Functional determinants

We are going to discuss situations where a functional determinant depends on some other field and so it cannot be absorbed into the overall normalization of the path integral.

Consider a theory of a complex scalar field:

\[ \mathcal{L} = -\partial^\mu \chi^\dagger \partial_\mu \chi - m^2 \chi^\dagger \chi + g\varphi \chi^\dagger \chi \]

we define the path integral:

\[ Z(\varphi) = \int D\chi^\dagger D\chi \ e^{i \int d^4x \mathcal{L}} \]

we fix \[ Z(0) = 1 \]

for the gaussian of \( n \) complex variables we found:

\[ \int d^n z \ d^n \bar{z} \ \exp \left( -i \bar{z}_i M_{ij} z_j \right) \propto (\det M)^{-1} \]

in the continuum limit the “matrix” in our case is:

\[ M(x, y) = \left[ -\partial_x^2 + m^2 - g\varphi(x) \right] \delta^4(x - y) \]

we need to calculate the determinant of this “matrix”
\[ M(x, y) = [-\partial_x^2 + m^2 - g\varphi(x)]\delta^4(x - y) \]

we can write it as a product of two matrices:

\[ M(x, z) = \int d^4y \, M_0(x, y) \tilde{M}(y, z) \]

where

\[ M_0(x, y) = (-\partial_x^2 + m^2)\delta^4(x - y), \]
\[ \tilde{M}(y, z) = \delta^4(y - z) - g\Delta(y - z)\varphi(z). \]

\[ (-\partial_y^2 + m^2)\Delta(y - z) = \delta^4(y - z) \]

now we can calculate the determinant:

\[ \det M = \det M_0 \det \tilde{M} \]

\[ \tilde{M} = I - G \]

\[ I(x, y) = \delta^4(x - y) \]
\[ G(x, y) = g\Delta(x - y)\varphi(y) \]

independent of the background field (can be absorbed into the overall normalization)

thus:

\[ Z(\varphi) = (\det \tilde{M})^{-1} \]
the path integral is given by:

\[ Z(\varphi) = (\text{det } \tilde{M})^{-1} \]

using \( \text{det } A = \exp \text{Tr } \ln A \) we get:

\[
\begin{align*}
\text{det } \tilde{M} &= \exp \text{Tr } \ln \tilde{M} \\
&= \exp \text{Tr } \ln (I - G) \\
&= \exp \text{Tr} \left[ -\sum_{n=1}^{\infty} \frac{1}{n} G^n \right]
\end{align*}
\]

thus we find:

\[ Z(\varphi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr } G^n \]

where

\[
\text{Tr } G^n = g^n \int d^4x_1 \ldots d^4x_n \Delta(x_1-x_2)\varphi(x_2) \ldots \Delta(x_n-x_1)\varphi(x_1)
\]

\[
Z(\varphi) = \int D\chi^\dagger D\chi \ e^{i \int d^4x \mathcal{L}}
\]

\[
\tilde{M} = I - G
\]

\[
I(x,y) = \delta^4(x-y)
\]

\[
G(x,y) = g\Delta(x-y)\varphi(y)
\]
we found:

\[ Z(\varphi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \ G^n \]

where

\[ \text{Tr} \ G^n = g^n \int d^4x_1 \ldots d^4x_n \Delta(x_1-x_2)\varphi(x_2) \ldots \Delta(x_n-x_1)\varphi(x_1) \]

It represents a connected diagram with \( n \) insertions of a vertex:

\[ \mathcal{L} = -\partial^\mu \chi^\dagger \partial_\mu \chi - m^2 \chi^\dagger \chi + g\varphi \chi^\dagger \chi \]

\[ \frac{1}{i} \Delta(x_1-x_2) \]

and the path integral is given by

\[ Z(\varphi) = \exp i\Gamma(\varphi) \]

\[ i\Gamma(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \ G^n \]

sum of all connected diagrams
Consider now a theory of a Dirac field:

\[ \mathcal{L} = i \overline{\psi} \partial \psi - m \overline{\psi} \psi + g \overline{\psi} \psi \]

