y))} \frac{\delta (\partial_\mu \varphi_b(y))}{\delta \varphi_a(x)} \right]
\]

\[
= \int d^4y \left[ \frac{\partial \mathcal{L}(y)}{\partial \varphi_b(y)} \delta_b \delta^4(y-x) + \frac{\partial \mathcal{L}(y)}{\partial (\partial_\mu \varphi_b(y))} \delta_b \partial_\mu \delta^4(y-x) \right]
\]

\[
= \frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} - \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \cdot \frac{\partial \mathcal{L}(x)}{\partial \varphi_a(x)} \rightarrow \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} + \frac{\delta S}{\delta \varphi_a(x)}
\]

setting \( \frac{\delta S}{\delta \varphi_a(x)} = 0 \) we would get equations of motion
if a set of infinitesimal transformations leaves the lagrangian unchanged, invariant, the Noether current is conserved!

\[
\delta L(x) = \frac{\partial L}{\partial \varphi_a(x)} \delta \varphi_a(x) + \frac{\partial L}{\partial (\partial_\mu \varphi_a(x))} (\partial_\mu \delta \varphi_a(x))
\]

thus we find:

\[
\delta L(x) = \partial_\mu \left( \frac{\partial L(x)}{\partial (\partial_\mu \varphi_a(x))} \delta \varphi_a(x) \right) + \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]

\[
J^\mu(x) = \frac{\partial L(x)}{\partial (\partial_\mu \varphi_a(x))} \delta \varphi_a(x)
\]

this is called Noether current; now we have:

\[
\partial_\mu J^\mu(x) = \delta L(x) - \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]

= 0 if eqs. of motion are satisfied

if a set of infinitesimal transformations leaves the lagrangian unchanged, invariant, \(\delta L = 0\), the Noether current is conserved!

\[
\partial_\lambda j^\lambda(0) + \nabla \cdot \mathbf{j}(x) = 0
\]

charge density current density
we set

\[ \hbar = c = 1 \]

it allows us to convert a time \( T \) to a length \( L \):

\[ T = c^{-1} L \]
a length to an inverse mass:

\[ L = \hbar c^{-1} M^{-1} \]

any quantity \( A \) has units of mass to some power that we call \([A]\), e.g:

\[
\begin{align*}
[m] &= +1 \\
[x^\mu] &= -1 \\
[\partial^\mu] &= +1 \\
[d^d x] &= -d
\end{align*}
\]

in \( d \) spacetime dimensions

the action appears in the exponential and so

\[ [S] = 0 \]

and for the lagrangian density we have:

\[ [\mathcal{L}] = d \]
from the kinetic term:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 + \frac{1}{6} g \varphi^3 \]

\[ [\varphi] = \frac{1}{2} (d - 2) \]

in 4 dimensions:

\[ [\varphi] = 1 \]

functional derivative:

\[ \frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2) \]

\[ \text{dim} = f \quad \text{dim} = l \]

\[ \text{dim} = -f + 1 \]
similarly, in 4d:

\[
\frac{\delta \rho_b(y)}{\delta \varphi_a(x)} = \delta_{ba} \delta^4(y-x)
\]

Noether current:

\[
\partial_\mu j^\mu(x) = \delta \mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x)
\]

and so on ...
Consider a theory of a complex scalar field:

\[ \mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \frac{1}{4} \lambda (\varphi^\dagger \varphi)^2 \]

clearly \( \mathcal{L} \) is left invariant by:

\[ \varphi(x) \rightarrow e^{-i\alpha} \varphi(x) \]

**U(1) transformation**

(transform transformation by a unitary 1x1 matrix)

in terms of two real scalar fields we get:

\[ \varphi = (\varphi_1 + i\varphi_2)/\sqrt{2} \]

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \]

and the **U(1) transformation** above is equivalent to:

\[ \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \rightarrow \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \]

**SO(2) transformation**

(transform transformation by an orthogonal 2x2 matrix with determinant = +1)
infinitesimal form of \( \varphi(x) \rightarrow e^{-i\alpha \varphi(x)} \) is:

\[
\varphi(x) \rightarrow \varphi(x) - i\alpha \varphi(x), \\
\varphi^\dagger(x) \rightarrow \varphi^\dagger(x) + i\alpha \varphi^\dagger(x),
\]

and the current is:

\[
\alpha j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^\dagger)} \delta \varphi^\dagger \\
= (-\partial^\mu \varphi^\dagger)(-i\alpha \varphi) + (-\partial^\mu \varphi)(+i\alpha \varphi^\dagger) \\
= \alpha \text{Im}(\varphi^\dagger \overleftarrow{\partial^\mu} \varphi),
\]

we treat \( \varphi \) and \( \varphi^\dagger \) as independent fields

\[
\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \frac{1}{4} \lambda (\varphi^\dagger \varphi)^2
\]

\[
j^\mu(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi^\dagger(a(x)))} \delta \varphi^\dagger(a(x))
\]

\[
A \overleftarrow{\partial^\mu} B = A \partial^\mu B - (\partial^\mu A)B
\]

\[
j^\mu = \text{Im}(\varphi^\dagger \overleftarrow{\partial^\mu} \varphi)
\]
repeating the same for the $SO(2)$ transformation:

\[ \delta \varphi_1 = +\alpha \varphi_2 \]

\[ \delta \varphi_2 = -\alpha \varphi_1 \]

the Noether current is:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \]

\[ j^\mu (x) \equiv \frac{\partial \mathcal{L} (x)}{\partial (\partial_\mu \varphi_1 (x))} \delta \varphi_1 (x) \]

\[ \alpha j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_1)} \delta \varphi_1 + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_2)} \delta \varphi_2 \]

\[ = (\partial^\mu \varphi_1)(+\alpha \varphi_2) + (-\partial^\mu \varphi_2)(-\alpha \varphi_1) \]

\[ = \alpha (\varphi_1 \overset{\to}{\partial}^\mu \varphi_2) \]

which is equivalent to

\[ j^\mu = \text{Im}(\varphi^\dagger \overset{\to}{\partial}^\mu \varphi) \]

\[ \varphi = (\varphi_1 + i\varphi_2)/\sqrt{2} \]
Let’s define the Noether charge:

$$ Q \equiv \int d^3x \, j^0(x) = \int d^3x \, \text{Im}(\varphi^\dagger \vec{\partial}^0 \varphi) $$

integrating $\frac{\partial}{\partial t} j^0(x) + \nabla \cdot j(x) = 0$ over $d^3x$, using Gauss’s law to write the volume integral of $\nabla \cdot j$ as a surface integral and assuming $j(x) = 0$ on that surface we find:

**Q is constant in time!**

using free field expansions, we get:

$$ \varphi(x) = \int \frac{dK}{2\pi} \left[ a(k)e^{ikx} + b^*(k)e^{-ikx} \right] $$

$$ \varphi^\dagger(x) = \int \frac{dK}{2\pi} \left[ b(k)e^{ikx} + a^*(k)e^{-ikx} \right] $$

for an interacting theory these formulas are valid at any given time

$$ Q = \int \frac{dk}{2\pi} \left[ a^*(k)a(k) - b(k)b^*(k) \right] $$

counts the number of a particles minus the number of b particles; it is time independent and so the scattering amplitudes do not change the value of Q; in Feynman diagrams Q is conserved in every vertex.
Another use of Noether current:

Consider a transformation of fields that change the lagrangian density by a total divergence:

$$\delta \mathcal{L}(x) = \partial_\mu K^\mu(x)$$

there is still a conserved current:

$$j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \delta \varphi_a(x) - K^\mu(x)$$

e.g. space-time translations:

$$\varphi_a(x) \to \varphi_a(x-a)$$
$$\varphi_a(x) \to \varphi_a(x) - a^\nu \partial_\nu \varphi_a(x)$$
$$\delta \varphi_a(x) = -a^\nu \partial_\nu \varphi_a(x)$$

we get:

$$j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} (-a^\nu \partial_\nu \varphi_a(x)) + a^\mu \mathcal{L}(x)$$

$$= a_\nu T^{\mu\nu}(x) , \quad T^{\mu\nu}(x) \equiv -\frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \partial^\nu \varphi_a(x) + g^{\mu\nu} \mathcal{L}(x)$$

stress-energy or energy-momentum tensor
for a theory of a set of real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_a \partial_\mu \varphi_a - V(\varphi) \]

we get:

\[ T^{\mu\nu}(x) \equiv -\frac{\partial \mathcal{L}(x)}{\partial (\partial_\mu \varphi_a(x))} \partial^\nu \varphi_a(x) + g^{\mu\nu} \mathcal{L}(x) \]

\[ T^{\mu\nu} = \partial^\mu \varphi_a \partial^\nu \varphi_a + g^{\mu\nu} \mathcal{L} \]

in particular:

\[ T^{00} = \frac{1}{2} \Pi_a^2 + \frac{1}{2} (\nabla \varphi_a)^2 + V(\varphi) \]

\[ \Pi_a = \partial_0 \varphi_a \]

then by Lorentz symmetry the momentum density must be:

\[ T^{0j} = \partial^0 \varphi_a \partial^j \varphi_a = -\Pi_a \nabla^j \varphi_a \]

plugging in the field expansions, we get:

\[ P^j = \int d^3x T^{0j}(x) = \int \tilde{d}k \ k^j \ a^\dagger_a(k) a_a(k) \]

as expected
The energy-momentum four-vector is:

\[ P^\mu = \int d^3x \, T^{0\mu}(x) \]

Recall, we defined the space-time translation operator

\[ T(a) \equiv \exp(-iP^\mu a_\mu) \]

so that

\[ T(a)^{-1} \varphi_a(x) T(a) = \varphi_a(x - a) \]

we can easily verify it; for an infinitesimal transformation it becomes:

\[ [\varphi_a(x), P^\mu] = \frac{1}{i} \partial^\mu \varphi_a(x) \]

it is straightforward to verify this by using the canonical commutation relations for \( \varphi_a(x) \) and \( \Pi_a(x) \).
The same procedure can be repeated for Lorentz transformations:

$$\varphi_a(x) \rightarrow \varphi_a(x + \delta \omega \cdot x)$$

the resulting conserved current is:

$$\mathcal{M}^{\mu \nu \rho}(x) = x^{\nu} T^{\mu \rho}(x) - x^{\rho} T^{\mu \nu}(x)$$

antisymmetric in the last two indices as a result of $\delta \omega^{\nu \rho}$ being antisymmetric

$$\partial_\mu \mathcal{M}^{\mu \nu \rho} = 0$$

the conserved charges associated with this current are:

$$M^{\nu \rho} = \int d^3x \, \mathcal{M}^{0 \nu \rho}(x)$$

again, one can check all the commutators...
Discrete symmetries: P, T, C and Z

Recall from S-2:

Infinitesimal Lorentz transformation:

\[ \Lambda_{\mu}^{\nu} = \delta_{\mu}^{\nu} + \delta \omega_{\mu}^{\nu} \]

not all LT can be obtained by compounding ILTs!

