Loop corrections in Yukawa theory

based on S-51

Let's consider the theory of a pseudoscalar field and a Dirac field:

$$P^{-1}\varphi(\mathbf{x},t)P = -\varphi(-\mathbf{x},t)$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \ ,$$

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}M^2\varphi^2 \ ,$$

$$\mathcal{L}_1 = iZ_g g\varphi\overline{\Psi}\gamma_5\Psi - \tfrac{1}{24}Z_\lambda\lambda\varphi^4 + \mathcal{L}_{\rm ct} \ ,$$

φ and φ^3 terms not allowed!

$$P^{-1}\big(\overline{\Psi}\Psi\big)P = +\overline{\Psi}\Psi \ ,$$

$$P^{-1}\big(\overline{\Psi}i\gamma_5\Psi\big)P = -\overline{\Psi}i\gamma_5\Psi \ ,$$

the only couplings allowed by symmetries!

$$\mathcal{L}_{\rm ct} = i(Z_\Psi-1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m-1)m\overline{\Psi}\Psi - \tfrac{1}{2}(Z_\varphi-1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_M-1)M^2\varphi^2$$

We will calculate 1-loop corrections in the OS renormalization scheme:

(the LSZ formula is valid as it is; the lagrangian mass is the physical mass; propagators have appropriate poles with unit residue)

For the scalar propagator we found:

we will assume $M < 2m$

$$\tilde{\mathbf{\Delta}}(k^2) = \frac{1}{k^2 + M^2 - i\epsilon} + \int_{M_{\rm th}^2}^{\infty} ds\, \frac{\rho(s)}{k^2 + s - i\epsilon}$$

$$\begin{aligned}\langle 0|\varphi(x)\varphi(y)|0\rangle &= \langle 0|\varphi(x)|0\rangle\langle 0|\varphi(y)|0\rangle \\ &+ \int \widetilde{dk}\, \langle 0|\varphi(x)|k\rangle\langle k|\varphi(y)|0\rangle \\ &+ \sum_n \int \widetilde{dk}\, \langle 0|\varphi(x)|k,n\rangle\langle k,n|\varphi(y)|0\rangle\end{aligned}$$

either $2m$ or $3M$

$$\tilde{\mathbf{\Delta}}(k^2)^{-1} = k^2 + M^2 - i\epsilon - \Pi(k^2)$$

A simple pole at $k^2 = -M^2$ with residue one implies:

$$\Pi(-M^2) = 0 \qquad\qquad \Pi'(-M^2) = 0$$

we use these conditions to fix Z_φ and Z_M.

Similarly, the exact Dirac propagator can be written as:

$$\tilde{\mathbf{S}}(\not{p}) = \frac{-\not{p}+m}{p^2+m^2-i\epsilon} + \int_{m_{\rm th}^2}^{\infty} ds\, \frac{-\not{p}\rho_1(s)+\sqrt{s}\,\rho_2(s)}{p^2+s-i\epsilon}$$

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \langle 0|\varphi(x)|0\rangle\langle 0|\varphi(y)|0\rangle + \int \widetilde{dk}\,\langle 0|\varphi(x)|k\rangle\langle k|\varphi(y)|0\rangle + \sum_n \int \widetilde{dk}\,\langle 0|\varphi(x)|k,n\rangle\langle k,n|\varphi(y)|0\rangle$$

$p^2 = -\not{p}\not{p}$

$$\int_{m_{\rm th}^2}^{\infty} ds\, \frac{-\not{p}\rho_1(s)+\sqrt{s}\,\rho_2(s)}{(-\not{p}+\sqrt{s}-i\epsilon)(\not{p}+\sqrt{s}-i\epsilon)}$$

$m+M$

a fermion plus a scalar

contains inverse matrices, but we can think of $\tilde{\mathbf{S}}(\not{p})$ as an analytic function of $\not{p}$

The exact fermion propagator can be written as:

$$\tilde{\mathbf{S}}(\not{p})^{-1} = \not{p} + m - i\epsilon - \Sigma(\not{p})$$

sum of 1PI diagrams with 2 external lines (and ext. propagators removed)

A simple pole at $\not{p} = -m$ with residue one implies:

$$\Sigma(-m) = 0 \qquad\qquad \Sigma'(-m) = 0$$

we use these conditions to fix Z_Ψ and Z_m .