we define the path integral:

\[ Z(\varphi) = \int D\overline{\psi} D\psi e^{i \int d^4x \mathcal{L}} \]

we fix \( Z(0) = 1 \)

for the gaussian of \( n \) complex Grassmann variables we found:

\[ \int d^n\overline{\psi} d^n\psi \exp \left( -i \overline{\psi}_i M_{ij} \psi_j \right) \propto \det M \]

in the continuum limit the “matrix” in this case is:

\[ M_{\alpha\beta}(x, y) = [-i \partial_x + m - g \varphi(x)]_{\alpha\beta} \delta^4(x - y) \]

we need to calculate the determinant of this “matrix”
\[ M(x, y) = [-\partial^2_x + m^2 - g\varphi(x)]\delta^4(x - y) \]

we can write it as a product of two matrices:

\[ M = M_0\widetilde{M} \]

\[ M_{\alpha\gamma}(x, z) = \int d^4y \, M_{0\alpha\beta}(x, y)\widetilde{M}_{\beta\gamma}(y, z) \]

where

\[ M_{0\alpha\beta}(x, y) = (-i\partial_x + m)_{\alpha\beta}\delta^4(x - y), \]

\[ \widetilde{M}_{\beta\gamma}(y, z) = \delta_{\beta\gamma}\delta^4(y - z) - gS_{\beta\gamma}(y - z)\varphi(z) \]

\[ (-i\partial_y + m)_{\alpha\beta}S_{\beta\gamma}(y - z) = \delta_{\alpha\gamma}\delta^4(y - z) \]

now we can calculate the determinant:

\[ \det M = \det M_0 \det \widetilde{M} \]

\[ \widetilde{M} = I - G \]

\[ I_{\alpha\beta}(x, y) = \delta_{\alpha\beta}\delta^4(x - y) \]

\[ G_{\alpha\beta}(x, y) = gS_{\alpha\beta}(x - y)\varphi(y) \]

independent of the background field (can be absorbed into the overall normalization)

thus:

\[ Z(\varphi) = \det \widetilde{M} \]
the path integral is given by:

\[ Z(\varphi) = \det \widetilde{M} \]

using \( \det A = \exp \text{Tr} \ln A \) we get:

\[
\det \widetilde{M} = \exp \text{Tr} \ln \widetilde{M} \\
= \exp \text{Tr} \ln(I - G) \\
= \exp \text{Tr} \left[ - \sum_{n=1}^{\infty} \frac{1}{n} G^n \right]
\]

thus we find:

\[ Z(\varphi) = \exp - \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} G^n \]

where

\[ \text{Tr} G^n = g^n \int d^4 x_1 \ldots d^4 x_n \text{tr} S(x_1 - x_2) \varphi(x_2) \ldots S(x_n - x_1) \varphi(x_1) \]
we found:
\[ Z(\varphi) = \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \, G^n \right) \]

where
\[ \text{Tr} \, G^n = g^n \int d^4x_1 \ldots d^4x_n \text{tr} \, S(x_1-x_2)\varphi(x_2) \ldots S(x_n-x_1)\varphi(x_1) \]

It represents a connected diagram with \( n \) insertions of a vertex:
\[ \mathcal{L} = i\bar{\Psi} \partial \Psi - m\bar{\Psi}\Psi + g\varphi\bar{\Psi}\Psi \]

and the path integral is given by
\[ Z(\varphi) = \exp i\Gamma(\varphi) \quad i\Gamma(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \, G^n \]

Note the trace and \(-1\) for fermion loop!
Quantum electrodynamics (QED)

Quantum electrodynamics is a theory of photons interacting with the electrons and positrons of a Dirac field:

\[ \mathcal{L} = -\frac{1}{4} F^\mu\nu F_{\mu\nu} + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi + e \bar{\Psi} \gamma^\mu \gamma^5 \Psi A_\mu \]

\[ e = -0.302822 \]
\[ \alpha = e^2 / 4\pi = 1 / 137.036 \]