\((\Lambda^{-1})_{\rho}^{\mu} = \Lambda_{\rho}^{\mu} \quad \Rightarrow \quad (\det \Lambda)^{-1} = \det \Lambda \quad \det \Lambda = \pm 1\)  

+1 proper

-1 improper

proper LTs form a subgroup of Lorentz group; ILTs are proper!

Another subgroup - orthochronous LTs,

\[ g_{\mu\nu} \Lambda_{\rho}^{\mu} \Lambda_{\sigma}^{\nu} = g_{\rho\sigma} \quad \Rightarrow \quad (\Lambda^0_0)^2 - \Lambda^i_0 \Lambda^i_0 = 1 \]  

ILTs are orthochronous!
When we say theory is Lorentz invariant we mean it is invariant under proper orthochronous subgroup only (those that can be obtained by compounding ILTs).

Transformations that take us out of proper orthochronous subgroup are parity and time reversal:

\[
\mathcal{P}^\mu{}_{\nu} = (\mathcal{P}^{-1})^\mu{}_{\nu} = \begin{pmatrix}
+1 & -1 \\
-1 & -1
\end{pmatrix} \quad \text{orthochronous but improper}
\]

\[
\mathcal{T}^\mu{}_{\nu} = (\mathcal{T}^{-1})^\mu{}_{\nu} = \begin{pmatrix}
-1 & +1 \\
+1 & +1
\end{pmatrix} \quad \text{nonorthochronous and improper}
\]

A quantum field theory doesn’t have to be invariant under P or T.
For every proper orthochronous LT there is a unitary operator:

$$U(\Lambda)^{-1} \varphi(x) U(\Lambda) = \varphi(\Lambda^{-1} x)$$

we expect the same for parity and time-reversal

$$P \equiv U(\mathcal{P})$$
$$T \equiv U(\mathcal{T})$$

so that

$$P^{-1} \varphi(x) P = \varphi(\mathcal{P} x)$$
$$T^{-1} \varphi(x) T = \varphi(\mathcal{T} x)$$

since $\mathcal{P}^2 = 1$ and $\mathcal{T}^2 = 1$ we need:

$$P^{-2} \varphi(x) P^2 = \varphi(x)$$
$$T^{-2} \varphi(x) T^2 = \varphi(x)$$

that can be also satisfied with:

$$P^{-1} \varphi(x) P = -\varphi(\mathcal{P} x)$$
$$T^{-1} \varphi(x) T = -\varphi(\mathcal{T} x)$$

scalars (even under parity)

pseudoscalars (odd under parity)
We can choose the transformation properties of fields. It is a part of specifying the theory. But if possible we want to have lagrangian density even under both parity and time-reversal,

\[
P^{-1} \mathcal{L}(x) P = + \mathcal{L}(\mathcal{P} x)
\]
\[
T^{-1} \mathcal{L}(x) T = + \mathcal{L}(\mathcal{T} x)
\]

so that parity and time-reversal are conserved.

Note: time-reversal operator must be antiunitary:

\[
T^{-1} i T = - i
\]

to see it, let’s look at transformations of the energy-momentum 4-vector:

\[
P^{-1} P^\mu P = \mathcal{P}^\mu _\nu P^\nu
\]
\[
T^{-1} P^\mu T = -T^\mu _\nu P^\nu
\]

can be checked directly using:

\[
P^\mu = \int d^3 x \ T^{0\mu}(x)
\]
\[
T^{00} = \frac{1}{2} \Pi^2_a + \frac{1}{2} (\nabla \varphi_a)^2 + V(\varphi)
\]
\[
T^{0j} = \partial^0 \varphi_a \partial^j \varphi_a = -\Pi_a \nabla^j \varphi_a
\]

the same result for scalar and pseudoscalar
for \( \mu = 0 \) we have:

\[
P^{-1} P^{\mu} P = P^{\mu, \nu} P^{\nu}
\]

\[
T^{-1} P^{\mu} T = -P^{\mu, \nu} P^{\nu}
\]

if \( T \) was unitary, \( T^{-1} P^{\mu} T = P^{\mu, \nu} P^{\nu} \) we would have \( T^{-1} H T = -H \) which is a DISASTER since hamiltonian is invariant under time-reversal only if \( H = -H \) and so \( H = 0 \).

