The Standard Model: Lepton sector

based on S-88

as far as we know there are three generations of leptons:

name	symbol	mass (MeV)	Q
electron	e	0.511	-1
electron neutrino	ν_e	0	0
muon	μ	105.7	-1
muon neutrino	ν_μ	0	0
tau	τ	1777	-1
tau neutrino	ν_τ	0	0

and they are described by introducing left-handed Weyl fields:

	$\mathrm{SU}(2) \times \mathrm{U}(1)_Y$
ℓ	$(2, -\frac{1}{2})$
$\bar{e}$	$(1, +1)$

just a name →

the covariant derivatives are:

$\ell \ (2, -\frac{1}{2})$

$$(D_\mu \ell)_i = \partial_\mu \ell_i - i g_2 A^a_\mu (T^a)_i{}^j \ell_j - i g_1 (-\tfrac{1}{2}) B_\mu \ell_i$$

$\bar{e} \ (1, +1)$

$$D_\mu \bar{e} = \partial_\mu \bar{e} - i g_1 (+1) B_\mu \bar{e}$$

and the kinetic term is the usual one for Weyl fields:

$$\mathcal{L}_{\text{kin}} = i \ell^{\dagger i} \bar{\sigma}^\mu (D_\mu \ell)_i + i \bar{e}^\dagger \bar{\sigma}^\mu D_\mu \bar{e}$$

there is no gauge group singlet contained in any of the products:

$$(2, -\tfrac{1}{2}) \otimes (2, -\tfrac{1}{2})$$
$$(2, -\tfrac{1}{2}) \otimes (1, +1)$$
$$(1, +1) \otimes (1, +1)$$

it is not possible to write any mass term!

Lagrangians for spinor fields

based on S-36

we want to find a suitable lagrangian for left- and right-handed spinor fields.

it should be:

- Lorentz invariant and hermitian
- quadratic in ψ_a and $\psi^\dagger_{\dot{a}}$

equations of motion will be linear with plane wave solutions
(suitable for describing free particles)

terms with no derivative:

$$\psi\psi = \psi^a\psi_a = \varepsilon^{ab}\psi_b\psi_a$$

+ h.c.

terms with derivatives:

$$\cancel{\partial^\mu\psi\partial_\mu\psi}$$

would lead to a hamiltonian unbounded from below to get a bounded hamiltonian the kinetic term has to contain both ψ_a and $\psi^\dagger_{\dot a}$, a candidate is:

$$i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi$$

is hermitian up to a total divergence

$$\begin{aligned}
(i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi)^\dagger &= (i\psi^\dagger_{\dot a}\,\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c)^\dagger \\
&= -i\partial_\mu\psi^\dagger_{\dot c}\,(\bar\sigma^{\mu a\dot c})^*\psi_a \\
&= -i\partial_\mu\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a \\
&= i\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\partial_\mu\psi_a - i\partial_\mu(\psi^\dagger_{\dot c}\,\bar\sigma^{\mu\dot c a}\psi_a). \\
&= i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - i\partial_\mu(\psi^\dagger\bar\sigma^\mu\psi)\ .
\end{aligned}$$

$\bar\sigma^{\mu\dot a a} = (I, -\vec\sigma)$

are hermitian

does not contribute to the action

Our complete lagrangian is:

$$\mathcal{L} = i\psi^\dagger\bar\sigma^\mu\partial_\mu\psi - \tfrac{1}{2}m\psi\psi - \tfrac{1}{2}m^*\psi^\dagger\psi^\dagger$$

the phase of m can be absorbed into the definition of fields

$$m = |m|e^{i\alpha} \qquad \psi = e^{-i\alpha/2}\,\tilde\psi$$

and so without loss of generality we can take m to be real and positive.