Noether current of the lagrangian for a free Dirac field

\[ j^\mu(x) = e \bar{\Psi}(x) \gamma^\mu \Psi(x) \]

\[ \partial_\mu j^\mu(x) = \delta \mathcal{L}(x) - \frac{\delta S}{\delta \varphi_a(x)} \delta \varphi_a(x) \]

we want the current to be conserved and so we need to enlarge the gauge transformation also to the Dirac field:

\[ A^\mu(x) \rightarrow A^\mu(x) - \partial^\mu \Gamma(x) \, , \]

\[ \Psi(x) \rightarrow \exp[-ie\Gamma(x)] \Psi(x) \, , \]

\[ \bar{\Psi}(x) \rightarrow \exp[+ie\Gamma(x)] \bar{\Psi}(x) \, . \]

global symmetry is promoted into local

\[ \Psi \rightarrow e^{-i\alpha} \Psi \]

\[ \bar{\Psi} \rightarrow e^{+i\alpha} \bar{\Psi} \]

symmetry of the lagrangian and so the current is conserved no matter if equations of motion are satisfied
We can write the QED lagrangian as:

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \overline{\Psi} \slashed{D} \Psi - m \overline{\Psi} \Psi \]

\[ D_\mu \equiv \partial_\mu - ieA_\mu \]

(covariant derivative
(the covariant derivative of a field transforms as the field itself)

\[ \Psi(x) \rightarrow \exp[-ie\Gamma(x)]\Psi(x) \]

\[ D_\mu \Psi(x) \rightarrow \exp[-ie\Gamma(x)]D_\mu \Psi(x) \]

and so the lagrangian is manifestly gauge invariant!

Proof:

\[ D_\mu \Psi \rightarrow \left( \partial_\mu - ie[A_\mu - \partial_\mu \Gamma] \right) \left( \exp[-ie\Gamma]\Psi \right) \]

\[ = \exp[-ie\Gamma] \left( \partial_\mu \Psi - ie(\partial_\mu \Gamma)\Psi - ie[A_\mu - \partial_\mu \Gamma]\Psi \right) \]

\[ = \exp[-ie\Gamma] \left( \partial_\mu - ieA_\mu \right) \Psi \]

\[ = \exp[-ie\Gamma] D_\mu \Psi . \]
We can also define the transformation rule for $D$:

$$D_\mu \rightarrow e^{-ie\Gamma}D_\mu e^{+ie\Gamma}$$

then

$$D_\mu \Psi \rightarrow (e^{-ie\Gamma}D_\mu e^{+ie\Gamma})(e^{-ie\Gamma}\Psi) = e^{-ie\Gamma}D_\mu \Psi,$$

as required.

Now we can express the field strength in terms of $D$'s:

$$D_\mu \equiv \partial_\mu - ieA_\mu$$

$$[D^\mu, D^\nu]\Psi(x) = -ieF^{\mu\nu}(x)\Psi(x)$$

$$F^{\mu\nu} = \frac{i}{e}[D^\mu, D^\nu]$$
Then we simply see:

\[ F^\mu\nu \rightarrow \frac{i}{e} \left[ e^{-ie\Gamma} D^\mu e^{+ie\Gamma}, e^{-ie\Gamma} D^\nu e^{+ie\Gamma} \right] \]

\[ = e^{-ie\Gamma} \left( \frac{i}{e} [D^\mu, D^\nu] \right) e^{+ie\Gamma} \]

\[ = e^{-ie\Gamma} F^\mu\nu e^{+ie\Gamma} \]

\[ = F^\mu\nu. \]