Let's trace the origin of antiunitarity:

the spacetime translation operator

\[
T(a) = \exp(-i P \cdot a)
\]

implies:

\[
U(\Lambda)^{-1} T(a) U(\Lambda) = T(\Lambda^{-1} a)
\]
\[ U(\Lambda)^{-1}T(a)U(\Lambda) = T(\Lambda^{-1}a) \]

\[ T(a) = \exp(-iP \cdot a) \]

for an infinitesimal translation we get:

\[ U(\Lambda)^{-1}(I - ia_\mu P^\mu)U(\Lambda) = I - i(\Lambda^{-1})_\mu^\nu a_\mu P^\nu \]

\[ = I - i\Lambda^\mu_\nu a_\mu P^\nu \]  

similarly for time-reversal:

\[ T^{-1}(I - ia_\mu P^\mu)T = I - iT^\mu_\nu a_\mu P^\nu \]  

comparing linear terms in \( a \) we see that in order to get

\[ T^{-1}P^\mu T = -T^\mu_\nu P^\nu \]

we need

\[ T^{-1}iT = -i \]

\( T \) is antiunitary
$Z_2$ symmetry:

we want to consider a possibility that the sign of a scalar field changes under a symmetry transformation (that does not act on spacetime arguments). The corresponding unitary operator is:

$$Z^{-1} \varphi_a(x) Z = \eta_a \varphi_a(x)$$

\[ \eta_a = +1 \text{ or } -1 \]

\[ Z^{-1} = Z \]

\[ Z^2 = 1 \]

$Z_2$ symmetry

e.g. a theory of a complex scalar field:

$\varphi = (\varphi_1 + i\varphi_2)/\sqrt{2}$

$$\mathcal{L} = -\partial^\mu \varphi^{\dagger} \partial_\mu \varphi - m^2 \varphi^{\dagger} \varphi - \frac{1}{4} \lambda (\varphi^{\dagger} \varphi)^2$$

$$= -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2$$

has U(1) symmetry: $\varphi(x) \to e^{-i\alpha} \varphi(x)$, equivalent to SO(2): \[
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\to
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\]

but also an additional discrete symmetry:

$\varphi(x) \leftrightarrow \varphi^{\dagger}(x)$

\[
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\to
\begin{pmatrix}
+1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\]

charge conjugation
Charge conjugation always occurs as a companion to a U(1) symmetry; it enlarges SO(2) symmetry (the group of 2x2 orthogonal matrices with determinant +1) into O(2) symmetry (the group of 2x2 orthogonal matrices).

\[
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\quad \quad
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
+1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\varphi_1(x) \\
\varphi_2(x)
\end{pmatrix}
\]

We can define the operator of charge conjugation:

\[ C^{-1} \varphi(x) C = \varphi^\dagger(x) \quad \text{or} \quad C^{-1} \varphi_1(x) C = +\varphi_1(x) \]
\[ C^{-1} \varphi_2(x) C = -\varphi_2(x) \]

and the charge conjugation is a symmetry of the theory:

\[ C^{-1} \mathcal{L}(x) C = \mathcal{L}(x) \]

Scattering amplitudes must be unchanged if we exchange all a-type particles (charge +1) with all b-type particles (charge -1). This is only possible if both particles have the same mass; we say particle b is antiparticle of a.
Another example of $Z_2$ symmetry:

consider $\varphi^4$ theory:

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{24} \lambda \varphi^4$$

this theory is obviously invariant under:

$$Z^{-1} \varphi(x) Z = -\varphi(x)$$

the ground state (if unique) must also be an eigenstate of $Z$; we can fix the phase of $Z$ via:

$$Z|0\rangle = Z^{-1}|0\rangle = +|0\rangle$$

and then we have:

$$\langle 0|\varphi(x)|0\rangle = \langle 0|ZZ^{-1}\varphi(x)ZZ^{-1}|0\rangle = -\langle 0|\varphi(x)|0\rangle .$$

the $Z_2$ symmetry implies that there is no need for a counterterm!
Nonabelian symmetries

Let’s generalize the theory of two real scalar fields:

\[
\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2
\]

to the case of \(N\) real scalar fields:

\[
\mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i - \frac{1}{2} m^2 \varphi_i \varphi_i - \frac{1}{16} \lambda (\varphi_i \varphi_i)^2
\]

the lagrangian is clearly invariant under the \(SO(N)\) transformation:

\[
\varphi_i(x) \rightarrow R_{ij} \varphi_j(x)
\]

orthogonal matrix with \(\det = 1\)

\[
R^T = R^{-1}
\]

\[
\det R = +1
\]

lagrangian has also the \(Z_2\) symmetry, \(\varphi_i(x) \rightarrow -\varphi_i(x)\), that enlarges \(SO(N)\) to \(O(N)\)
infinitesimal $\text{SO}(N)$ transformation:

$$R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2)$$

antisymmetric

$$R^T = R^{-1}$$

$$R_{ij}^T = \delta_{ij} + \theta_{ji}$$

$$R_{ij}^{-1} = \delta_{ij} - \theta_{ij}$$

real

$$\text{Im}(R^{-1}R)_{ij} = \text{Im}\sum_k R_{ki}R_{kj} = 0$$

$$(N^2 \text{ linear combinations of } \text{Im parts } = 0)$$

there are $\frac{1}{2}N(N-1)$ linearly independent real antisymmetric matrices, and we can write:

$$\theta_{jk} = -i\theta^a(T^a)_{jk}$$

or

$$R = e^{-i\theta^a T^a}.$$
e.g. $SO(3)$:

\[(T^a)_{ij} = -i\varepsilon^{aij}\]

\[[T^a, T^b] = i\varepsilon^{abc}T^c\]

