Setting $g = 0$ and matching coefficients of e^{-ikx} we find:

$$i[Q_{\rm B}, A^a_\mu(x)] = D^{ab}_\mu c^b(x) \;,$$
$$i\{Q_{\rm B}, c^a(x)\} = -\tfrac{1}{2} g f^{abc} c^b(x) c^c(x) \;,$$
$$i\{Q_{\rm B}, \bar{c}^a(x)\} = B^a(x) \;,$$
$$i[Q_{\rm B}, B^a(x)] = 0 \;,$$
$$i[Q_{\rm B}, \phi_i(x)]_\pm = igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x) \;.$$

$$A^\mu(x) = \sum_{\substack{\lambda=>,<,\\+,-}} \int \widetilde{dk} \left[\varepsilon^{\mu*}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \varepsilon^\mu_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-ikx}\right]$$
$$c(x) = \int \widetilde{dk} \left[c(\mathbf{k}) e^{ikx} + c^\dagger(\mathbf{k}) e^{-ikx}\right] \;,$$
$$\bar{c}(x) = \int \widetilde{dk} \left[b(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}\right] \;,$$
$$\phi(x) = \int \widetilde{dk} \left[a_\phi(\mathbf{k}) e^{ikx} + a^\dagger_\phi(\mathbf{k}) e^{-ikx}\right] \;,$$

$$\varepsilon^\mu_>(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,1) \;,$$
$$\varepsilon^\mu_<(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,-1) \;,$$
$$\varepsilon^\mu_+(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,-i,0) \;,$$
$$\varepsilon^\mu_-(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,+i,0) \;.$$

$$[Q_{\rm B}, a^\dagger_\lambda(\mathbf{k})] = \sqrt{2}\omega\, \delta_{\lambda>}\, c^\dagger(\mathbf{k}) \;,$$
$$\{Q_{\rm B}, c^\dagger(\mathbf{k})\} = 0 \;,$$
$$\{Q_{\rm B}, b^\dagger(\mathbf{k})\} = \xi^{-1}\sqrt{2}\omega\, a^\dagger_<(\mathbf{k}) \;,$$
$$[Q_{\rm B}, a^\dagger_\phi(\mathbf{k})] = 0 \;.$$

we also use EM to eliminate B:

$$\frac{\partial(\delta_{\rm B}\mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A^a_\mu(x) = 0$$

Consider a normalized state in the cohomology: $\langle\psi|\psi\rangle = 1,\, Q_{\rm B}|\psi\rangle = 0,$

if we add a photon with polarization >, the state $a^\dagger_>(\mathbf{k})|\psi\rangle$ is not annihilated by Q and so it is not in the cohomology

$$[Q_{\rm B}, a^\dagger_\lambda(\mathbf{k})] = \sqrt{2}\omega\, \delta_{\lambda >}\, c^\dagger(\mathbf{k}) \;,$$

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the state $a^\dagger_<(\mathbf{k})|\psi\rangle$ is
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the state $a^\dagger_<(\mathbf{k})|\psi\rangle$ is proportional to $Q_{\rm B}b^\dagger(\mathbf{k})|\psi\rangle$ and so it is not in the cohomology

states: $a^\dagger_+(\mathbf{k})|\psi\rangle$, $a^\dagger_-(\mathbf{k})|\psi\rangle$ and $a^\dagger_\phi(\mathbf{k})|\psi\rangle$ are annihilated by Q but cannot be written as Q acting on some state and so they are in the cohomology!

the vacuum is also in the cohomology

Consider a normalized state in the cohomology: $\langle\psi|\psi\rangle = 1,\, Q_{\rm B}|\psi\rangle = 0,$

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$$[Q_{\rm B}, a^\dagger_\phi(\mathbf{k})] = 0 \ .$$

the state $b^\dagger(\mathbf{k})|\psi\rangle$ is not annihilated by Q and so it is not in the cohomology

the state $a^\dagger_<(\mathbf{k})|\psi\rangle$ is proportional to $Q_{\rm B} b^\dagger(\mathbf{k})|\psi\rangle$ and so it is not in the cohomology

states: $a^\dagger_+(\mathbf{k})|\psi\rangle$, $a^\dagger_-(\mathbf{k})|\psi\rangle$ and $a^\dagger_\phi(\mathbf{k})|\psi\rangle$ are annihilated by Q but cannot be written as Q acting on some state and so they are in the cohomology!

the vacuum is also in the cohomology

Thus we found:

we can build an initial state of widely separated particles that is in the cohomology only with matter particles and photons with polarizations + and -. No ghosts or >, < polarized photons can be produced in the scattering process (a state in the cohomology will evolve to another state in the cohomology).