The field strength is gauge invariant as we already knew.
Nonabelian symmetries

Let’s generalize the theory of two real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_1 \partial_\mu \varphi_1 - \frac{1}{2} \partial^\mu \varphi_2 \partial_\mu \varphi_2 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{1}{16} \lambda (\varphi_1^2 + \varphi_2^2)^2 \]

to the case of \( N \) real scalar fields:

\[ \mathcal{L} = -\frac{1}{2} \partial^\mu \varphi_i \partial_\mu \varphi_i - \frac{1}{2} m^2 \varphi_i \varphi_i - \frac{1}{16} \lambda (\varphi_i \varphi_i)^2 \]

the lagrangian is clearly invariant under the SO(\( N \)) transformation:

\[ \varphi_i(x) \rightarrow R_{ij} \varphi_j(x) \]

\[ R^T = R^{-1} \]

\[ \det R = +1 \]

lagrangian has also the \( Z_2 \) symmetry, \( \varphi_i(x) \rightarrow -\varphi_i(x) \), that enlarges SO(\( N \)) to O(\( N \))
infinitesimal $\text{SO}(N)$ transformation:

\[ R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2) \]

\[ R^T = R^{-1} \]
\[ R_{ij}^T = \delta_{ij} + \theta_{ij} \]
\[ R_{ij}^{-1} = \delta_{ij} - \theta_{ij} \]

there are $\frac{1}{2}N(N-1)$ linearly independent real antisymmetric matrices, and we can write:

\[ \theta_{jk} = -i\theta^a(T^a)_{jk} \]

or

\[ R = e^{-i\theta^a T^a} \].

The commutator of two generators is a lin. comb. of generators:

\[ [T^a, T^b] = i f^{abc} T^c \]

we choose normalization:

\[ \text{Tr}(T^a T^b) = 2\delta^{ab} \]

\[ f^{abd} = -\frac{1}{2} i \text{Tr}([T^a, T^b]T^d) \]

structure constants of the $\text{SO}(N)$ group
e.g. $SO(3)$:

$$(T^a)_{ij} = -i\epsilon^{aij}$$

$$[T^a, T^b] = i\epsilon^{abc} T^c$$

$\epsilon^{123} = +1$

Levi-Civita symbol
consider now a theory of \( N \) complex scalar fields:

\[
L = -\partial^\mu \varphi_i^\dagger \partial_\mu \varphi_i - m^2 \varphi_i^\dagger \varphi_i - \frac{1}{4} \lambda (\varphi_i^\dagger \varphi_i)^2
\]

the lagrangian is clearly invariant under the \( U(N) \) transformation:

\[
\varphi_i(x) \rightarrow U_{ij} \varphi_j(x)
\]

\[
U^\dagger = U^{-1}
\]

we can always write \( U_{ij} = e^{-i\theta \tilde{U}_{ij}} \) so that \( \det \tilde{U} = +1 \).

actually, the lagrangian has larger symmetry, \( SO(2N) \):

\[
\varphi_j = (\varphi_{j1} + i\varphi_{j2})/\sqrt{2}
\]

\[
\varphi_j^\dagger \varphi_j = \frac{1}{2} (\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2)
\]

\( U(N) = U(1) \times SU(N) \)

\( SU(N) \) - group of special unitary \( NxN \) matrices

\( \text{group of unitary } NxN \text{ matrices} \)
infinitesimal $\text{SU}(N)$ transformation:

$$\tilde{U}_{ij} = \delta_{ij} - i\theta^a(T^a)_{ij} + O(\theta^2)$$

or $\tilde{U} = e^{-i\theta^a T^a}$.

there are $N^2 - 1$ linearly independent traceless hermitian matrices:

- e.g. $\text{SU}(2)$ - 3 Pauli matrices
- $\text{SU}(3)$ - 8 Gell-Mann matrices

the structure coefficients are $f^{abc} = 2\epsilon^{abc}$, the same as for $\text{SO}(3)$.
Nonabelian gauge theory

Consider a theory of $N$ scalar or spinor fields that is invariant under:

$$\phi_i(x) \rightarrow U_{ij} \phi_j(x)$$

for $SU(N)$: a special unitary $N \times N$ matrix
for $SO(N)$: a special orthogonal $N \times N$ matrix

