The magnetic moment of the electron

based on S-64

For the vertex function we have found (at one loop):

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^{\mu}(p',p)u_s(\mathbf{p}) = e\overline{u}'\left[F_1(q^2)\gamma^{\mu} - \frac{i}{m}F_2(q^2)S^{\mu\nu}q_{\nu}\right]u$$

momentum of an incoming photon

$q = p' - p$

where the form factors for $q^2 = 0$ are:

$$F_1(0) = 1 \quad \text{exactly,}$$

$$F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2) \; .$$

We can obtain the vertex function from the quantum action:

$$\Gamma = \int d^4x \, \left[eF_1(0)\overline{\Psi}\not{A}\Psi + \frac{e}{2m}F_2(0)F_{\mu\nu}\overline{\Psi}S^{\mu\nu}\Psi + \ldots\right]$$

incoming photon

$A_{\mu} \sim \varepsilon^*_{\mu}e^{iqx}$

$F_{\mu\nu} \sim i(q_{\mu}\varepsilon^*_{\nu} - q_{\nu}\varepsilon^*_{\mu})e^{iqx}$

We define the magnetic moment in the following way:

we take the photon field to be a classical field that corresponds to a constant magnetic field in the z direction:

$$A^0 = 0 \qquad \mathbf{A} = (0, Bx, 0) \quad \longrightarrow \quad F_{12} = -F_{21} = B$$

all other components are zero

the magnetic moment of a normalized quantum state with definite angular momentum in the $\mathbf{B}$ direction is defined as:

$$\mu B \equiv -\langle e | H_1 | e \rangle$$

$$\Gamma = \int d^4x \left[eF_1(0)\overline{\Psi}\not{A}\Psi + \frac{e}{2m}F_2(0)F_{\mu\nu}\overline{\Psi}S^{\mu\nu}\Psi + \ldots \right]$$

$F_1(0) = 1$ exactly,

$F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2)$.

$$H_1 \equiv -eB \int d^3x\, \overline{\Psi} \left[x\gamma^2 + \frac{\alpha}{2\pi m} S^{12} \right] \Psi$$

normalized state of an electron at rest, with spin along the z axis:

$$|e\rangle \equiv \int \widetilde{dp}\, f(\mathbf{p}) b_+^\dagger(\mathbf{p}) |0\rangle$$

$$f(\mathbf{p}) \sim \exp(-a^2\mathbf{p}^2/2)$$

the wave packet (rotationally invariant) and sharply peaked at $\mathbf{p} = 0$, with $a \ll 1/m$

$$\int \widetilde{dp}\, |f(\mathbf{p})|^2 = 1$$

$$\langle e | e \rangle = 1$$

$$\mu B \equiv -\langle e|H_1|e\rangle$$

$$H_1 \equiv -eB\int d^3x\,\overline{\Psi}\left[x\gamma^2 + \frac{\alpha}{2\pi m}S^{12}\right]\Psi$$

Let's evaluate it now:

$$|e\rangle \equiv \int \widetilde{dp}\, f(\mathbf{p}) b_+^\dagger(\mathbf{p})|0\rangle$$

$$\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\left[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + b_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}\right],$$

$$\overline{\Psi}(y) = \sum_{s'=\pm}\int \widetilde{dp}'\left[b_{s'}^\dagger(\mathbf{p}')\overline{u}_{s'}(\mathbf{p}')e^{-ip'y} + b_{s'}(\mathbf{p}')\overline{v}_{s'}(\mathbf{p}')e^{ip'y}\right]$$

$$\langle 0|b_+(\mathbf{p}')\overline{\Psi}_\alpha(x)\Psi_\beta(x)b_+^\dagger(\mathbf{p})|0\rangle = \overline{u}_+(\mathbf{p}')_\alpha u_+(\mathbf{p})_\beta\, e^{i(p-p')x}$$

$$\langle e|H_1|e\rangle = -eB\int \widetilde{dp}\,\widetilde{dp}'\, d^3x\, e^{i(p-p')x}$$
$$\times f^*(\mathbf{p}')\overline{u}_+(\mathbf{p}')\left[x\gamma^2 + \frac{\alpha}{2\pi m}S^{12}\right]u_+(\mathbf{p})f(\mathbf{p})$$

$$(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})$$

$$-i\partial_{p_1}\, e^{i(p-p')x}$$

and integrate by parts (the surface term will vanish thanks to wave packets)

we find:

$$\langle e|H_1|e\rangle = -eB\int \frac{\widetilde{dp}}{2\omega}\, f^*(\mathbf{p})\overline{u}_+(\mathbf{p})\left[i\gamma^2\partial_{p_1} + \frac{\alpha}{2\pi m}S^{12}\right]u_+(\mathbf{p})f(\mathbf{p})$$

$$\langle e|H_1|e\rangle = -eB\int \widetilde{\frac{dp}{2\omega}}\, f^*(\mathbf{p})\overline{u}_+(\mathbf{p})\left[i\gamma^2\partial_{p_1} + \frac{\alpha}{2\pi m}S^{12}\right]u_+(\mathbf{p})f(\mathbf{p})$$

f is rotationally invariant and so the derivative is odd in p_1 ; in addition it is also odd in p_2 due to $\overline{u}_+(\mathbf{p})\gamma^i u_+(\mathbf{p}) = 2p^i$;

thus this term doesn't contribute!

$$\partial_{p_1}u_+(\mathbf{p})\Big|_{\mathbf{p}=0} = \frac{i}{m}K^1 u_+(\mathbf{0})$$
$$= -\frac{1}{2m}\gamma^1\gamma^0 u_+(\mathbf{0})$$
$$= -\frac{1}{2m}\gamma^1 u_+(\mathbf{0}) ,$$

$\gamma^0 u_s(\mathbf{0}) = u_s(\mathbf{0})$

$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\cdot\mathbf{K})u_s(\mathbf{0})$

$K^j = S^{j0} = \frac{i}{2}\gamma^j\gamma^0$

$\eta = \sinh^{-1}(|\mathbf{p}|/m)$

$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$

expand for small p, take derivative and set $\mathbf{p} = 0$

$$\overline{u}_+(\mathbf{p})i\gamma^2\partial_{p_1}u_+(\mathbf{p})\Big|_{\mathbf{p}=0} = \overline{u}_+(\mathbf{0})\frac{-i}{2m}\gamma^2\gamma^1 u_+(\mathbf{0})$$
$$= \frac{1}{m}\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})$$

$$\langle e|H_1|e\rangle = -eB\int \frac{\widetilde{dp}}{2\omega}\, f^*(\mathbf{p})\overline{u}_+(\mathbf{p})\left[i\gamma^2\partial_{p_1} + \frac{\alpha}{2\pi m}S^{12}\right]u_+(\mathbf{p})f(\mathbf{p})$$

$$\overline{u}_+(\mathbf{p})i\gamma^2\partial_{p_1}u_+(\mathbf{p})\Big|_{\mathbf{p}=0} = \overline{u}_+(\mathbf{0})\frac{-i}{2m}\gamma^2\gamma^1 u_+(\mathbf{0})$$
$$= \frac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})$$

we find:

$$\langle e|H_1|e\rangle = -eB\int \frac{\widetilde{dp}}{2\omega}\,|f(\mathbf{p})|^2\left(1+\frac{\alpha}{2\pi}\right)\frac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})$$

$$= -\frac{eB}{2m^2}\left(1+\frac{\alpha}{2\pi}\right)\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})\ .$$

$\int \widetilde{dp}\,|f(\mathbf{p})|^2 = 1$

$S^{12}u_\pm(\mathbf{0}) = \pm\frac{1}{2}u_\pm(\mathbf{0})$

$\overline{u}_\pm(\mathbf{0})u_\pm(\mathbf{0}) = 2m$

$$\langle e|H_1|e\rangle = -\frac{eB}{2m}\left(1+\frac{\alpha}{2\pi}\right)$$

$$\mu B \equiv -\langle e|H_1|e\rangle$$

$$\langle e|H_1|e\rangle = -\frac{eB}{2m}\left(1+\frac{\alpha}{2\pi}\right)$$

we find that the magnetic moment of the electron is:

$$\mu = g\,\frac{1}{2}\,\frac{e}{2m}$$

Landé g factor

Bohr magneton

$$g = 2\left(1+\frac{\alpha}{2\pi}+O(\alpha^2)\right)$$

corrections of order α^4 were calculated!

anomalous magnetic moment of the electron:

$$a = \frac{g-2}{2}$$

$$a = \frac{\alpha}{2\pi} \approx .0011614$$

exp. value is: .0011659208 (6)

anomalous magnetic moment of the electron:	α^{-1} = 137.035 999 070 (98)
Atom-recoil measurements:	α^{-1} = 137.035 998 78 (91)
Neutron Compton wavelength:	α^{-1} = 137.036 010 1 (5 4)
Hyperfine splitting in muonium:	α^{-1} = 137.035 994 (18)
Lamb shift:	α^{-1} = 137.036 8 (7)
Positronium:	α^{-1} = 137.034 (16)
...	
Condensed matter systems:	
quantum Hall effect:	α^{-1} = 137.035 997 9 (3 2)
AC Josephson effect:	α^{-1} = 137.035 977 0 (7 7)