Spontaneous breaking of continuous symmetries

based on S-32

Consider the theory of a complex scalar field:

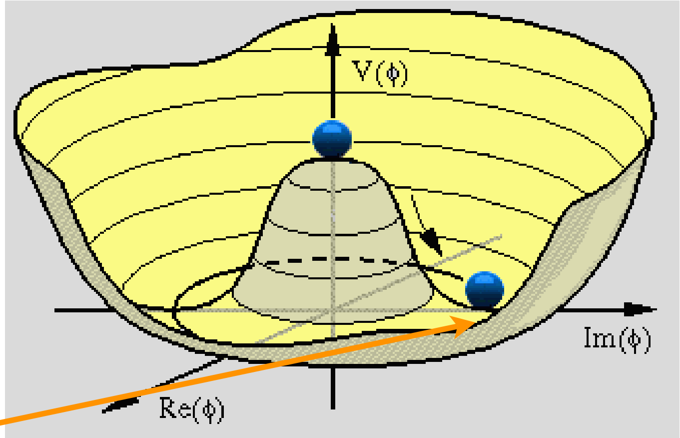
$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2$$

the lagrangian is invariant under the U(1) transformation:

$$\varphi(x) \to e^{-i\alpha}\varphi(x)$$

for m^2 negative, the minimum of the potential

$$V(\varphi) = m^2 \varphi^\dagger \varphi + \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2$$



is achieved for

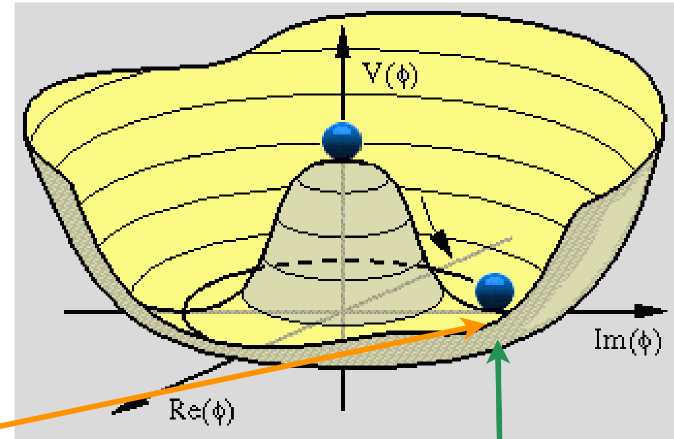
$$\varphi(x) = \frac{1}{\sqrt{2}} v e^{-i\theta}$$

$$v = (4|m^2|/\lambda)^{1/2}$$

we have a continuous family of minima (ground states):

$$\langle \theta | \varphi(x) | \theta \rangle = \frac{1}{\sqrt{2}} v e^{-i\theta}$$

$$V(\varphi) = m^2\varphi^\dagger\varphi + \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2$$



is achieved for

$$\varphi(x) = \frac{1}{\sqrt{2}} v e^{-i\theta}$$

$$v = (4|m^2|/\lambda)^{1/2}$$

we have a continuous family of minima (ground states):

$$\langle\theta|\varphi(x)|\theta\rangle = \frac{1}{\sqrt{2}} v e^{-i\theta}$$

all the minima are equivalent, there is a flat direction in the field space (we can move along without changing the energy)

for $\theta = 0$ we can write:

real scalar fields

$$\varphi(x) = \frac{1}{\sqrt{2}}[v + a(x) + ib(x)]$$

and we get:

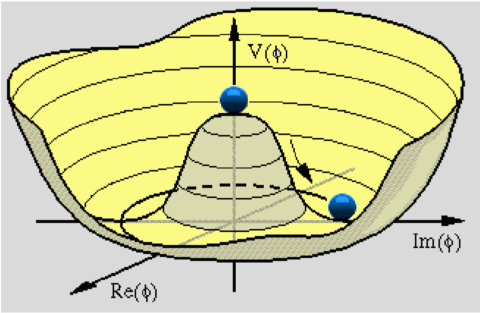
$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu a\partial_\mu a - \tfrac{1}{2}\partial^\mu b\partial_\mu b$$
$$- |m^2|a^2 - \tfrac{1}{2}\lambda^{1/2}|m|a(a^2+b^2) - \tfrac{1}{16}\lambda(a^2+b^2)^2$$

$\frac{1}{2}m_a^2 = |m^2|$

b is massless!

the physical consequence of a flat direction is the existence of a massless particle - Goldstone boson

It is also instructive to use a different parameterization:



real scalar fields

$$\varphi(x) = \frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v}$$

$$\varphi(x) \to e^{-i\alpha}\varphi(x)$$

the U(1) transformation takes a simple form:

$$\chi(x) \to \chi(x) + \alpha$$

χ parameterizes the flat direction

now we get:

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2$$

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \rho \partial_\mu \rho - \tfrac{1}{2}\left(1 + \frac{\rho}{v}\right)^2 \partial^\mu \chi \partial_\mu \chi$$

$$- |m^2|\rho^2 - \tfrac{1}{2}\lambda^{1/2}|m|\rho^3 - \tfrac{1}{16}\lambda\rho^4$$

$$\tfrac{1}{2}m_\rho^2 = |m^2|$$

χ is massless!

spectrum and scattering amplitudes are independent of field redefinitions!

The Goldstone boson remains massless even after including loop corrections!

To see it, note that if the GB remains massless, the exact propagator:

$$\tilde{\mathbf{\Delta}}_\chi(k^2) \;=\; 1/[k^2 \;-\; \Pi_\chi(k^2)]$$

has to have a pole at $k^2 = 0$.

$$\Pi_\chi(0) = 0$$

we can calculate it by summing 1PI diagrams with two external χ lines with $k = 0$.

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - \tfrac{1}{2}\Big(1+\frac{\rho}{v}\Big)^2\partial^\mu\chi\partial_\mu\chi - |m^2|\rho^2 - \tfrac{1}{2}\lambda^{1/2}|m|\rho^3 - \tfrac{1}{16}\lambda\rho^4$$

The vertices are proportional to momenta of χ (because of derivatives) and so they vanish.

Note: the theory expanded around the minimum of the potential has interactions that were not present in the original theory. However it turns out that divergent parts of the counterterms for the theory with $m^2 > 0$ (where the symmetry is unbroken) will also serve to cancel the divergencies in the theory with $m^2 < 0$ where the symmetry is spontaneously broken. This is a general rule for theories with spontaneous symmetry breaking.

Let's now generalize our results for a nonabelian case:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \tfrac{1}{2}m^2\varphi_i\varphi_i - \tfrac{1}{16}\lambda(\varphi_i\varphi_i)^2$$

the lagrangian is invariant under the SO(N) transformation:

$$\delta\varphi_i = -i\theta^a(T^a)_{ij}\varphi_j$$

set of $\frac{1}{2}N(N-1)$ infinitesimal parameters

set of antisymmetric generator matrices with a single nonzero entry -i above the main diagonal

for $m^2 < 0$, the minimum of the potential is achieved for:

$$\varphi_i(x) = v_i \qquad \text{with} \qquad v^2 = v_i v_i = 4|m^2|/\lambda$$

the vacuum expectation value, N-component vector with arbitrary direction

we can choose the coordinate system (SO(N) rotation) so that

$$v_i = v\delta_{iN}$$

Let's make an infinitesimal SO(N) transformation; this changes the VEV:

$$v_i = v\delta_{iN}$$

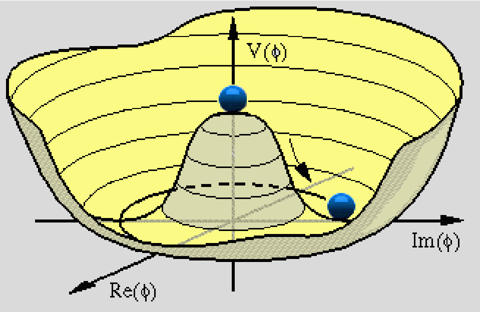
$$v_i \rightarrow v_i - i\theta^a (T^a)_{ij} v_j$$

