$$-\tfrac{1}{2}g^2\,T(\mathrm{A})\delta^{ab}\,\tilde{\mu}^{\varepsilon}\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\,\frac{N^{\mu\nu}}{(q^2+D)^2}$$

$$q=\ell+xk$$

$$D=x(1-x)k^2$$

$$N^{\mu\nu}\to-\tfrac{9}{2}q^2g^{\mu\nu}-(5-2x+2x^2)k^2g^{\mu\nu}+(2+10x-10x^2)k^\mu k^\nu$$

$$q^2\to(\tfrac{2}{d}-1)^{-1}D$$

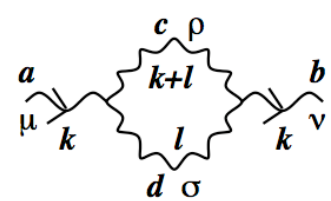
$$N^{\mu\nu}\to-(5-11x+11x^2)k^2g^{\mu\nu}+(2+10x-10x^2)k^\mu k^\nu$$

$$\tilde{\mu}^{\varepsilon}\int\frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2}=\frac{i}{8\pi^2\varepsilon}+O(\varepsilon^0)$$

we get:

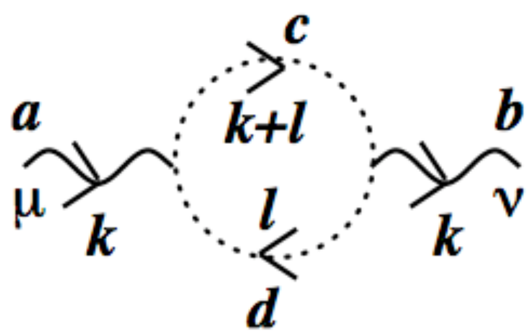
$$-\frac{ig^2}{16\pi^2}T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\int_0^1 dx\,N^{\mu\nu}+O(\varepsilon^0)$$

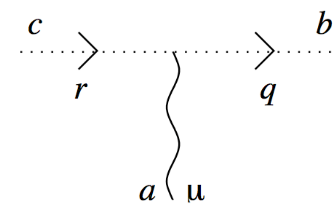
integrating over x, we get the result for the divergent part of



$$-\frac{ig^2}{16\pi^2}T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{19}{6}k^2g^{\mu\nu}+\tfrac{11}{3}k^\mu k^\nu\right)$$

now let's calculate the ghost loop:





$$i\mathbf{V}^{abc}_{\mu}(q,r) = i(gf^{abc})(-iq_\mu)$$
$$= gf^{abc}q_\mu \,.$$

extra -1 for closed ghost loop

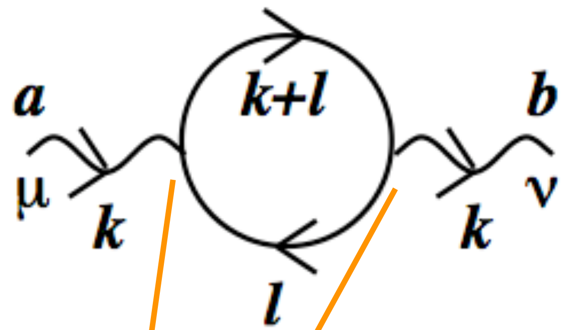
$$i\Pi^{\mu\nu ab}(k) = (-1)g^2 f^{acd} f^{bdc} \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell+k)^\mu \ell^\nu}{\ell^2(\ell+k)^2}$$

$$f^{acd} f^{bdc} = -T(\mathrm{A})\delta^{ab}$$

combining denominators,
shifting the numerator,
performing integrals, ...

$$-\frac{ig^2}{8\pi^2} T(\mathrm{A})\delta^{ab} \frac{1}{\varepsilon} \left(-\tfrac{1}{12}k^2 g^{\mu\nu} - \tfrac{1}{6}k^\mu k^\nu\right)$$

finally, let’s calculate the fermion loop:



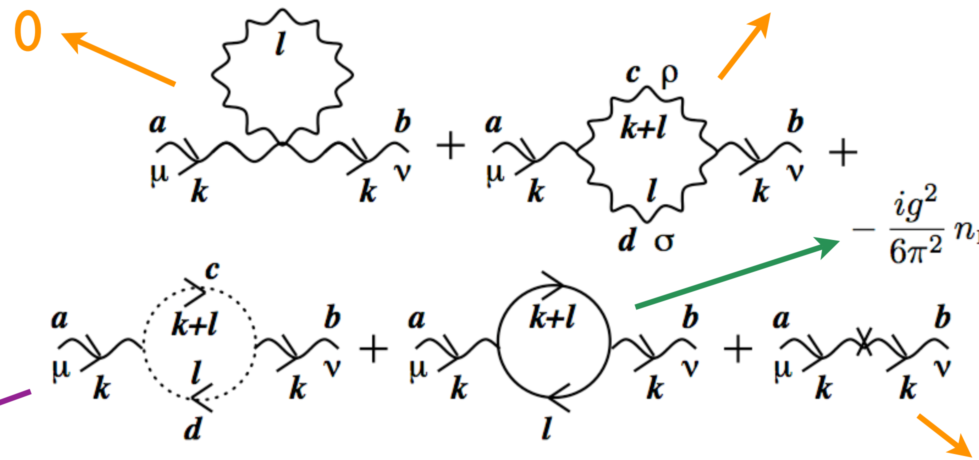
the same calculation as in QED, except for the color factor:

$$\mathrm{Tr}(T^a T^b) = T(\mathrm{R})\delta^{ab}$$

$$-\frac{ig^2}{6\pi^2} n_{\mathrm{F}} T(\mathrm{R})\delta^{ab} \frac{1}{\varepsilon} \left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$

the number of flavors

Putting pieces together:



$$-\frac{ig^2}{16\pi^2}T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{19}{6}k^2g^{\mu\nu}+\tfrac{11}{3}k^\mu k^\nu\right)$$

$$-\frac{ig^2}{6\pi^2}n_{\mathrm{F}}T(\mathrm{R})\delta^{ab}\,\frac{1}{\varepsilon}\left(k^2g^{\mu\nu}-k^\mu k^\nu\right)$$

$$-\frac{ig^2}{8\pi^2}T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{1}{12}k^2g^{\mu\nu}-\tfrac{1}{6}k^\mu k^\nu\right)$$

$$-\,i(Z_3-1)(k^2g^{\mu\nu}-k^\mu k^\nu)\,\delta^{ab}$$

we find:

$$\Pi^{\mu\nu ab}(k)=\Pi(k^2)(k^2g^{\mu\nu}-k^\mu k^\nu)\delta^{ab}$$

gluon self-energy is transverse

$$\Pi(k^2)_{\mathrm{div}}=-(Z_3-1)+\left[\tfrac{5}{3}T(\mathrm{A})-\tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon}+O(g^4)$$

and so:

$$Z_3=1+\left[\tfrac{5}{3}T(\mathrm{A})-\tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon}+O(g^4)$$

We found: in Feynman gauge and the $\overline{\mathrm{MS}}$ scheme

not equal!

$$Z_1 = 1 - \left[C(\mathrm{R}) + T(\mathrm{A})\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \ ,$$

$$Z_2 = 1 - C(\mathrm{R}) \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \ ,$$

$$Z_3 = 1 + \left[\tfrac{5}{3} T(\mathrm{A}) - \tfrac{4}{3} n_{\mathrm{F}} T(\mathrm{R})\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \ ,$$

Let's calculate the beta function; define:

$$\alpha \equiv \frac{g^2}{4\pi}$$

the dictionary:

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^{\varepsilon} = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2 \tilde{\mu}^{\varepsilon} = \frac{Z_{3g}^2}{Z_3^3} g^2 \tilde{\mu}^{\varepsilon} = \frac{Z_{4g}}{Z_3^2} g^2 \tilde{\mu}^{\varepsilon}$$

$$\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3} \alpha \tilde{\mu}^{\varepsilon}$$