Loop corrections in scalar electrodynamics

based on S-65

Let's outline the calculation of loop corrections in scalar electrodynamics:

(in the $\overline{\text{MS}}$ scheme)

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \ ,$$

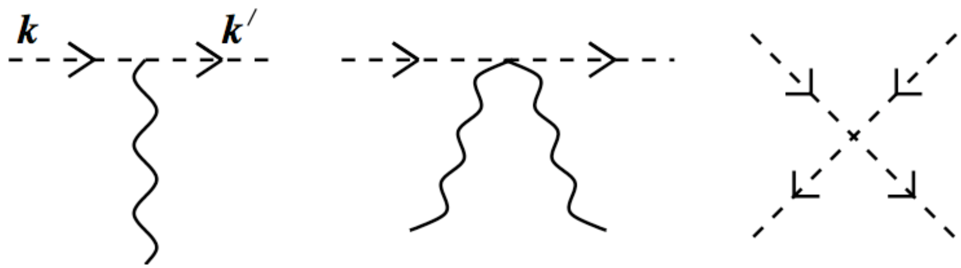
$$\mathcal{L}_0 = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \ ,$$

$$\mathcal{L}_1 = i Z_1 e [\varphi^\dagger \partial^\mu \varphi - (\partial^\mu \varphi^\dagger)\varphi] A_\mu - Z_4 e^2 \varphi^\dagger \varphi A^\mu A_\mu - \tfrac{1}{4} Z_\lambda \lambda (\varphi^\dagger \varphi)^2 + \mathcal{L}_{\text{ct}} \ ,$$

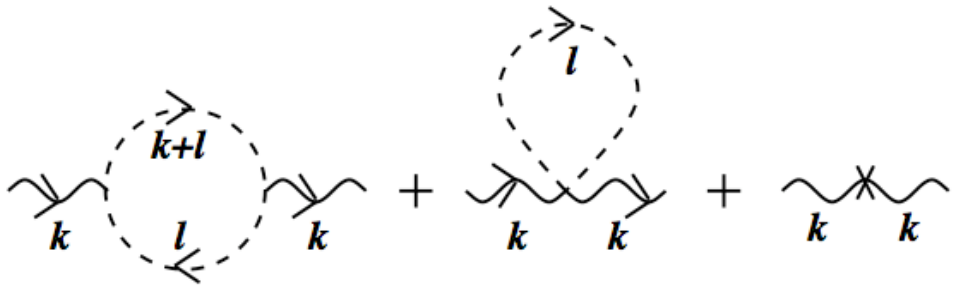
$$\mathcal{L}_{\text{ct}} = -(Z_2 - 1)\partial^\mu \varphi^\dagger \partial_\mu \varphi - (Z_m - 1) m^2 \varphi^\dagger \varphi - \tfrac{1}{4}(Z_3 - 1) F^{\mu\nu} F_{\mu\nu}$$

new vertices:

$ie(k+k')_\mu$ $\qquad -2ie^2 g_{\mu\nu}$ $\qquad -i\lambda$



The one-loop corrections to the photon propagator:



$$i\Pi^{\mu\nu}(k) = (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{(2\ell+k)^\mu (2\ell+k)^\nu}{((\ell+k)^2+m^2)(\ell^2+m^2)}$$

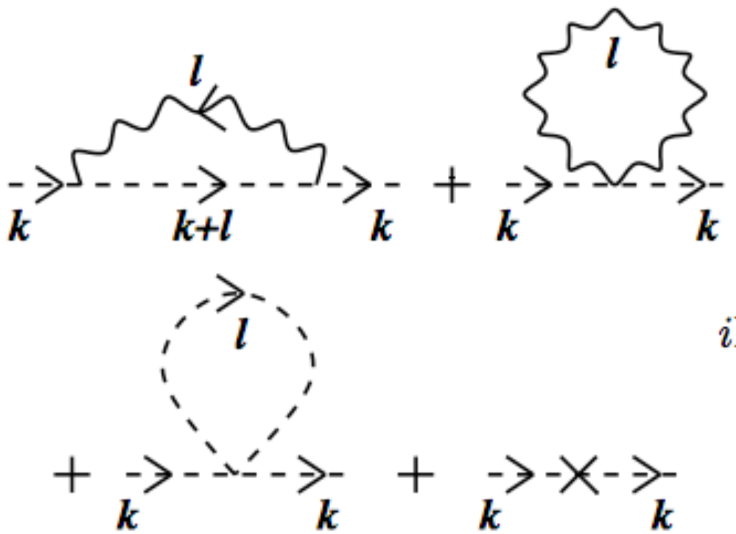
$$+ (-2iZ_4)e^2 g^{\mu\nu} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+m^2}$$