Equation of motion:

$$0 = -\frac{\delta S}{\delta\psi^\dagger} = -i\bar\sigma^\mu\partial_\mu\psi + m\psi^\dagger$$

$$0 = -i\bar\sigma^{\mu\dot a c}\partial_\mu\psi_c + m\psi^{\dagger\dot a}$$

Taking hermitian conjugate:

$\bar\sigma^{\mu\dot a a} = (I, -\vec\sigma)$

are hermitian

$\bar\sigma^{\mu\dot a a} \equiv \varepsilon^{ab}\varepsilon^{\dot a\dot b}\sigma^\mu_{b\dot b}$

$$\begin{aligned} 0 &= +i(\bar\sigma^{\mu a\dot c})^*\,\partial_\mu\psi^\dagger_{\dot c} + m\psi^a \\ &= +i\bar\sigma^{\mu\dot c a}\partial_\mu\psi^\dagger_{\dot c} + m\psi^a \\ &= -i\sigma^\mu_{a\dot c}\,\partial_\mu\psi^{\dagger\dot c} + m\psi_a\ . \end{aligned}$$

however we can write a Yukawa term of the form:

$$\mathcal{L}_{\text{Yuk}} = -y\varepsilon^{ij}\varphi_i \ell_j \bar{e} + \text{h.c.}$$

$$(2, -\tfrac{1}{2}) \otimes (2, -\tfrac{1}{2}) \otimes (1, +1) = \underline{(1, 0)} \oplus (3, 0)$$

there are no other terms that have mass dimension four or less!

in the unitary gauge we have:

$$\varphi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}$$

and the Yukawa term becomes:

$$\mathcal{L}_{\text{Yuk}} = -\frac{1}{\sqrt{2}} y(v + H)(\ell_2 \bar{e} + \text{h.c.})$$

$$\mathcal{L}_{\rm Yuk} = -\frac{1}{\sqrt{2}} y(v+H)(\ell_2 \bar{e} + {\rm h.c.})$$

it is convenient to label the components of the lepton doublet as:

$$\ell = \begin{pmatrix} \nu \\ e \end{pmatrix}$$

then we have:

$$\begin{aligned} \mathcal{L}_{\rm Yuk} &= -\frac{1}{\sqrt{2}} y(v+H)(e\bar{e} + \bar{e}^\dagger e^\dagger) \\ &= -\frac{1}{\sqrt{2}} y(v+H)\overline{\mathcal{E}}\mathcal{E} \end{aligned}$$

we define a Dirac field for the electron $\mathcal{E} \equiv \begin{pmatrix} e \\ \bar{e}^\dagger \end{pmatrix}$

and we see that the electron has acquired a mass:

$$m_e = \frac{yv}{\sqrt{2}}$$

and the neutrino remained massless!

consider a theory of two left-handed spinor fields:

$$\mathcal{L} = i\psi_i^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i - \tfrac{1}{2} m \psi_i \psi_i - \tfrac{1}{2} m \psi_i^\dagger \psi_i^\dagger$$

i = 1,2

the lagrangian is invariant under the SO(2) transformation:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

it can be written in the form that is manifestly U(1) symmetric:

$$\chi = \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$$

$$\xi = \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$$

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi - m\chi\xi - m\xi^\dagger \chi^\dagger$$

$$\chi \to e^{-i\alpha}\chi$$

$$\xi \to e^{+i\alpha}\xi$$

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

Equations of motion for this theory:

$$\begin{pmatrix} m\delta_a{}^c & -i\sigma^\mu_{a\dot{c}}\,\partial_\mu \\ -i\bar{\sigma}^{\mu\dot{a}c}\,\partial_\mu & m\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix} \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix} = 0$$

we can define a four-component Dirac field:

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$(-i\gamma^\mu\partial_\mu + m)\Psi = 0$$

Dirac equation

we want to write the lagrangian in terms of the Dirac field:

$$\Psi^\dagger = (\chi^\dagger_{\dot{a}}\,,\ \xi^a)$$

$$\beta \equiv \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$$

Let’s define:

$$\overline{\Psi} \equiv \Psi^\dagger\beta = (\xi^a,\ \chi^\dagger_{\dot{a}})$$

numerically

$$\beta = \gamma^0$$

but different spinor index structure

Then we find:

$$\overline{\Psi} \equiv \Psi^\dagger \beta = (\xi^a,\, \chi^\dagger_{\dot a})$$

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger \dot c} \end{pmatrix}$$

$$\overline{\Psi}\Psi = \xi^a \chi_a + \chi^\dagger_{\dot a} \xi^{\dagger \dot a}$$

$$\overline{\Psi}\gamma^\mu \partial_\mu \Psi = \xi^a \sigma^\mu_{a\dot c}\, \partial_\mu \xi^{\dagger \dot c} + \chi^\dagger_{\dot a}\, \bar\sigma^{\mu \dot a c}\, \partial_\mu \chi_c$$