Beta functions in quantum electrodynamics

based on S-66

Let's calculate the beta function in QED:

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_1 = Z_1 e\overline{\Psi}A\!\!\!/\Psi + \mathcal{L}_{\rm ct} \; ,$$

$$\mathcal{L}_{\rm ct} = i(Z_2-1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m-1)m\overline{\Psi}\Psi - \tfrac{1}{4}(Z_3-1)F^{\mu\nu}F_{\mu\nu}$$

the dictionary:

$$e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \, \tilde{\mu}^{\varepsilon/2} e$$

$$\alpha = e^2/4\pi$$

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \, \tilde{\mu}^{\varepsilon} \alpha$$

$$Z_1 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_2 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

Note $Z_1 = Z_2$!

following the usual procedure:

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \, \tilde{\mu}^{\varepsilon} \alpha$$

$$\ln \alpha_0 = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu}$$

$$\ln\left(Z_3^{-1} Z_2^{-2} Z_1^2\right) = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n}$$

$$Z_1 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_2 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$E_1(\alpha) = \frac{2\alpha}{3\pi} + O(\alpha^2)$$

we find:

$$\beta(\alpha) = \alpha^2 E_1'(\alpha)$$

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3)$$

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3)$$

or equivalently:

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5)$$

$$\alpha = e^2/4\pi$$

$$\dot{\alpha} = e\dot{e}/2\pi$$

For a theory with N Dirac fields with charges $Q_i e$:

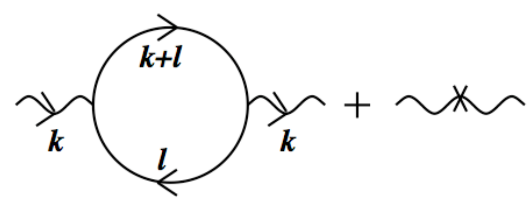
$$Z_1 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_2 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2)$$

Z_{1i}/Z_{2i} = 1

$\sum_i Q_i^2 \alpha$



we find:

$$\beta(e) = \frac{\sum_{i=1}^{N} Q_i^2}{12\pi^2}\, e^3 + O(e^5)$$

following the usual procedure:

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \, \tilde{\mu}^{\varepsilon} \alpha$$

$$\ln \alpha_0 = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu}$$

$$\ln\left(Z_3^{-1} Z_2^{-2} Z_1^2\right) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n}$$

$$Z_1 = 1 - \left[C(\mathrm{R}) + T(\mathrm{A})\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \,,$$

$$Z_2 = 1 - C(\mathrm{R}) \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \,,$$

$$Z_3 = 1 + \left[\tfrac{5}{3} T(\mathrm{A}) - \tfrac{4}{3} n_{\mathrm{F}} T(\mathrm{R})\right] \frac{g^2}{8\pi^2} \frac{1}{\varepsilon} + O(g^4) \,,$$

$$G_1(\alpha) = -\left[\tfrac{11}{3} T(\mathrm{A}) - \tfrac{4}{3} n_{\mathrm{F}} T(\mathrm{R})\right] \frac{\alpha}{2\pi} + O(\alpha^2)$$

we find:

$$\beta(\alpha) = \alpha^2 G_1'(\alpha)$$

$$\beta(\alpha) = -\left[\tfrac{11}{3} T(\mathrm{A}) - \tfrac{4}{3} n_{\mathrm{F}} T(\mathrm{R})\right] \frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$\beta(\alpha) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{\alpha^2}{2\pi} + O(\alpha^3)$$

$$\alpha = g^2/4\pi$$

$$\dot{\alpha} = g\dot{g}/2\pi$$

or equivalently:

$$\beta(g) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^3}{16\pi^2} + O(g^5)$$

For QCD:

$T(\mathrm{A}) = 3$

$T(\mathrm{R}) = \frac{1}{2}$

$$11 - \frac{2}{3}n_{\mathrm{F}}$$

beta function is negative for $n_{\mathrm{F}} \leq 16$,
the gauge coupling gets weaker at higher energies!