In the case of $U(1)$ we could promote the symmetry to local symmetry but we had to include a gauge field $A_\mu(x)$ and promote ordinary derivative to covariant derivative:

$$\phi(x) \rightarrow U(x) \phi(x) \quad U(x) = \exp[-ie\Gamma(x)] \quad D_\mu = \partial_\mu - ieA_\mu$$

$$D_\mu \rightarrow U(x) D_\mu U^\dagger(x)$$

then the kinetic terms and mass terms: $-(D_\mu \varphi)^\dagger D^\mu \varphi$, $m^2 \varphi^\dagger \varphi$, $i\overline{\Psi} D \Psi$ and $m \overline{\Psi} \Psi$, are gauge invariant. The transformation of covariant derivative in general implies that the gauge field transforms as:

$$A_\mu(x) \rightarrow U(x) A_\mu(x) U^\dagger(x) + \frac{i}{e} U(x) \partial_\mu U^\dagger(x)$$

for $U(1)$: $A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \Gamma(x)$
Now we can easily generalize this construction for SU(N) or SO(N):

an infinitesimal SU(N) transformation:

\[ U_{jk}(x) = \delta_{jk} - i g \theta^a(x) (T^a)_{jk} + O(\theta^2) \]

\[ \text{generator matrices (hermitian and traceless):} \]
\[ [T^a, T^b] = i f^{abc} T^c \]
\[ \text{structure constants (completely antisymmetric)} \]
\[ \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \]

the SU(N) gauge field is a traceless hermitian N\times N matrix transforming as:

\[ A_\mu(x) \rightarrow U(x) A_\mu(x) U^\dagger(x) + \frac{i}{q} U(x) \partial_\mu U^\dagger(x) \]
\[ U(x) = \exp[-ig \Gamma^a(x) T^a] \]
the covariant derivative is:

\[ D_\mu = \partial_\mu - igA_\mu(x) \]

or acting on a field:

\[(D_\mu \phi)_j(x) = \partial_\mu \phi_j(x) - igA_\mu(x)_{jk} \phi_k(x) \]

using covariant derivative we get a gauge invariant lagrangian

We define the field strength (kinetic term for the gauge field) as:

\[ F_{\mu\nu}(x) \equiv \frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \]

it transforms as:

\[ F_{\mu\nu}(x) \rightarrow U(x)F_{\mu\nu}(x)U^\dagger(x) \]

and so the gauge invariant kinetic term can be written as:

\[ \mathcal{L}_{\text{kin}} = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu}) \]
we can expand the gauge field in terms of the generator matrices:

\[ A_\mu(x) = A_\mu^a(x)T^a \]

that can be inverted:

\[ A_\mu^a(x) = 2 \text{Tr} A_\mu(x)T^a \]

similarly:

\[ F_{\mu\nu}(x) = F_{\mu\nu}^a T^a , \]

\[ F_{\mu\nu}^a(x) = 2 \text{Tr} F_{\mu\nu}T^a . \]

\[ F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu] \]

\[ = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \rightarrow F_{\mu\nu}^c T^c = (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c)T^c - ig A_\mu^a A_\nu^b [T^a , T^b] \]

\[ = (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + gf^{abc} A_\mu^a A_\nu^b)T^c . \]

\[ [T^a , T^b] = if^{abc}T^c \]

thus we have:

\[ F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + gf^{abc} A_\mu^a A_\nu^b \]
the kinetic term can be also written as:

$$L_{\text{kin}} = -\frac{1}{4} F_{\mu\nu}^c F_{\mu\nu}^c$$

$$F_{\mu\nu}^c(x) = F_{\mu\nu}^a T^a$$

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$$

$$F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b$$

Example, quantum chromodynamics - QCD:

$$L = i \bar{\psi}_i \gamma_\mu \psi_j - m_I \bar{\psi}_I \psi_I - \frac{1}{2} \text{Tr}(F_{\mu\nu}^c F_{\mu\nu}^c)$$

flavor index: up, down, strange, charm, top, bottom

$1, \ldots, 8$ gluons (massless spin 1 particles)

color index: 1, 2, 3

in general, scalar and spinor fields can be in different representations of the group, $T^a_R$; gauge invariance requires that the gauge fields transform independently of the representation.
A representation of a group is specified by a set of hermitian matrices that obey:

\[ [T_R^a, T_R^b] = i f^{abc} T_R^c \]