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot c} \\ \bar\sigma^{\mu \dot a c} & 0 \end{pmatrix}$$

$$A\partial B = -(\partial A)B + \partial(AB)$$

$$\xi^a \sigma^\mu_{a\dot c}\, \partial_\mu \xi^{\dagger \dot c} = -(\partial_\mu \xi^a)\sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c} + \partial_\mu(\xi^a \sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c})$$

$$-(\partial_\mu \xi^a)\sigma^\mu_{a\dot c}\, \xi^{\dagger \dot c} = +\xi^{\dagger \dot c} \sigma^\mu_{a\dot c}\, \partial_\mu \xi^a = +\xi^\dagger_{\dot c}\, \bar\sigma^{\mu \dot c a} \partial_\mu \xi_a$$

$$\bar\sigma^{\mu \dot a a} \equiv \varepsilon^{ab} \varepsilon^{\dot a \dot b} \sigma^\mu_{b \dot b}$$

Thus we have:

$$\overline{\Psi}\gamma^\mu \partial_\mu \Psi = \chi^\dagger \bar\sigma^\mu \partial_\mu \chi + \xi^\dagger \bar\sigma^\mu \partial_\mu \xi + \partial_\mu(\xi \sigma^\mu \xi^\dagger)$$

Thus the lagrangian can be written as:

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi + i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi - m\chi\xi - m\xi^\dagger\chi^\dagger$$

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\overline{\Psi}\Psi$$

The U(1) symmetry is obvious:

$$\Psi \rightarrow e^{-i\alpha}\,\Psi$$

$$\overline{\Psi} \rightarrow e^{+i\alpha}\,\overline{\Psi}$$

The Nether current associated with this symmetry is:

$$j^\mu(x) \equiv \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\varphi_a(x))}\,\delta\varphi_a(x)$$

$$j^\mu = \overline{\Psi}\gamma^\mu\Psi = \chi^\dagger\bar{\sigma}^\mu\chi - \xi^\dagger\bar{\sigma}^\mu\xi$$

later we will see that this is the electromagnetic current

If we want to go back from 4-component Dirac or Majorana fields to the two-component Weyl fields, it is useful to define a projection matrix:

$$\gamma_5 \equiv \begin{pmatrix} -\delta_a{}^c & 0 \\ 0 & +\delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}$$

just a name

We can define left and right projection matrices:

$$P_{\rm L} \equiv \tfrac{1}{2}(1-\gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_{\rm R} \equiv \tfrac{1}{2}(1+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot{a}}{}_{\dot{c}} \end{pmatrix}$$

And for a Dirac field we find:

$$P_{\rm L}\Psi = \begin{pmatrix} \chi_c \\ 0 \end{pmatrix}$$

$$\Psi \equiv \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

$$P_{\rm R}\Psi = \begin{pmatrix} 0 \\ \xi^{\dagger\dot{c}} \end{pmatrix}$$

We can describe the neutrino with a Majorana field:

$$\mathcal{N} \equiv \begin{pmatrix} \nu \\ \nu^\dagger \end{pmatrix}$$

and it is also convenient to define:

a "Dirac field" for the neutrino

$$\mathcal{N}_\mathrm{L} \equiv P_\mathrm{L} \mathcal{N} = \begin{pmatrix} \nu \\ 0 \end{pmatrix}$$

$$P_\mathrm{L} = \tfrac{1}{2}(1-\gamma_5)$$

because then the kinetic term:

$$i\nu^\dagger \bar{\sigma}^\mu \partial_\mu \nu$$

can be written as:

$$i\overline{\mathcal{N}_\mathrm{L}} \partial\!\!\!/ \mathcal{N}_\mathrm{L}$$

Now we have to figure out lepton-lepton-gauge boson interaction terms:

(we want to write covariant derivatives in terms of $W^{\pm}_{\mu}$, Z_{μ} and A_{μ})

$$(D_\mu \ell)_i = \partial_\mu \ell_i - i g_2 A^a_\mu (T^a)_i{}^j \ell_j - i g_1 (-\tfrac{1}{2}) B_\mu \ell_i$$

$$D_\mu \bar{e} = \partial_\mu \bar{e} - i g_1 (+1) B_\mu \bar{e}$$

$$W^{\pm}_{\mu} \equiv \tfrac{1}{\sqrt{2}}(A^1_\mu \mp i A^2_\mu)$$

$$Z_\mu \equiv c_{\rm W} A^3_\mu - s_{\rm W} B_\mu$$

$$A_\mu \equiv s_{\rm W} A^3_\mu + c_{\rm W} B_\mu$$

we have

$$g_2 A^1_\mu T^1 + g_2 A^2_\mu T^2 = \frac{g_2}{\sqrt{2}} \begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix}$$

and

$$\begin{aligned} g_2 A^3_\mu T^3 + g_1 B_\mu Y &= \tfrac{e}{s_{\rm W}}(s_{\rm W} A_\mu + c_{\rm W} Z_\mu) T^3 + \tfrac{e}{c_{\rm W}}(c_{\rm W} A_\mu - s_{\rm W} Z_\mu) Y \\ &= e(A_\mu + \cot\theta_{\rm W} Z_\mu) T^3 + e(A_\mu - \tan\theta_{\rm W} Z_\mu) Y \\ &= e\underline{(T^3 + Y)} A_\mu + e(\cot\theta_{\rm W} T^3 - \tan\theta_{\rm W} Y) Z_\mu \end{aligned}$$

↙ we identify it with electric charge: $Q = T^3 + Y$

$$\begin{aligned} &= eQA_\mu + e[(\cot\theta_{\rm W} + \tan\theta_{\rm W}) T^3 - \tan\theta_{\rm W} Q] Z_\mu \\ &= eQA_\mu + \tfrac{e}{s_{\rm W} c_{\rm W}} (T^3 - s^2_{\rm W} Q) Z_\mu \ . \end{aligned}$$

$$\ell\ (2,-\tfrac{1}{2})$$
$$\bar{e}\ (1,+1)$$
$$\ell=\begin{pmatrix}\nu\\ e\end{pmatrix}$$

since we have:

$$T^3\nu=+\tfrac{1}{2}\nu\,,\quad T^3 e=-\tfrac{1}{2}e\,,\quad T^3\bar{e}=0\,,$$
$$Y\nu=-\tfrac{1}{2}\nu\,,\quad Ye=-\tfrac{1}{2}e\,,\quad Y\bar{e}=+\bar{e}\,,$$

the electric charge assignments are as expected:

$$Q=T^3+Y$$

$$Q\nu=0\,,\qquad Qe=-e\,,\qquad Q\bar{e}=+\bar{e}$$

covariant derivatives in terms of the four-component fields:

$$g_2A^3_\mu T^3+g_1B_\mu Y=eQA_\mu+\frac{e}{s_{\rm W}c_{\rm W}}(T^3-s^2_{\rm W}Q)Z_\mu$$

$$(g_2A^3_\mu T^3+g_1B_\mu Y)\mathcal{E}\ =\Big[-eA_\mu+\frac{e}{s_{\rm W}c_{\rm W}}(-\tfrac{1}{2}P_{\rm L}+s^2_{\rm W})Z_\mu\Big]\mathcal{E}$$
$$(g_2A^3_\mu T^3+g_1B_\mu Y)\mathcal{N}_{\rm L}=\frac{e}{s_{\rm W}c_{\rm W}}(+\tfrac{1}{2})Z_\mu\mathcal{N}_{\rm L}\,.$$

$$\mathcal{E}\equiv\begin{pmatrix}e\\ \bar{e}^\dagger\end{pmatrix}$$

$$\mathcal{N}_{\rm L}\equiv P_{\rm L}\mathcal{N}=\begin{pmatrix}\nu\\ 0\end{pmatrix}$$

Putting pieces together,

$$\mathcal{E} \equiv \begin{pmatrix} e \\ \bar{e}^\dagger \end{pmatrix}$$

$$\mathcal{L}_{\rm kin} = i\ell^{\dagger i}\bar{\sigma}^\mu (D_\mu \ell)_i + i\bar{e}^\dagger \bar{\sigma}^\mu D_\mu \bar{e}$$

$$\mathcal{N}_{\rm L} \equiv P_{\rm L}\mathcal{N} = \begin{pmatrix} \nu \\ 0 \end{pmatrix}$$

$$(D_\mu \ell)_i = \partial_\mu \ell_i - ig_2 A^a_\mu (T^a)_i{}^j \ell_j - ig_1(-\tfrac{1}{2})B_\mu \ell_i$$