BRST symmetry

based on S-74

We are going to show that the gauge-fixed lagrangian:

$$\mathcal{L} \equiv \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh}$$

has a residual form of the gauge symmetry - Becchi-Rouet-Stora-Tyutin symmetry

Consider an infinitesimal transformation for a non-abelian gauge theory:

$$\delta A^a_\mu(x) = -D^{ab}\theta^b(x)$$

$$\delta\phi_i(x) = -ig\theta^a(x)(T^a_{\rm R})_{ij}\phi_j(x)$$

scalars or spinors in representation R

The BRST transformation is defined as:

$$\delta_{\rm B} A^a_\mu(x) \equiv D^{ab}_\mu c^b(x)$$

$$= \partial_\mu c^a(x) - gf^{abc}A^c_\mu(x)c^b(x)$$

$$\delta_{\rm B}\phi_i(x) \equiv igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x)$$

we use the ghost field (scalar Grassmann field) instead of $-\theta^a(x)$.

Anything that is gauge invariant is automatically BRST invariant, in particular $\delta_{\rm B}\mathcal{L}_{\rm YM} = 0$!

Now we are going to require that $\delta_{\rm B}\,\delta_{\rm B} = 0$:

this requirement will determine the BRST transformation of the ghost field.

$$\delta_{\rm B}\phi_i(x) \equiv igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x)$$

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^a)(T^a_{\rm R})_{ij}\phi_j - igc^a(T^a_{\rm R})_{ij}\delta_{\rm B}\phi_j$$

-1 for $\delta_{\rm B}$ acting as an anticommuting object

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^a)(T^a_{\rm R})_{ij}\phi_j - g^2c^ac^b(T^a_{\rm R}T^b_{\rm R})_{ik}\phi_k$$

$$c^bc^a = -c^ac^b$$

$$\tfrac{1}{2}[T^a_{\rm R}, T^b_{\rm R}] = \tfrac{i}{2}f^{abc}T^c_{\rm R}$$

Thus we have:

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^c + \tfrac{1}{2}gf^{abc}c^ac^b)(T^c_{\rm R})_{ij}\phi_j$$

that will vanish for all $\phi_j(x)$ if and only if:

$$\delta_{\rm B}c^c(x) = -\tfrac{1}{2}gf^{abc}c^a(x)c^b(x)$$

Now we have to check that $\delta_{\rm B}\,\delta_{\rm B} = 0$ for the gauge field:

$$\delta_{\rm B} A^a_\mu(x) \equiv D^{ab}_\mu c^b(x)$$
$$= \partial_\mu c^a(x) - g f^{abc} A^c_\mu(x) c^b(x)$$

$$\begin{aligned}\delta_{\rm B}(\delta_{\rm B} A^a_\mu) &= (\delta^{ab}\partial_\mu - g f^{abc} A^c_\mu)(\delta_{\rm B} c^b) - g f^{abc}(\delta_{\rm B} A^c_\mu) c^b \\ &= D^{ab}_\mu(\delta_{\rm B} c^b) - g f^{abc}(D^{cd}_\mu c^d) c^b \\ &= D^{ab}_\mu(\delta_{\rm B} c^b) - g f^{abc}\underline{(\partial_\mu c^c) c^b} + g^2 \underline{f^{abc} f^{cde}} A^e_\mu c^d c^b\end{aligned}$$

$$c^b c^a = -c^a c^b$$

$$\begin{aligned}\tfrac{1}{2}(\partial_\mu c^{[c}) c^{b]} &\equiv \tfrac{1}{2}(\partial_\mu c^c) c^b - \tfrac{1}{2}(\partial_\mu c^b) c^c \\ &= \tfrac{1}{2}(\partial_\mu c^c) c^b + \tfrac{1}{2} c^c(\partial_\mu c^b) \\ &= \tfrac{1}{2}\partial_\mu(c^c c^b)\,.\end{aligned}$$