$$- i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\mu) + \ldots\ ,$$

in the $\overline{\mathrm{MS}}$ scheme we find:

$$Z_3 = 1 - \frac{e^2}{24\pi^2}\frac{1}{\varepsilon} + \ldots$$

The one-loop corrections to the scalar propagator:



$$
\begin{aligned}
i\Pi_\varphi(k^2) = {} & (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{P_{\mu\nu}(\ell)(\ell+2k)^\mu(\ell+2k)^\nu}{\ell^2((\ell+k)^2+m^2)} \\
& + (-2iZ_4 e^2 g^{\mu\nu})\left(\tfrac{1}{i}\right) \int \frac{d^4\ell}{(2\pi)^4} \frac{P_{\mu\nu}(\ell)}{\ell^2+m_\gamma^2} \\
& + (-i\lambda)\left(\tfrac{1}{i}\right) \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+m^2} \\
& - i(Z_2-1)k^2 - i(Z_m-1)m^2 + \ldots\ .
\end{aligned}
$$

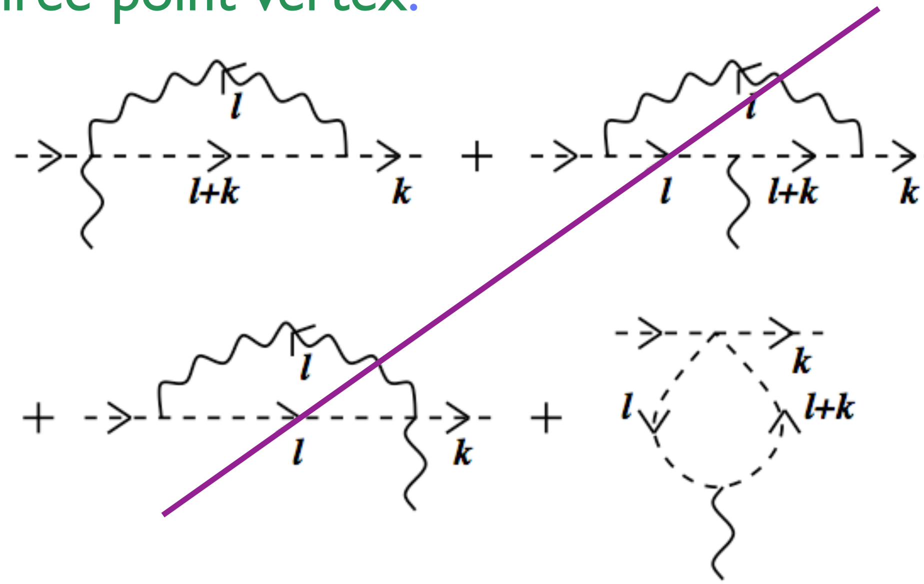
in the $\overline{\mathrm{MS}}$ scheme we find:

$$Z_2 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon} + \ldots$$

$$Z_m = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\varepsilon} + \ldots$$

The one-loop corrections to the three-point vertex:

it is convenient to work in the Lorentz gauge and choosing the momentum of incoming scalar = 0 if we are interested in the divergent part only



$$i\mathbf{V}_3^\mu(k,0) = ieZ_1k^\mu$$

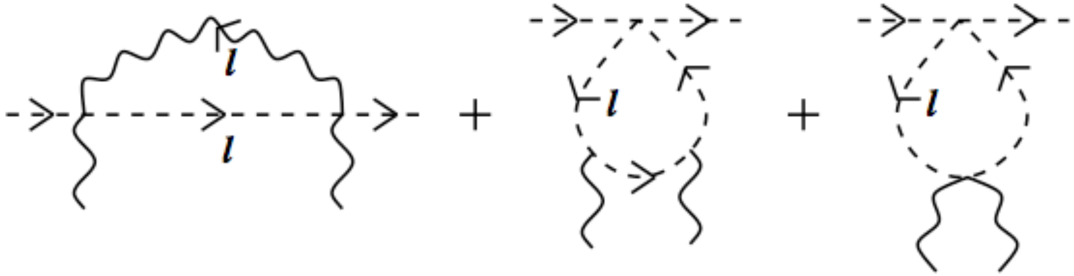
$$+ (iZ_1e)(-2iZ_4e^2g^{\mu\nu})\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{P_{\nu\rho}(\ell)(\ell+2k)^\rho}{\ell^2((\ell+k)^2+m^2)}$$