$$D_\mu \bar{e} = \partial_\mu \bar{e} - ig_1(+1)B_\mu \bar{e}$$

$$g_2 A^1_\mu T^1 + g_2 A^2_\mu T^2 = \frac{g_2}{\sqrt{2}} \begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix}$$

$$\ell = \begin{pmatrix} \nu \\ e \end{pmatrix}$$

$$\begin{aligned} (g_2 A^3_\mu T^3 + g_1 B_\mu Y)\mathcal{E} &= \Big[-eA_\mu + \tfrac{e}{s_{\rm W}c_{\rm W}}(-\tfrac{1}{2}P_{\rm L} + s^2_{\rm W})Z_\mu\Big]\mathcal{E} \\ (g_2 A^3_\mu T^3 + g_1 B_\mu Y)\mathcal{N}_{\rm L} &= \tfrac{e}{s_{\rm W}c_{\rm W}}(+\tfrac{1}{2})Z_\mu \mathcal{N}_{\rm L}\ . \end{aligned}$$

we find the interaction lagrangian:

$$\mathcal{L}_{\rm int} = \frac{1}{\sqrt{2}}g_2 W^+_\mu J^{-\mu} + \frac{1}{\sqrt{2}}g_2 W^-_\mu J^{+\mu} + \frac{e}{s_{\rm W}c_{\rm W}}Z_\mu J^\mu_{\rm Z} + eA_\mu J^\mu_{\rm EM}$$

where

$$\begin{aligned} J^{+\mu} &\equiv \overline{\mathcal{E}}_{\rm L}\gamma^\mu \mathcal{N}_{\rm L}\ , \\ J^{-\mu} &\equiv \overline{\mathcal{N}}_{\rm L}\gamma^\mu \mathcal{E}_{\rm L}\ , \\ J^\mu_{\rm Z} &\equiv J^\mu_3 - s^2_{\rm W} J^\mu_{\rm EM}\ , \\ J^\mu_3 &\equiv \tfrac{1}{2}\overline{\mathcal{N}}_{\rm L}\gamma^\mu \mathcal{N}_{\rm L} - \tfrac{1}{2}\overline{\mathcal{E}}_{\rm L}\gamma^\mu \mathcal{E}_{\rm L}\ , \\ J^\mu_{\rm EM} &\equiv -\overline{\mathcal{E}}\gamma^\mu \mathcal{E}\ . \end{aligned}$$

The connection with Fermi theory:

if we want to calculate some scattering amplitude (or a decay rate) of leptons with momenta well below masses of W and Z, the propagators effectively become:

$$g^{\mu\nu}/M^2_{\mathrm{W,Z}}$$

$$\mathcal{L}_{\mathrm{int}} = \tfrac{1}{\sqrt{2}} g_2 W^+_\mu J^{-\mu} + \tfrac{1}{\sqrt{2}} g_2 W^-_\mu J^{+\mu} + \frac{e}{s_{\mathrm{W}} c_{\mathrm{W}}} Z_\mu J^\mu_{\mathrm{Z}} + e A_\mu J^\mu_{\mathrm{EM}}$$

$$\mathcal{L}_{\mathrm{mass}} = -M^2_{\mathrm{W}} W^{+\mu} W^-_\mu - \tfrac{1}{2} M^2_{\mathrm{Z}} Z^\mu Z_\mu$$

and we get 4-fermion-interaction terms:

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} &= \frac{g_2^2}{2M^2_{\mathrm{W}}} J^{+\mu} J^-_\mu + \frac{e^2}{2 s^2_{\mathrm{W}} c^2_{\mathrm{W}} M^2_{\mathrm{Z}}} J^\mu_{\mathrm{Z}} J_{\mathrm{Z}\mu} \\ &= \frac{e^2}{2 s^2_{\mathrm{W}} M^2_{\mathrm{W}}} (J^{+\mu} J^-_\mu + J^\mu_{\mathrm{Z}} J_{\mathrm{Z}\mu}) \\ &= 2\sqrt{2}\, G_{\mathrm{F}} (J^{+\mu} J^-_\mu + J^\mu_{\mathrm{Z}} J_{\mathrm{Z}\mu}) \;.\end{aligned}$$

$$e = g_2 \sin\theta_{\mathrm{W}}$$

$$M_{\mathrm{W}} = M_{\mathrm{Z}} \cos\theta_{\mathrm{W}}$$

where the Fermi constant is:

$$G_{\mathrm{F}} \equiv \frac{e^2}{4\sqrt{2} \sin^2\theta_{\mathrm{W}} M^2_{\mathrm{W}}}$$