$$= v\delta_{iN} - i\theta^a (T^a)_{iN} v$$

zero for all the generators with no non-zero entry in the last column - UNBROKEN GENERATORS

$$(T^a)_{ij} v_j = 0$$

these do not change the VEV of the field, and correspond to the unbroken symmetry



non-zero for N-1 generators with a non-zero entry in the last column - BROKEN GENERATORS

$$(T^a)_{ij} v_j \neq 0$$

these change the VEV of the field but not the energy; each of them corresponds to a flat direction in field space; each flat direction implies the existence of a massless particle - Goldstone boson.

For SO(N) we have

$$\tfrac{1}{2}N(N-1) - (N-1) = \tfrac{1}{2}(N-1)(N-2)$$

unbroken generators ⟶ SO(N-1) - obvious symmetry of the lagrangian!

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i - \tfrac{1}{2} m^2 \varphi_i \varphi_i - \tfrac{1}{16}\lambda(\varphi_i\varphi_i)^2$$

Let's work it out in detail; we can write our lagrangian as:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i - V(\varphi) \quad \text{where} \quad V(\varphi) = \tfrac{1}{16}\lambda(\varphi_i\varphi_i - v^2)^2$$

summed from 1 to N

$$v = (4|m^2|/\lambda)^{1/2}$$

Let's shift the field by the VEV and define:

$$\varphi_N(x) = v + \rho(x)$$

we get:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \varphi_i \partial_\mu \varphi_i - \tfrac{1}{2}\partial^\mu \rho \partial_\mu \rho - V(\rho, \varphi)$$

summed from 1 to N-1

N-1 massless particles
- Goldstone bosons

where:

$$\begin{aligned} V(\rho,\varphi) &= \tfrac{1}{16}\lambda[(v+\rho)^2 + \varphi_i\varphi_i - v^2]^2 \\ &= \tfrac{1}{16}\lambda(2v\rho + \rho^2 + \varphi_i\varphi_i)^2 \\ &= \tfrac{1}{4}\lambda v^2\rho^2 + \tfrac{1}{4}\lambda v\rho(\rho^2 + \varphi_i\varphi_i) + \tfrac{1}{16}\lambda(\rho^2 + \varphi_i\varphi_i)^2 \end{aligned}$$

SO(N-1) - manifest symmetry of the lagrangian!

Spontaneous breaking of gauge symmetries

based on S-84,85

Consider scalar electrodynamics:

$$\mathcal{L} = -(D^\mu \varphi)^\dagger D_\mu \varphi - V(\varphi) - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$D_\mu = \partial_\mu - igA_\mu$$

where

$$V(\varphi) = m^2 \varphi^\dagger \varphi + \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2$$

for $m^2 < 0$, the minimum of the potential is achieved for:

$$\langle 0|\varphi(x)|0\rangle = \frac{1}{\sqrt{2}} v$$

$$V(\varphi) = \tfrac{1}{4}\lambda(\varphi^\dagger \varphi - \tfrac{1}{2}v^2)^2$$

$$v = (4|m^2|/\lambda)^{1/2}$$

we write:

$$\varphi(x) = \frac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v}$$

and the scalar potential becomes:

$$V(\varphi) = \tfrac{1}{4}\lambda v^2 \rho^2 + \tfrac{1}{4}\lambda v \rho^3 + \tfrac{1}{16}\lambda \rho^4$$

$$\mathcal{L} = -(D^\mu\varphi)^\dagger D_\mu\varphi - V(\varphi) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$D_\mu = \partial_\mu - igA_\mu$$

$$\varphi(x) = \tfrac{1}{\sqrt{2}}(v+\rho(x))e^{-i\chi(x)/v}$$

The gauge symmetry allows us to shift the phase of $\varphi(x)$ by an arbitrary spacetime function; we can set:

$$\chi(x) = 0$$

Unitary gauge

with this choice the kinetic term becomes:

$$-(D^\mu\varphi)^\dagger D_\mu\varphi = -\tfrac{1}{2}(\partial^\mu\rho + ig(v+\rho)A^\mu)(\partial_\mu\rho - ig(v+\rho)A_\mu)$$
$$= -\tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - \tfrac{1}{2}g^2(v+\rho)^2A^\mu A_\mu\ .$$

$M = gv$ mass term for the gauge field!

$$-\tfrac{1}{2}M^2A^\mu A_\mu$$

Higgs mechanism: the Goldstone boson disappears and the gauge field acquires a mass. The Goldstone boson has become the longitudinal polarization of the massive gauge field.