$\varepsilon^{123} = +1$

Levi-Civita symbol
consider now a theory of $N$ complex scalar fields:

$$\mathcal{L} = -\partial^\mu \varphi_i^\dagger \partial_\mu \varphi_i - m^2 \varphi_i^\dagger \varphi_i - \frac{1}{4} \lambda (\varphi_i^\dagger \varphi_i)^2$$

the lagrangian is clearly invariant under the $U(N)$ transformation:

$$\varphi_i(x) \to U_{ij} \varphi_j(x)$$

$$U^\dagger = U^{-1}$$

we can always write $U_{ij} = e^{-i\theta} \tilde{U}_{ij}$ so that $\det \tilde{U} = +1$.

actually, the lagrangian has larger symmetry, $SO(2N)$:

$$\varphi_j = (\varphi_{j1} + i\varphi_{j2})/\sqrt{2}$$

$$\varphi_j^\dagger \varphi_j = \frac{1}{2} (\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2)$$

$U(N) = U(1) \times SU(N)$
infinitesimal $\text{SU}(N)$ transformation:

$$\tilde{U}_{ij} = \delta_{ij} - i\theta^a(T^a)_{ij} + O(\theta^2)$$

or $\tilde{U} = e^{-i\theta^a T^a}$.

there are $N^2 - 1$ linearly independent traceless hermitian matrices:

$\text{SU}(2)$ - 3 Pauli matrices

$\text{SU}(3)$ - 8 Gell-Mann matrices

the structure coefficients are $f^{abc} = 2\epsilon^{abc}$, the same as for $\text{SO}(3)$
Plan for the rest of the semester

\[ \varphi \rightarrow e^{-i\alpha} \varphi(x) \]

\[ \varphi(x) \rightarrow e^{-i\alpha(x)} \varphi(x) \]
We defined a unitary operator that implemented a Lorentz transformation on a scalar field:

\[ U(\Lambda)^{-1} \varphi(x) U(\Lambda) = \varphi(\Lambda^{-1} x) \]

and then a derivative transformed as:

\[ U(\Lambda)^{-1} \partial^\mu \varphi(x) U(\Lambda) = \Lambda^\mu_\rho \bar{\partial}^\rho \varphi(\Lambda^{-1} x) \]

it suggests, we could define a vector field that would transform as:

\[ U(\Lambda)^{-1} A^\mu(x) U(\Lambda) = \Lambda^\mu_\rho A^\rho(\Lambda^{-1} x) \]

and a tensor field \( B^{\mu\nu}(x) \) that would transform as:

\[ U(\Lambda)^{-1} B^{\mu\nu}(x) U(\Lambda) = \Lambda^\mu_\rho \Lambda^\nu_\sigma B^{\rho\sigma}(\Lambda^{-1} x) \]
for symmetric $B^{\mu\nu}(x) = B^{\nu\mu}(x)$ and antisymmetric $B^{\mu\nu}(x) = -B^{\nu\mu}(x)$ tensors, the symmetry is preserved by Lorentz transformations.

In addition, the trace $T(x) \equiv g_{\mu\nu}B^{\mu\nu}(x)$ transforms as a scalar:

$$U(\Lambda)^{-1}T(x)U(\Lambda) = T(\Lambda^{-1}x)$$

Thus a general tensor field can be written as:

$$B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4}g^{\mu\nu}T(x)$$

where different parts do not mix with each other under LT!
How do we find the smallest (irreducible) representations of the Lorentz group for a field with $n$ vector indices?

Let’s start with a field carrying a generic Lorentz index:

$$U(\Lambda)^{-1} \varphi_A(x) U(\Lambda) = L_A^B(\Lambda) \varphi_B(\Lambda^{-1} x)$$

matrices that depend on $\Lambda$, they must obey the group composition rule

$$L_A^B(\Lambda') L_B^C(\Lambda) = L_A^C(\Lambda' \Lambda)$$

we say these matrices form a representation of the Lorentz group.
For an infinitesimal transformation we had:

\[ U(1+\delta\omega) = I + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \]

where the generators of the Lorentz group satisfied:

\[ [M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\nu\sigma} - (\mu\leftrightarrow\nu)\right) - (\rho\leftrightarrow\sigma) \]