Generalization to three lepton generations:

$$\mathcal{L}_{\text{kin}} = i\ell^{\dagger i}_I \bar{\sigma}^\mu (D_\mu)_i{}^j \ell_{jI} + i\bar{e}^\dagger_I \bar{\sigma}^\mu D_\mu \bar{e}_I$$

$$I = 1, 2, 3$$

the most general Yukawa term:

$$\mathcal{L}_{\text{Yuk}} = -\varepsilon^{ij} \varphi_i \ell_{jI} y_{IJ} \bar{e}_J + \text{h.c.}$$

a complex 3x3 matrix;
it can be diagonalized (with positive or zero entries on the diagonal) by a bi-unitary transformation

$$\begin{aligned} \ell_I &\rightarrow L_{IJ}\ell_J \\ \bar{e}_I &\rightarrow \bar{E}_{IJ}\bar{e}_J \end{aligned}$$

currents remain diagonal,
just add the generation index

$$L^{\mathrm{T}} y \bar{E}$$

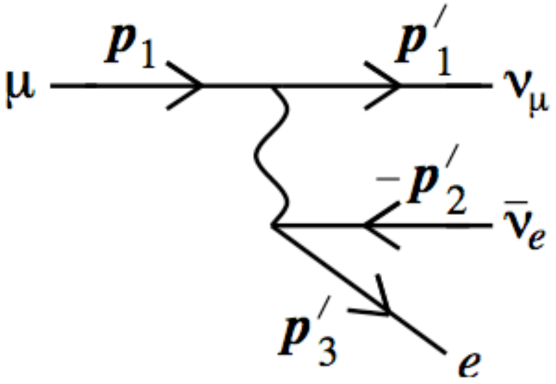
$$\begin{aligned} J^{+\mu} &\equiv \overline{\mathcal{E}}_{\mathrm{L}} \gamma^\mu \mathcal{N}_{\mathrm{L}} \ , \\ J^{-\mu} &\equiv \overline{\mathcal{N}}_{\mathrm{L}} \gamma^\mu \mathcal{E}_{\mathrm{L}} \ , \\ J^\mu_{\mathrm{Z}} &\equiv J^\mu_3 - s^2_{\mathrm{W}} J^\mu_{\mathrm{EM}} \ , \\ J^\mu_3 &\equiv \tfrac{1}{2}\overline{\mathcal{N}}_{\mathrm{L}} \gamma^\mu \mathcal{N}_{\mathrm{L}} - \tfrac{1}{2}\overline{\mathcal{E}}_{\mathrm{L}} \gamma^\mu \mathcal{E}_{\mathrm{L}} \ , \\ J^\mu_{\mathrm{EM}} &\equiv -\overline{\mathcal{E}} \gamma^\mu \mathcal{E} \ . \end{aligned}$$

fermion masses are then given by:

$$m_{e_I} = y_I v / \sqrt{2}$$

diagonal entries

We can calculate e.g. muon decay:



$$\mu^- \rightarrow e^- \overline{\nu}_e \nu_\mu$$

the relevant part of the charged current:

$$J^{+\mu} = \overline{\mathcal{E}}_\text{L} \gamma^\mu \mathcal{N}_{e\text{L}} + \overline{\mathcal{M}}_\text{L} \gamma^\mu \mathcal{N}_{m\text{L}}$$

$$J^{-\mu} = \overline{\mathcal{N}}_{e\text{L}} \gamma^\mu \mathcal{E}_\text{L} + \overline{\mathcal{N}}_{m\text{L}} \gamma^\mu \mathcal{M}_\text{L}$$

we get the effective lagrangian:

$$\mathcal{L}_\text{eff} = 2\sqrt{2}\, G_\text{F} (\overline{\mathcal{E}}_\text{L} \gamma^\mu \mathcal{N}_{e\text{L}})(\overline{\mathcal{N}}_{m\text{L}} \gamma_\mu \mathcal{M}_\text{L})$$

and the rest is straightforward...