$$\begin{aligned}&= \tfrac{1}{2}(f^{abc} f^{cde} - f^{adc} f^{cbe}) \\ &= -\tfrac{1}{2}[(T^b_{\rm A})^{ac}(T^d_{\rm A})^{ce} - (T^d_{\rm A})^{ac}(T^b_{\rm A})^{ce}] \\ &= -\tfrac{1}{2} i f^{bdh}(T^h_{\rm A})^{ae} \\ &= -\tfrac{1}{2} f^{bdh} f^{hae}\,,\end{aligned}$$

Now we have to check that $\delta_{\rm B}\,\delta_{\rm B} = 0$ for the gauge field:

$$\delta_{\rm B} A_\mu^a(x) \equiv D_\mu^{ab} c^b(x)$$
$$= \partial_\mu c^a(x) - g f^{abc} A_\mu^c(x) c^b(x)$$

$$\delta_{\rm B}(\delta_{\rm B} A_\mu^a) = (\delta^{ab}\partial_\mu - g f^{abc} A_\mu^c)(\delta_{\rm B} c^b) - g f^{abc}(\delta_{\rm B} A_\mu^c) c^b$$
$$= D_\mu^{ab}(\delta_{\rm B} c^b) - g f^{abc}(D_\mu^{cd} c^d) c^b$$
$$= D_\mu^{ab}(\delta_{\rm B} c^b) - g f^{abc}(\partial_\mu c^c) c^b + g^2 f^{abc} f^{cde} A_\mu^e c^d c^b$$

$$= \tfrac{1}{2}\partial_\mu(c^c c^b) \qquad\qquad = -\tfrac{1}{2} f^{bdh} f^{hae}$$

$$\delta_{\rm B}(\delta_{\rm B} A_\mu^a) = D_\mu^{ab}(\delta_{\rm B} c^b) - \tfrac{1}{2} g f^{abc}(\partial_\mu c^c c^b) - \tfrac{1}{2} g^2 f^{bdh} f^{hae} A_\mu^e c^d c^b$$
$$= D_\mu^{ah}(\delta_{\rm B} c^h) - (\delta^{ah}\partial_\mu - g f^{ahe} A_\mu^e)\tfrac{1}{2} g f^{bch} c^c c^b$$
$$= D_\mu^{ah}(\delta_{\rm B} c^h + \tfrac{1}{2} g f^{bch} c^b c^c) \,.$$

vanishes for the variation of the ghost field we found before:

$$\delta_{\rm B} c^c(x) = -\tfrac{1}{2} g f^{abc} c^a(x) c^b(x)$$

The BRST transformation of the antighost field is defined as:

we treat ghost and antighost fields as independent fields

$$\delta_B \bar{c}^a(x) = B^a(x)$$

B is a scalar field

Lautrup-Nakanishi auxiliary field

then $\delta_B \, \delta_B \; = 0$ implies:

$$\delta_B B^a(x) = 0$$

What is it good for?

We can add to the lagrangian any term that is the BRST variation of some object:

corresponds to fixing a gauge

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \delta_B \mathcal{O}$$

BRST invariant because it is gauge invariant

BRST invariant because $\delta_B(\delta_B \mathcal{O}) \; = 0$

Let's choose:

$$\mathcal{O}(x) = \bar{c}^a(x)\left[\tfrac{1}{2}\xi B^a(x) - G^a(x)\right]$$

gauge-fixing function

$$G^a(x) = \partial^\mu A^a_\mu(x)$$

we will get the R_ξ gauge

then

$$\delta_{\mathrm{B}}\mathcal{O} = (\delta_{\mathrm{B}}\bar{c}^a)\left[\tfrac{1}{2}\xi B^a - \partial^\mu A^a_\mu\right] - \bar{c}^a\left[\tfrac{1}{2}\xi(\delta_{\mathrm{B}}B^a) - \partial^\mu(\delta_{\mathrm{B}}A^a_\mu)\right]$$