The scalar field that is used to break the gauge symmetry is called the Higgs field.

Part of the lagrangian quadratic in gauge fields:

$$\mathcal{L}_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}M^2 A^\mu A_\mu$$

the equation of motion:

$$[(-\partial^2 + M^2)g^{\mu\nu} + \partial^\mu\partial^\nu]A_\nu = 0$$

acting on both sides with ∂_μ we find:

$$M^2\partial^\nu A_\nu = 0$$

this is not a gauge-fixing condition!

thus each component obeys the Klein-Gordon equation:

$$(-\partial^2 + M^2)A_\nu = 0$$

the general solution is:

$$A^\mu(x) = \sum_{\lambda=-,0,+}\int \widetilde{dk}\left[\varepsilon_\lambda^{\mu *}(k)a_\lambda(k)e^{ikx} + \varepsilon_\lambda^\mu(k)a_\lambda^\dagger(k)e^{-ikx}\right]$$

with

$$k_\mu\varepsilon_\lambda^\mu(k) = 0$$

the general solution is:

$$A^\mu(x) = \sum_{\lambda=-,0,+} \int \widetilde{dk} \left[\varepsilon_\lambda^{\mu*}(k) a_\lambda(k) e^{ikx} + \varepsilon_\lambda^\mu(k) a_\lambda^\dagger(k) e^{-ikx}\right]$$

$$k_\mu \varepsilon_\lambda^\mu(k) = 0$$

in the rest frame $k = (M, 0, 0, 0)$ we can choose the polarization vector to correspond to definite spin along the z-axis:

$$\begin{aligned} \varepsilon_+(0) &= \tfrac{1}{\sqrt{2}}(0, 1, -i, 0) \ , \\ \varepsilon_-(0) &= \tfrac{1}{\sqrt{2}}(0, 1, +i, 0) \ , \\ \varepsilon_0(0) &= (0, 0, 0, 1) \ . \end{aligned}$$

polarization vectors together with the timelike unit vector k^μ/M form an orthonormal and complete set:

$$k\cdot\varepsilon_\lambda(k) = 0 \ ,$$

$$\varepsilon_{\lambda'}(k)\cdot\varepsilon_\lambda^*(k) = \delta_{\lambda'\lambda} \ ,$$

$$\sum_{\lambda=-,0,+} \varepsilon_\lambda^{\mu*}(k)\varepsilon_\lambda^\nu(k) = g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \ .$$

$$A^\mu(x) = \sum_{\lambda=-,0,+} \int \widetilde{dk} \left[\varepsilon_\lambda^{\mu *}(k) a_\lambda(k) e^{ikx} + \varepsilon_\lambda^\mu(k) a_\lambda^\dagger(k) e^{-ikx}\right]$$

$$k \cdot \varepsilon_\lambda^\mu(k) = 0 \,,$$
$$\varepsilon_{\lambda'}(k) \cdot \varepsilon_\lambda^*(k) = \delta_{\lambda'\lambda} \,,$$
$$\sum_{\lambda=-,0,+} \varepsilon_\lambda^{\mu *}(k) \varepsilon_\lambda^\nu(k) = g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \,.$$

In analogy with the scalar field theory or QED the propagator for the "massive" gauge field is:

$$i\langle 0|\mathrm{T} A^\mu(x) A^\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + M^2 - i\epsilon} \sum\nolimits_\lambda \varepsilon_\lambda^{\mu *}(k) \varepsilon_\lambda^\nu(k)$$
$$= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + M^2 - i\epsilon} \left(g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right).$$

and interactions can be read off of:

$$-(D^\mu \varphi)^\dagger D_\mu \varphi = -\tfrac{1}{2}(\partial^\mu \rho + ig(v+\rho)A^\mu)(\partial_\mu \rho - ig(v+\rho)A_\mu)$$
$$= -\tfrac{1}{2}\partial^\mu \rho \partial_\mu \rho - \tfrac{1}{2} g^2 (v+\rho)^2 A^\mu A_\mu \,.$$

and

$$V(\varphi) = \tfrac{1}{4}\lambda v^2 \rho^2 + \tfrac{1}{4}\lambda v \rho^3 + \tfrac{1}{16}\lambda \rho^4$$