(the original set of \( N \times N \) dimensional matrices for \( SU(N) \) or \( SO(N) \) corresponds to the fundamental representation)

Taking the complex conjugate we see that \( -(T_R^a)^* \) is also a representation!

\[ R \text{ is real if } -(T_R^a)^* = T_R^a \]

or if there is a unitary transformation \( T_R^a \rightarrow U^{-1}T_R^aU \) that makes \( -(T_R^a)^* = T_R^a \)

\[ R \text{ is pseudoreal if it is not real but there is a transformation such that } -(T_R^a)^* = V^{-1}T_R^aV \]

\[ -(1/2 \sigma^a)^* \neq 1/2 \sigma^a \]

\[ -(1/2 \sigma^a)^* = V^{-1}(1/2 \sigma^a)V, \quad V = \sigma_2 \]

\[ R \text{ is complex if } R \text{ is not real or pseudoreal then it is complex conjugate representation } R \overline{R} \text{ is specified by: } T_R^a = -(T_R^a)^* \]
The adjoint representation $\mathbf{A}$:

\[(T^a_A)^{bc} = -i f^{abc}\]

$A$ is a real representation

\[-(T^a_A)^* = T^a_A\]

the dimension of the adjoint representation, $D(A) = \# \text{ of generators} = \text{the dimension of the group}$

to see that $T^a_A$ s satisfy commutation relations we use the Jacobi identity:

\[f^{abcd} f^{dce} + f^{bcde} f^{dae} + f^{cad} f^{dbe} = 0\]

follows from:

\[\text{Tr} T^e \left( [T^a_a, T^b], T^c_c \right) + \left[ [T^b_b, T^c_c], T^a_a \right] + \left[ [T^c_c, T^a_a], T^b_b \right] = 0\]

$[T^a_a, T^b_b] = i f^{abc} T^c_c$

$\text{Tr}(T^a_a T^b_b) = \frac{1}{2} \delta^{ab}$

$[T^a_a, T^c_c] = i f^{acd} T^d_d$
The index of a representation $T(R)$:

$$\text{Tr}(T_R^a T_R^b) = T(R)\delta^{ab}$$

The quadratic Casimir $C(R)$:

$$T_R^a T_R^a = C(R)$$

multiplies the identity matrix
commutes with every generator, homework S-69.2

Useful relation:

$$T(R) D(A) = C(R) D(R)$$

SU(N):

\begin{align*}
T(N) &= \frac{1}{2} \\
T(A) &= N \\
D(A) &= N^2 - 1
\end{align*}

SO(N):

\begin{align*}
T(N) &= 2 \\
T(A) &= 2N - 4 \\
D(A) &= \frac{1}{2}N(N-1)
\end{align*}
A representation is **reducible** if there is a unitary transformation

\[ T_R^a \rightarrow U^{-1}T_R^a U \]

that brings all the generators to the same block diagonal form (with at least two blocks); otherwise it is **irreducible**.

For example, consider a reducible representation \( R \) that can put into two blocks, then \( R \) is a **direct sum representation**:

\[ R = R_1 \oplus R_2 \]

and we have:

\[ D(R_1 \oplus R_2) = D(R_1) + D(R_2) \]

\[ \text{Tr}(T_R^a T_R^b) = T(R) \delta^{ab} \]

\[ T(R_1 \oplus R_2) = T(R_1) + T(R_2) \]
Consider a field that carries two group indices $\varphi_{iI}(x)$:

then the field is in the **direct product representation**:

$$R_1 \otimes R_2$$

The corresponding generator matrix is:

$$(T_{R_1 \otimes R_2}^a)_{iI,jJ} = (T_{R_1}^a)_{iJ} \delta_{IJ} + \delta_{ij} (T_{R_2}^a)_{IJ}$$

and we have:

$$D(R_1 \otimes R_2) = D(R_1)D(R_2)$$

$$T(R_1 \otimes R_2) = T(R_1)D(R_2) + D(R_1)T(R_2)$$

**to prove this we use the fact that** $(T_{R}^a)_{ii} = 0$. 
We will use the following notation for indices of a complex representation:

\[ \varphi_i \quad \text{for} \quad i = 1, 2, \ldots, D(R) \]

Hermitian conjugation changes \( R \) to \( \overline{R} \) and for a field in the conjugate representation we will use the upper index

\[ (\varphi_i)^\dagger = \varphi^\dagger i \]

We write generators as:

\[ (T^a_R)_{i,j} \]

Indices are contracted only if one is up and one is down!