\text{Lie algebra of the Lorentz group}

or in components (angular momentum and boost),

\[ J_i \equiv \frac{1}{2}\varepsilon_{ijk}M^{jk} \]
\[ K_i \equiv M^{i0} \]

we have found:

\[ [J_i, J_j] = i\hbar\varepsilon_{ijk}J_k \]
\[ [J_i, K_j] = i\hbar\varepsilon_{ijk}K_k \]
\[ [K_i, K_j] = -i\hbar\varepsilon_{ijk}J_k \]
In a similar way, for an infinitesimal transformation we also define:

\[ \Lambda_{\mu \nu} = \delta_{\mu \nu} + \delta \omega_{\mu \nu} \]
\[ U(1 + \delta \omega) = I + \frac{i}{2} \delta \omega_{\mu \nu} M^{\mu \nu} \]

and we find:

\[ L_A^B (1 + \delta \omega) = \delta_A^B + \frac{i}{2} \delta \omega_{\mu \nu} (S^{\mu \nu})_A^B \]

\[ U(\Lambda)^{-1} \varphi_A(x) U(\Lambda) = L_A^B (\Lambda) \varphi_B (\Lambda^{-1} x) \]

and we find:

\[ [\varphi_A(x), M^{\mu \nu}] = \mathcal{L}^{\mu \nu} \varphi_A(x) + (S^{\mu \nu})_A^B \varphi_B(x) \]

\[ \mathcal{L}^{\mu \nu} \equiv \frac{1}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu) \]

also it is possible to show that \( \mathcal{L}^{\mu \nu} \) and \((S^{\mu \nu})_A^B\) obey the same commutation relations as the generators

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i \left( g^{\mu \rho} M^{\nu \sigma} - (\mu \leftrightarrow \nu) \right) - (\rho \leftrightarrow \sigma) \]
How do we find all possible sets of matrices that satisfy
\[ [M^{\mu\nu}, M^{\rho\sigma}] = i \left( g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu) \right) - (\rho \leftrightarrow \sigma) \]

\[ [J_i, J_j] = i \hbar \varepsilon_{ijk} J_k , \]
\[ [J_i, K_j] = i \hbar \varepsilon_{ijk} K_k , \]
\[ [K_i, K_j] = -i \hbar \varepsilon_{ijk} J_k \]

the first one is just the usual set of commutation relations for angular momentum in QM:

for given \( j \) (0, 1/2, 1,...) we can find three \((2j+1)\times(2j+1)\) hermitian matrices \( J_1, J_2 \) and \( J_3 \) that satisfy the commutation relations and the eigenvalues of \( J_3 \) are \(-j, -j+1, ..., +j\).

such matrices constitute all of the inequivalent, irreducible representations of the Lie algebra of \( SO(3) \)

not related by a unitary transformation

equivalent to the Lie algebra of \( SU(2) \)

cannot be made block diagonal by a unitary transformation
Crucial observation:

\[
\begin{align*}
[J_i, J_j] &= i\hbar \varepsilon_{ijk} J_k , \\
[J_i, K_j] &= i\hbar \varepsilon_{ijk} K_k , \\
[K_i, K_j] &= -i\hbar \varepsilon_{ijk} J_k , \\
N_i &= \frac{1}{2}(J_i - iK_i) , \\
N_i^\dagger &= \frac{1}{2}(J_i + iK_i) , \\
[N_i, N_j] &= i\varepsilon_{ijk} N_k , \\
[N_i^\dagger, N_j^\dagger] &= i\varepsilon_{ijk} N_k^\dagger , \\
[N_i, N_j^\dagger] &= 0 .
\end{align*}
\]

The Lie algebra of the Lorentz group splits into two different SU(2) Lie algebras that are related by hermitian conjugation!

A representation of the Lie algebra of the Lorentz group can be specified by two integers or half-integers:

\[(2n+1, 2n'+1)\]

there are \((2n+1)(2n'+1)\) different components of a representation, they can be labeled by their angular momentum representations: since \(J_i = N_i + N_i^\dagger\), for given \(n\) and \(n'\) the allowed values of \(j\) are

\[|n-n'|, |n-n'|+1, \ldots, n+n'\]

(the standard way to add angular momenta, each value appears exactly once)
The simplest representations of the Lie algebra of the Lorentz group are:

\[(2n+1, 2n'+1)\]

\[
(1, 1) = \text{scalar or singlet}
\]

\[
(2, 1) = \text{left-handed spinor}
\]

\[
(1, 2) = \text{right-handed spinor}
\]

\[
(2, 2) = \text{vector}
\]

\[j = 0 \text{ and } 1\]
Left- and Right-handed spinor fields

Let's start with a left-handed spinor field (left-handed Weyl field) $\psi_a(x)$:

under Lorentz transformation we have:

$$U(\Lambda)^{-1}\psi_a(x)U(\Lambda) = L_a^b(\Lambda)\psi_b(\Lambda^{-1}x)$$

matrices in the $(2,1)$ representation, that satisfy the group composition rule:

$$L_a^b(\Lambda')L_b^c(\Lambda) = L_a^c(\Lambda'\Lambda)$$

For an infinitesimal transformation we have:

$$L_a^b(1+\delta\omega) = \delta_a^b + \frac{i}{2}\delta\omega_{\mu\nu}(S^\mu_{\nu\rho})_a^b$$

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \delta\omega^\mu_\nu$$

$$S^\mu_{\nu\rho} = -(S^\rho_{\mu\nu})_a^b$$

$$[S^\mu_{\nu}, S^\rho_{\sigma}] = i\left(g^{\mu\rho}S^\nu_{\sigma\rho} - (\mu\leftrightarrow\nu)\right) - (\rho\leftrightarrow\sigma)$$
Using

\[ U(1 + \delta \omega) = I + \frac{i}{2} \delta \omega_{\mu\nu} M^{\mu\nu} \]

we get

\[ U(\Lambda)^{-1} \psi_a(x) U(\Lambda) = L_a^b(\Lambda) \psi_b(\Lambda^{-1} x) \]

\[
[\psi_a(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu} \psi_a(x) + (S^{\mu\nu}_L)_a^b \psi_b(x)
\]

\[ \mathcal{L}^{\mu\nu} = \frac{1}{i} (x^\mu \partial^\nu - x^\nu \partial^\mu) \]

present also for a scalar field to simplify the formulas, we can evaluate everything at space-time origin, \( x^\mu = 0 \).