There are complications with calculating loop diagrams!

taking the unitary gauge, $\chi(x) = 0$, corresponds to inserting a functional delta function:

$$\prod_x \delta(\chi(x))$$

to the path integral. Changing integration variables from $\mathrm{Re}\,\varphi$ and $\mathrm{Im}\,\varphi$ to ρ and χ we must include the functional determinant:

$$\varphi(x) = \frac{1}{\sqrt{2}}(v+\rho(x))e^{-i\chi(x)/v}$$

$$\prod_x \Big(v+\rho(x)\Big) = \det(v+\rho)$$

$$\propto \det(1+v^{-1}\rho)$$

$$\propto \int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{-im_{\rm gh}^2 \int d^4x\; \bar{c}(1+v^{-1}\rho)c}$$

$$\int d^n\bar{\psi}\, d^n\psi\, \exp\left(-i\bar{\psi}_i M_{ij}\psi_j\right) \propto \det M$$

arbitrary constant

ghost-ghost-scalar vertex: $-im_{\rm gh}^2 v^{-1}$

ghost propagator: $\tilde{\Delta}(k^2) = 1/m_{\rm gh}^2$

no ghost-gauge field interactions

the ghost propagator doesn't contain momentum and so loop diagrams with arbitrarily many external lines diverge more and more (also the gauge field propagator scales like $1/M^2$ for large momenta); difficult to establish renormalizability.

The R_ξ gauge doesn't suffer from this problem:

real scalar fields

$$\varphi = \frac{1}{\sqrt{2}}(v + h + ib)$$

with this parameterization, the potential is:

$$V(\varphi) = \tfrac{1}{4}\lambda v^2 h^2 + \tfrac{1}{4}\lambda v h(h^2 + b^2) + \tfrac{1}{16}\lambda(h^2 + b^2)^2$$

$$V(\varphi) = \tfrac{1}{4}\lambda(\varphi^\dagger\varphi - \tfrac{1}{2}v^2)^2$$

$$\mathcal{L} = -(D^\mu\varphi)^\dagger D_\mu\varphi - V(\varphi) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$D_\mu = \partial_\mu - igA_\mu$$

and the covariant derivative is:

$$D_\mu\varphi = \frac{1}{\sqrt{2}}\Big[(\partial_\mu h + gbA_\mu) + i(\partial_\mu b - g(v{+}h)A_\mu)\Big]$$

and so the kinetic term can be written as:

$$-(D^\mu\varphi)^\dagger D_\mu\varphi = -\tfrac{1}{2}(\partial_\mu h + gbA_\mu)^2 - \tfrac{1}{2}(\partial_\mu b - g(v{+}h)A_\mu)^2$$

rearranging the kinetic term: $-(D^\mu\varphi)^\dagger D_\mu\varphi = -\frac{1}{2}(\partial_\mu h + gbA_\mu)^2 - \frac{1}{2}(\partial_\mu b - g(v+h)A_\mu)^2$

$$-(D^\mu\varphi)^\dagger D_\mu\varphi = -\tfrac{1}{2}\partial_\mu h\partial_\mu h - \tfrac{1}{2}\partial_\mu b\partial_\mu b - \tfrac{1}{2}g^2v^2A^\mu A_\mu + \underline{gvA^\mu\partial_\mu b}$$
$$+ gA^\mu(h\partial_\mu b - b\partial_\mu h)$$
$$- gvhA^\mu A_\mu - \tfrac{1}{2}g^2(h^2+b^2)A^\mu A_\mu\ .$$

cross term between the vector field and the b-field

For abelian gauge theory in the absence of spontaneous symmetry breaking we fixed the gauge by adding the gauge-fixing and ghost terms:

$$\mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} = -\tfrac{1}{2}\xi^{-1}G^2 - \bar{c}\,\frac{\delta G}{\delta\theta}\,c$$

$G = \partial^\mu A_\mu$

$A_\mu \to A_\mu - \partial_\mu\theta$

$\varphi \to \varphi - ig\theta\varphi$

$\delta G/\delta\theta = -\partial^2$

ghost fields have no interactions and can be ignored.