-1 for δ_{B} acting as an anticommuting object

$$\begin{aligned}\delta_{\mathrm{B}}A^a_\mu(x) &\equiv D^{ab}_\mu c^b(x)\\ &= \partial_\mu c^a(x) - gf^{abc}A^c_\mu(x)c^b(x)\end{aligned}$$

$$\delta_{\mathrm{B}}\bar{c}^a(x) = B^a(x)$$

$$\delta_{\mathrm{B}}\mathcal{O} = \tfrac{1}{2}\xi B^a B^a - B^a\partial^\mu A^a_\mu + \bar{c}^a\partial^\mu D^{ab}_\mu c^b$$

or

$$\delta_{\mathrm{B}}\mathcal{O} \to \tfrac{1}{2}\xi B^a B^a - B^a\partial^\mu A^a_\mu - \partial^\mu\bar{c}^a D^{ab}_\mu c^b$$

$$\delta_{\rm B}\mathcal{O} \to \frac{1}{2}\xi B^a B^a - B^a \partial^\mu A^a_\mu - \partial^\mu \bar{c}^a D^{ab}_\mu c^b$$

now we can easily perform the path integral over B:

it is equivalent to solving the classical equation of motion,

$$\frac{\partial(\delta_{\rm B}\mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A^a_\mu(x) = 0$$

and substituting the result back to the formula:

$$\delta_{\rm B}\mathcal{O} \to -\frac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu - \partial^\mu \bar{c}^a D^{ab}_\mu c^b$$

we obtained the gauge fixing lagrangian and the ghost lagrangian

$$S = \int d^4x\, \mathcal{L}$$

$$\mathcal{L} = \mathcal{L}_{\rm YM} + \delta_{\rm B}\mathcal{O}$$

$$\delta_{\rm B}\mathcal{O} \to -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu - \partial^\mu \bar{c}^a D^{ab}_\mu c^b$$

Symmetries of the complete action:

- Lorentz invariance
- Parity, Time reversal, Charge conjugation
- Global invariance under a given (non-abelian) symmetry group
- BRST invariance
- ghost number conservation (+1 for ghost and -1 for antighost)
- antighost translation invariance $\bar{c}^a(x) \to \bar{c}^a(x) + \chi$

The lagrangian already includes all the terms allowed by these symmetries!

this means that all the divergencies can be absorbed by the Zs of these terms, BRST symmetry requires that the gauge coupling renormalize in the same way in each term.

There is the Noether current associated with the BRST symmetry:

$$j_{\mathrm{B}}^{\mu}(x) = \sum_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_I(x))} \delta_{\mathrm{B}} \Phi_I(x)$$

all the fields in the theory

and the corresponding BRST charge:

$$Q_{\mathrm{B}} = \int d^3x \, j_{\mathrm{B}}^0(x)$$

it is hermitian

the BRST charge generates a BRST transformation:

$$i[Q_{\mathrm{B}}, A_\mu^a(x)] = D_\mu^{ab} c^b(x) \,, \qquad \delta_{\mathrm{B}} A_\mu^a(x) \equiv D_\mu^{ab} c^b(x)$$

$$i\{Q_{\mathrm{B}}, c^a(x)\} = -\tfrac{1}{2} g f^{abc} c^b(x) c^c(x) \,, \qquad \delta_{\mathrm{B}} c^c(x) = -\tfrac{1}{2} g f^{abc} c^a(x) c^b(x)$$

$$i\{Q_{\mathrm{B}}, \bar{c}^a(x)\} = B^a(x) \,, \qquad \delta_{\mathrm{B}} \bar{c}^a(x) = B^a(x)$$