$$+ (-iZ_\lambda\lambda)(iZ_1e)\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{(2\ell+k)^\mu}{(\ell^2+m^2)((\ell+k)^2+m^2)}$$

in the $\overline{\mathrm{MS}}$ scheme we find:

$$Z_1 = 1 + \frac{3e^2}{8\pi^2}\,\frac{1}{\varepsilon} + \ldots$$

The one-loop corrections to the scalar-scalar-photon-photon vertex:



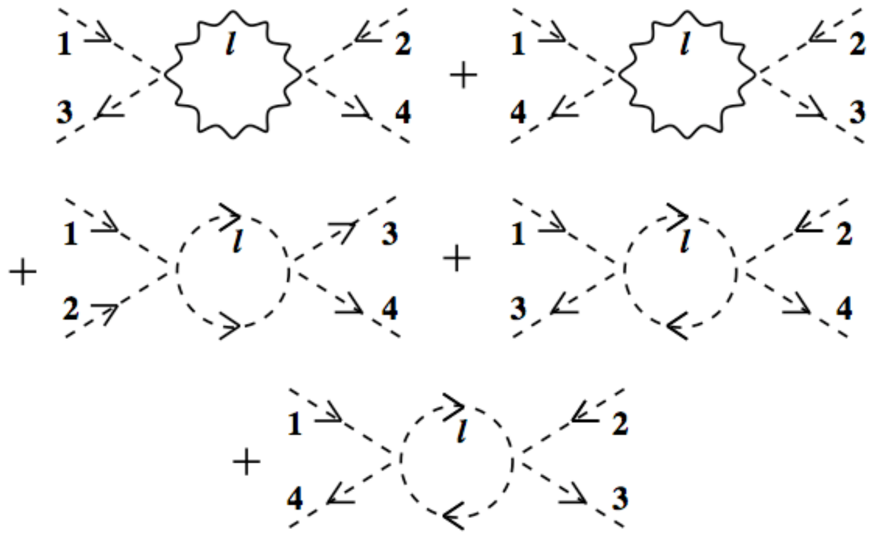
plus many diagrams that do not contribute in the Lorentz gauge with external momenta = 0

$$
\begin{aligned}
i\mathbf{V}_4^{\mu\nu}(0,0,0) = & -2iZ_4 e^2 g^{\mu\nu} \\
& + (-2iZ_4e^2)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{g^{\mu\rho}P_{\rho\sigma}(\ell)g^{\sigma\nu}}{\ell^2(\ell^2+m^2)} + (\mu\leftrightarrow\nu) \\
& + (iZ_1e)^2(-iZ_\lambda\lambda)\left(\tfrac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{(2\ell)^\mu(2\ell)^\nu}{(\ell^2+m^2)^3} + (\mu\leftrightarrow\nu) \\
& + (-iZ_\lambda\lambda)(-2iZ_4e^2g^{\mu\nu})\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2+m^2)^2}
\end{aligned}
$$

in the $\overline{\mathrm{MS}}$ scheme we find:

$$
Z_4 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon} + \ldots
$$

The one-loop corrections to the four-scalar vertex:



plus many diagrams that do not contribute in the Lorentz gauge with external momenta = 0

$$
\begin{aligned}
i\mathbf{V}_{4\varphi}(0,0,0) &= -iZ_\lambda\lambda \\
&\quad + \left(\tfrac{1}{2}+\tfrac{1}{2}\right)(-2iZ_4e^2)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{g^{\mu\nu}P_{\nu\rho}(\ell)g^{\rho\sigma}P_{\rho\mu}(\ell)}{(\ell^2+m_\gamma^2)^2} \\
&\quad + \left(\tfrac{1}{2}+1+1\right)(-iZ_\lambda\lambda)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2+m^2)^2}
\end{aligned}
$$

in the $\overline{\rm MS}$ scheme we find:

$$
Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\varepsilon} + \dots
$$