An infinitesimal group transformation of \( \varphi_i \) is:

\[ \varphi_i \rightarrow (1 - i\theta^a T^a_R)_{i,j} \varphi_j \]

\[ = \varphi_i - i\theta^a (T^a_R)_{i,j} \varphi_j \]
generator matrices for $\mathbf{R}$ are then given by
\[
(T^a_R)_i^j = -(T^a_R)_j^i \quad \quad T^a_R = -(T^a_R)^* \\
\text{we trade complex conjugation for transposition}
\]
and an infinitesimal group transformation of $\varphi^{\dagger i}$ is:
\[
\varphi^{\dagger i} \rightarrow (1 - i\theta^a T^a_R)_j^i \varphi^{\dagger j} = \varphi^{\dagger i} - i\theta^a (T^a_R)_j^i \varphi^{\dagger j}
\]
\[
(T^a_R)_j^i = -(T^a_R)_i^j \rightarrow \varphi^{\dagger i} + i\theta^a (T^a_R)_j^i \varphi^{\dagger j}
\]
$\varphi_i \rightarrow (1 - i\theta^a T^a_R)_i^j \varphi_j = \varphi_i - i\theta^a (T^a_R)_i^j \varphi_j$

is invariant!
Consider the Kronecker delta symbol

\[
\delta_{ij} \rightarrow (1 + i\theta^a T^a_R)_i^k (1 + i\theta^a T^a_R)_j^l \delta_{kl}
\]

\[
= (1 + i\theta^a T^a_R)_i^k \delta_{kl} (1 - i\theta^a T^a_R)_j^l
\]

\[
= \delta_{ij} + O(\theta^2).
\]

is an invariant symbol of the group!

this means that the product of the representations $R$ and $\overline{R}$ must contain the singlet representation $1$, specified by $T^a_1 = 0$.

Thus we can write:

\[
R \otimes \overline{R} = 1 \oplus \ldots
\]
Another invariant symbol:

\[
(T^b_R)_i^j \rightarrow (1 - i \theta^a T^a_R)_i^k (1 - i \theta^a T^a_R)_j^l (1 - i \theta^a T^a_A)^{bc} (T^c_R)_k^l
\]

\[
R = (T^b_R)_i^j - i \theta^a [(T^a_R)_i^k (T^b_R)_k^j + (T^a_R)_j^l (T^b_R)_l^i + (T^a_A)^{bc} (T^c_R)_i^j]
\]

\[
+ O(\theta^2).
\]

\[
(T^a_R)_i^j = -(T^a_R)_j^i \quad (T^a_A)^{bc} = -i f^{abc}
\]

\[
[...] = (T^a_R)_i^k (T^b_R)_k^j - (T^a_R)_l^j (T^b_R)_l^i - i f^{abc} (T^c_R)_i^j
\]

\[
= (T^a_R T^b_R)_i^j - (T^b_R T^a_R)_i^j - i f^{abc} (T^c_R)_i^j
\]

\[
= 0,
\]

this implies that:

\[
R \otimes \overline{R} \otimes A = 1 \oplus \ldots
\]

must contain the singlet representation!
multiplying by $A$ we find:

$$R \otimes \overline{R} \otimes A = 1 \oplus \ldots$$

$$R \otimes \overline{R} = A \oplus \ldots$$

combining it with a previous result we get

$$R \otimes \overline{R} = 1 \oplus \ldots$$

the product of a representation with its complex conjugate is always reducible into a sum that contains at least the singlet and the adjoint representations!