For spontaneously broken theory we choose:

$$G = \partial^\mu A_\mu - \xi gvb$$

and we get:

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1}\partial^\mu A_\mu\partial^\nu A_\nu + gvb\partial^\mu A_\mu - \tfrac{1}{2}\xi g^2v^2b^2$$
$$= -\tfrac{1}{2}\xi^{-1}\partial^\mu A_\nu\partial^\nu A_\mu - \underline{gvA_\mu\partial^\mu b} - \tfrac{1}{2}\xi g^2v^2b^2$$

cancel

integration by parts

mass term for b

Let's choose:

$$\mathcal{O}(x) = \bar{c}^a(x)\left[\tfrac{1}{2}\xi B^a(x) - G^a(x)\right]$$

gauge-fixing function

$$G^a(x) = \partial^\mu A^a_\mu(x)$$

we will get the R_ξ gauge

then

$$\delta_{\rm B}\mathcal{O} = (\delta_{\rm B}\bar{c}^a)\left[\tfrac{1}{2}\xi B^a - \partial^\mu A^a_\mu\right] - \bar{c}^a\left[\tfrac{1}{2}\xi(\delta_{\rm B}B^a) - \partial^\mu(\delta_{\rm B}A^a_\mu)\right]$$

-1 for $\delta_{\rm B}$ acting as an anticommuting object

$$\delta_{\rm B}A^a_\mu(x) \equiv D^{ab}_\mu c^b(x)$$
$$= \partial_\mu c^a(x) - gf^{abc}A^c_\mu(x)c^b(x)$$
$$\delta_{\rm B}\bar{c}^a(x) = B^a(x)$$

$$\delta_{\rm B}\mathcal{O} = \tfrac{1}{2}\xi B^a B^a - B^a\partial^\mu A^a_\mu + \bar{c}^a\partial^\mu D^{ab}_\mu c^b$$

or

$$\delta_{\rm B}\mathcal{O} \to \tfrac{1}{2}\xi B^a B^a - B^a\partial^\mu A^a_\mu - \partial^\mu\bar{c}^a D^{ab}_\mu c^b$$

$$\delta_{\mathrm{B}}\mathcal{O} \to \tfrac{1}{2}\xi B^a B^a - B^a \partial^\mu A^a_\mu - \partial^\mu \bar{c}^a D^{ab}_\mu c^b$$

now we can easily perform the path integral over B:

it is equivalent to solving the classical equation of motion,

$$\frac{\partial(\delta_{\mathrm{B}}\mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A^a_\mu(x) = 0$$

and substituting the result back to the formula:

$$\delta_{\mathrm{B}}\mathcal{O} \to -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu - \partial^\mu \bar{c}^a D^{ab}_\mu c^b$$

we obtained the gauge fixing lagrangian and the ghost lagrangian

$$\mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} = -\tfrac{1}{2}\xi^{-1}G^2 - \bar{c}\,\frac{\delta G}{\delta\theta}\,c$$

$$G = \partial^\mu A_\mu$$

$$A_\mu \to A_\mu - \partial_\mu\theta$$

$$\varphi \to \varphi - ig\theta\varphi$$

$$\delta G/\delta\theta = -\partial^2$$

For spontaneously broken theory we choose:

$$G = \partial^\mu A_\mu - \xi g v b$$

$$\varphi = \tfrac{1}{\sqrt{2}}(v + h + ib)$$

$$h \to h + g\theta b\,,$$

$$b \to b - g\theta(v+h)$$

$$\frac{\delta G}{\delta\theta} = -\partial^2 + \xi g^2 v(v+h)$$

and for the ghost term we find:

$$\mathcal{L}_{\mathrm{gh}} = -\bar{c}\Big[-\partial^2 + \xi g^2 v(v+h)\Big]c$$

$$= -\partial^\mu\bar{c}\partial_\mu c - \xi g^2 v^2 \bar{c}c - \xi g^2 v h \bar{c}c$$

ghost has the same mass as the b-field!