and since \( M^{ij} = \varepsilon^{ijk} J_k \), we have:

\[ \varepsilon^{ijk} [\psi_a(0), J_k] = (S^{ij}_L)_a^b \psi_b(0) \]

so that for \( i=1 \) and \( j=2 \):

\[
(S^{12}_L)_a^b = \frac{1}{2} \varepsilon^{12k} \sigma_k = \frac{1}{2} \sigma_3
\]

\[
(S^{12}_L)_1^1 = \frac{1}{2}, (S^{12}_L)_2^2 = -\frac{1}{2}
\]

\[
(S^{12}_L)_1^2 = (S^{12}_L)_2^1 = 0
\]

standard convention

\[ (S^{ij}_L)_a^b = \frac{1}{2} \varepsilon^{ijk} \sigma_k \]

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
Once we set the representation matrices for the angular momentum operator, those for boosts $K_k = M^{k0}$ follow from:

\[ J_k = N_k + N_k^\dagger \]
\[ K_k = i(N_k - N_k^\dagger) \]

$N_k^\dagger$ do not contribute when acting on a field in (2,1) representation and so the representation matrices for $K_k$ are $i$ times those for $J_k$:

\[ (S_{k0}^{(k)})_{\alpha}{}^{\beta} = \frac{1}{2}i\sigma_k \]
\[ (S_{ij}^{(k)})_{\alpha}{}^{\beta} = \frac{1}{2}\varepsilon^{ijk}\sigma_k \]
Let's consider now a hermitian conjugate of a left-handed spinor field $\psi_a(x)$ (a hermitian conjugate of a $(2,1)$ field should be a field in the $(1,2)$ representation) = right-handed spinor field (right-handed Weyl field) $\psi\dagger_a(x)$.

$$[\psi_a(x)]^\dagger = \psi^\dagger_{\dot{a}}(x)$$

we use dotted indices to distinguish $(2,1)$ from $(1,2)$!

under Lorentz transformation we have:

$$U(\Lambda)^{-1}\psi_{\dot{a}}^\dagger(x)U(\Lambda) = R_{\dot{a}}^\dot{b}(\Lambda)\psi_{\dot{b}}^\dagger(\Lambda^{-1}x)$$

matrices in the $(1,2)$ representation, that satisfy the group composition rule:

$$R_{\dot{a}}^\dot{b}(\Lambda')R_{\dot{b}}^\dot{c}(\Lambda) = R_{\dot{a}}^\dot{c}(\Lambda'\Lambda)$$

For an infinitesimal transformation we have:

$$R_{\dot{a}}^\dot{b}(1+\delta \omega) = \delta_{\dot{a}}^\dot{b} + \frac{i}{2}\delta \omega_{\mu\nu}(S_{\mu\nu}^R)_{\dot{a}}^\dot{b}$$

$$(S_{\mu\nu}^R)_{\dot{a}}^\dot{b} = -(S_{\mu\nu}^R)_{\dot{a}}^\dot{b}$$
in the same way as for the left-handed field we find:

\[ [\psi_a(x), M^\mu\nu] = \mathcal{L}^{\mu\nu} \psi_a(x) + (S_L^{\mu\nu})_{a}^{\ b} \psi_b(x) \]

in the same way as for the left-handed field we find:

\[ [\psi_\alpha^\dagger(0), M^\mu\nu] = (S_R^{\mu\nu})_{\alpha}^{\ b} \psi_\beta^\dagger(0) \]

taking the hermitian conjugate,

\[ [M^\mu\nu, \psi_a(0)] = [(S_R^{\mu\nu})_{\alpha}^{\ b}]^* \psi_b(0) \]

we find:

\[ (S_R^{\mu\nu})_{\alpha}^{\ b} = -[(S_L^{\mu\nu})_{a}^{\ b}]^* \]
Let's consider now a field that carries two \((2,1)\) indices. Under Lorentz transformation we have:

\[
U(\Lambda)^{-1}C_{ab}(x)U(\Lambda) = L_a^c(\Lambda)L_b^d(\Lambda)C_{cd}(\Lambda^{-1}x)
\]

Can we group 4 components of \(C\) into smaller sets that do not mix under Lorentz transformation?