For the fundamental representation $N$ of $SU(N)$ we have:

$$N \otimes \overline{N} = 1 \oplus A$$

$$D(1) = 1$$

$$D(N) = D(\overline{N}) = N$$

$$D(A) = N^2 - 1$$

(no room for anything else)
Consider a real representation $\mathbf{R}$:

\[
\mathbf{R} \otimes \bar{\mathbf{R}} = 1 \oplus \mathbf{A} \oplus \ldots
\]

implies the existence of an invariant symbol with two R indices

\[
\delta_{ij} \rightarrow (1 - i\theta^a T^a_R)_i^k (1 - i\theta^a T^a_R)_j^l \delta_{kl}
\]

\[
= \delta_{ij} - i\theta^a [(T^a_R)_{ij} + (T^a_R)_{ji}] + O(\theta^2)
\]

For the fundamental representation $\mathbf{N}$ of $\text{SO}(N)$ we have:

\[
\mathbf{N} \otimes \mathbf{N} = 1_\mathbf{S} \oplus \mathbf{A}_\mathbf{A} \oplus \mathbf{S}_\mathbf{S}
\]

\[
D(1) = 1 \\
D(\mathbf{N}) = N \\
D(\mathbf{A}) = \frac{1}{2}N(N-1) \\
D(\mathbf{S}) = \frac{1}{2}N(N+1)-1
\]

$\delta_{ij} = \delta_{ji}$

corresponds to a field with a symmetric traceless pair of fundamental indices

$\varphi_{ij} = \varphi_{ji}$
Consider now a pseudoreal representation:

$$R \otimes R = 1 \oplus A \oplus \ldots$$

still holds but the Kronecker delta is not the corresponding invariant symbol:

$$\delta_{ij} \rightarrow (1 - i\theta^a T_R^a)^i (1 - i\theta^a T_R^a)^j \delta_{kl}$$

$$= \delta_{ij} - i\theta^a [(T_R^a)_{ij} + (T_R^a)_{ji}] + O(\theta^2)$$

the only alternative is to have the singlet appear in the antisymmetric part of the product. For SU(N) another invariant symbol is the Levi-Civita symbol with \(N\) indices:

$$\varepsilon_{i_1\ldots i_N} \rightarrow U_{i_1 j_1} \ldots U_{i_N j_N} \varepsilon_{j_1 \ldots j_N}$$

$$= (\det U) \varepsilon_{i_1 \ldots i_N}.$$

Similarly for \(\varepsilon^{i_1 \ldots i_N} \).

For SU(2):

$$2 \otimes 2 = 1_A \oplus 3_S$$

we can use \(\varepsilon^{ij}\) and \(\varepsilon_{ij}\) to raise and lower SU(2) indices; if \(\varphi_i\) is in the 2 representation, then we can get a field in the \(\overline{2}\) representation by raising the index: \(\varphi^i = \varepsilon^{ij} \varphi_j\).
Another invariant symbol of interest is \( f^{abc} \):

\[
(T^a_A)^{bc} = -i f^{abc}
\]

generator matrices in any rep. are invariant, or

\[
T(R)f^{abc} = -i \text{Tr}(T^a_R T^b_R T^c_R)
\]

the right-hand side is obviously invariant.

Very important invariant symbol is the anomaly coefficient of the rep.:

\[
A(R) a^{abc} \equiv \frac{1}{2} \text{Tr}(T^a_R \{T^b_R, T^c_R\})
\]

is completely symmetric normalized so that \( A(N) = 1 \) for SU(N) with \( N \geq 3 \).

Since \((T^a_R)^{ij} = -(T^a_R)_{ji}\) we have:

\[
A(R) = -A(R)
\]

for real or pseudoreal representations \( A(R) = 0 \).

e.g. for SU(2), all representation are real or pseudoreal and \( A(R) = 0 \) for all of them

we also have:

\[
A(R_1 \oplus R_2) = A(R_1) + A(R_2)
\]

\[
A(R_1 \otimes R_2) = A(R_1) D(R_2) + D(R_1) A(R_2)
\]