$$i[Q_{\mathrm{B}}, B^a(x)] = 0 \,, \qquad \delta_{\mathrm{B}} B^a(x) = 0$$

$$i[Q_{\mathrm{B}}, \phi_i(x)]_{\pm} = igc^a(x)(T_{\mathrm{R}}^a)_{ij}\phi_j(x) \,. \qquad \delta_{\mathrm{B}} \phi_i(x) \equiv igc^a(x)(T_{\mathrm{R}}^a)_{ij}\phi_j(x)$$

commutator for scalars and anticommutator for spinors

The energy-momentum four-vector is:

$$P^\mu = \int d^3x \, T^{0\mu}(x)$$

Recall, we defined the space-time translation operator

$$T(a) \equiv \exp(-iP^\mu a_\mu)$$

so that

$$T(a)^{-1} \varphi_a(x) T(a) = \varphi_a(x-a)$$

we can easily verify it; for an infinitesimal transformation it becomes:

$$[\varphi_a(x), P^\mu] = \frac{1}{i} \partial^\mu \varphi_a(x)$$

it is straightforward to verify this by using the canonical commutation relations for $\varphi_a(x)$ and $\Pi_a(x)$.

Since $\delta_B \, \delta_B = 0$ we have:

$$Q_B^2 = 0$$

Consider states for which:

$$Q_B|\psi\rangle = 0$$

such states are said to be in the Kernel of Q_B .

Cohomology of Q_B :

$$|\psi\rangle \neq Q_B|\chi\rangle$$

Image of Q_B :

$$|\psi\rangle = Q_B|\chi\rangle$$

we identify two states as a single element of the cohomology if their difference is in the image:

$$|\psi'\rangle = |\psi\rangle + Q_B|\zeta\rangle$$

zero norm states:

$$\langle\psi|\psi\rangle = \langle\psi|Q_B|\chi\rangle = 0$$

Q is hermitian: $\langle\psi|Q_B = 0$.

Consider a normalized state in the cohomology:

$$\langle\psi|\psi\rangle = 1,\, Q_{\rm B}|\psi\rangle = 0,\, |\psi\rangle \neq Q_{\rm B}|\chi\rangle$$

since the lagrangian is BRST invariant:

$$[H, Q_{\rm B}] \;=\; 0$$

and so the time evolved state is still annihilated by $Q_{\rm B}$:

$$Q_{\rm B} e^{-iHt}|\psi\rangle = e^{-iHt} Q_{\rm B}|\psi\rangle = 0$$

(in addition, a unitary time evolution does not change the norm of a state)

the time-evolved stay must still be in the cohomology!

We will see shortly that the physical states of the theory correspond to the cohomology of $Q_{\rm B}$!

All states in the theory can be generated from creation operators (we start with widely separated wave packets, and so we can neglect interaction):

$$A^\mu(x) = \sum_{\substack{\lambda=>,<,\\ +,-}} \int \widetilde{dk} \left[\varepsilon_\lambda^{\mu*}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \varepsilon_\lambda^{\mu}(\mathbf{k}) a_\lambda^\dagger(\mathbf{k}) e^{-ikx}\right]$$

we are neglecting group indices

$$c(x) = \int \widetilde{dk} \left[c(\mathbf{k}) e^{ikx} + c^\dagger(\mathbf{k}) e^{-ikx}\right] ,$$

$$\bar{c}(x) = \int \widetilde{dk} \left[b(\mathbf{k}) e^{ikx} + b^\dagger(\mathbf{k}) e^{-ikx}\right] ,$$

$$\phi(x) = \int \widetilde{dk} \left[a_\phi(\mathbf{k}) e^{ikx} + a_\phi^\dagger(\mathbf{k}) e^{-ikx}\right] ,$$

represents matter fields

for $k^\mu = (\omega, \mathbf{k}) = \omega(1,0,0,1)$ four polarization vectors can be chosen as:

$$\varepsilon_>^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,1) ,$$

$$\varepsilon_<^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(1,0,0,-1) ,$$

$$\varepsilon_+^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,-i,0) ,$$

$$\varepsilon_-^\mu(\mathbf{k}) = \tfrac{1}{\sqrt{2}}(0,1,+i,0) .$$