Recall from QM that two spin 1/2 particles can be in a state of total spin 0 or 1:

\[2 \otimes 2 = 1_A \oplus 3_S\]

1 antisymmetric spin 0 state
3 symmetric spin 1 states

Thus for the Lorentz group we have:

\[(2,1) \otimes (2,1) = (1,1)_A \oplus (3,1)_S\]

and we should be able to write:

\[C_{ab}(x) = \epsilon_{ab}D(x) + G_{ab}(x)\]

\[\epsilon_{ab} = -\epsilon_{ba}\]

\[G_{ab}(x) = G_{ba}(x)\]
\[ C_{ab}(x) = \epsilon_{ab} D(x) + G_{ab}(x) \]

D is a scalar

\[ \epsilon_{ab} = -\epsilon_{ba} \]
\[ \epsilon_{21} = -\epsilon_{12} = +1 \]

\[ U(\Lambda)^{-1} C_{ab}(x) U(\Lambda) = L_a^c(\Lambda) L_b^d(\Lambda) C_{cd}(\Lambda^{-1} x) \]

\[ L_a^c(\Lambda) L_b^d(\Lambda) \epsilon_{cd} = \epsilon_{ab} \]

similar to

\[ \Lambda^\rho_\mu \Lambda^\sigma_\nu g_{\rho\sigma} = g_{\mu\nu} \]

is an invariant symbol of the Lorentz group

(does not change under a Lorentz transformation that acts on all of its indices)

We can use it, and its inverse to raise and lower left-handed spinor indices:

\[ \epsilon^{12} = \epsilon_{21} = +1, \quad \epsilon^{21} = \epsilon_{12} = -1 \]

\[ \epsilon_{ab} \epsilon^{bc} = \delta^c_a, \quad \epsilon^{ab} \epsilon_{bc} = \delta^a_c \]

to raise and lower left-handed spinor indices:

\[ \psi^a(x) \equiv \epsilon^{ab} \psi_b(x) \]
\[ \varepsilon_{ab} \varepsilon^{bc} = \delta_a^c, \quad \varepsilon^{ab} \varepsilon_{bc} = \delta^a_c \]

\[ \psi^a(x) \equiv \varepsilon^{ab} \psi_b(x) \]

We also have:

\[ \psi_a = \varepsilon_{ab} \psi^b = \varepsilon_{ab} \varepsilon^{bc} \psi_c = \delta_a^c \psi_c \]

we have to be careful with the minus sign, e.g.:

\[ \psi^a = \varepsilon^{ab} \psi_b = -\varepsilon^{ba} \psi_b = -\psi_b \varepsilon^{ba} = \psi_b \varepsilon^{ab} \]

or when contracting indices:

\[ \psi^a \chi_a = \varepsilon^{ab} \psi_b \chi_a = -\varepsilon^{ba} \psi_b \chi_a = -\psi_b \chi^b \]

Exactly the same discussion applies to two (1,2) indices:

\[ (1, 2) \otimes (1, 2) = (1, 1)_A \oplus (1, 3)_S \]

with \( \varepsilon_{\dot{a}\dot{b}} \) defined in the same way as \( \varepsilon_{ab} : \quad \varepsilon_{\dot{a}\dot{b}} = -\varepsilon_{\dot{b}\dot{a}} \), .....
Finally, let’s consider a field that carries one undotted and one dotted index; it is in the (2,2) representation (vector):

\[ A_{a\dot{a}}(x) = \sigma^\mu_{a\dot{a}} A_\mu(x) \]

It is an invariant symbol, so we can deduce its existence from

\[(2,1) \otimes (1,2) \otimes (2,2) = (1,1) \oplus \ldots .\]

A consistent choice with what we have already set for \( S^\mu_{\nu} \) and \( S^{\mu\nu}_R \) is:

\[ \sigma^\mu_{a\dot{a}} = (I, \vec{\sigma}) \]

homework
In general, whenever the product of a set of representations includes the singlet, there is a corresponding invariant symbol, e.g. the existence of $g_{\mu\nu} = g_{\nu\mu}$ follows from

$$(2, 2) \otimes (2, 2) = (1, 1)_S \oplus (1, 3)_A \oplus (3, 1)_A \oplus (3, 3)_S$$

another invariant symbol we will use is completely antisymmetric Levi-Civita symbol:

$$(2, 2) \otimes (2, 2) \otimes (2, 2) \otimes (2, 2) = (1, 1)_A \oplus \ldots$$

$$ \varepsilon^{\mu\nu\rho\sigma} \quad \varepsilon^{0123} = +1$$

$$\Lambda^\mu_{\alpha} \Lambda^\nu_{\beta} \Lambda^\rho_{\gamma} \Lambda^\sigma_{\delta} \varepsilon^{\alpha\beta\gamma\delta}$$ is antisymmetric on exchange of any two of its uncontracted indices, and therefore must be proportional to $\varepsilon^{\mu\nu\rho\sigma}$, the constant of proportionality is $\det \Lambda$ which is $+1$ for proper Lorentz transformations.
Comparing the formula for a general field with two vector indices

\[ B^{\mu\nu}(x) = A^{\mu\nu}(x) + S^{\mu\nu}(x) + \frac{1}{4} g^{\mu\nu} T(x) \]

with

\[ (2, 2) \otimes (2, 2) = (1, 1)_S \oplus (1, 3)_A \oplus (3, 1)_A \oplus (3, 3)_S \]

we see that \( A \) is not irreducible and, since \((3, 1)\) corresponds to a symmetric part of undotted indices,

\[ 2 \otimes 2 = 1_A \oplus 3_S \]

\[ C_{ab}(x) = \epsilon_{ab} D(x) + G_{ab}(x) \]

we should be able to write it in terms of \( G \) and its hermitian conjugate.

see Srednicki