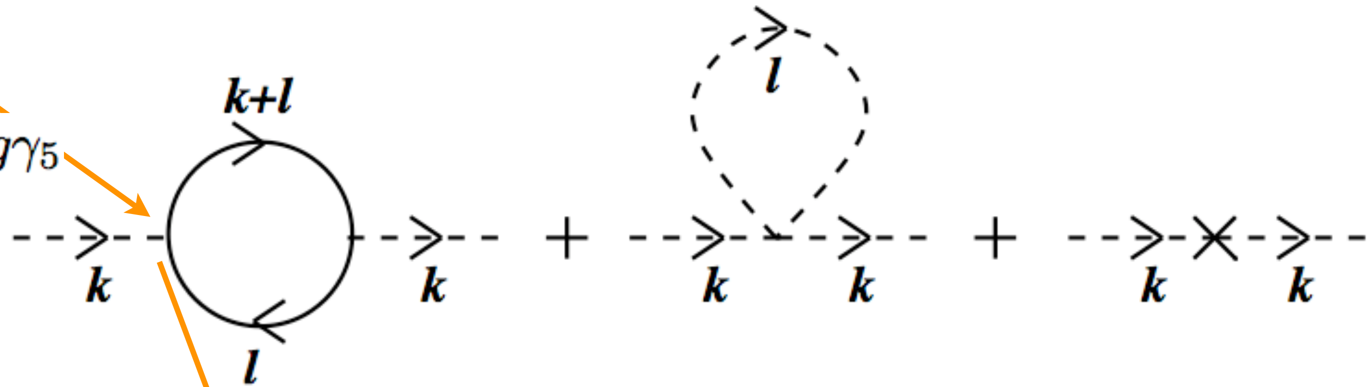
Let's evaluate diagrams contributing to the scalar propagator:

$$\mathcal{L}_1 = iZ_g g\varphi\overline{\Psi}\gamma_5\Psi - \tfrac{1}{24}Z_\lambda\lambda\varphi^4 + \mathcal{L}_{\rm ct}$$

vertex factor:

$$i(iZ_g g)\gamma_5 = -Z_g g\gamma_5$$

$$Z_g = 1 + O(g^2)$$



$$i\Pi_{\Psi\,\rm loop}(k^2) = (-1)(-g)^2\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,{\rm Tr}\Big[\tilde{S}(\not\ell+\not k)\gamma_5\tilde{S}(\not\ell)\gamma_5\Big]$$

extra -1 for fermion loop (Homework S-51.1); and the trace

$$\tilde{S}(\not p) = \frac{-\not p + m}{p^2+m^2-i\epsilon}$$

$$\Pi_{\varphi\,\rm loop}(k^2) = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon} + \tfrac{1}{2} - \tfrac{1}{2}\ln(M^2/\mu^2)\right]M^2$$

was your homework

$$\Pi_{\rm ct}(k^2) = -(Z_\varphi - 1)k^2 - (Z_M - 1)M^2$$

$$i\Pi_{\Psi\,\text{loop}}(k^2) = (-1)(-g)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\text{Tr}\Big[\tilde{S}(\not{\ell}+\not{k})\gamma_5\tilde{S}(\not{\ell})\gamma_5\Big]$$

Let's evaluate the fermion loop:

$$\tilde{S}(\not{p}) = \frac{-\not{p}+m}{p^2+m^2-i\epsilon}$$

numerator:

$$\text{Tr}[(-\not{\ell}-\not{k}+m)\gamma_5(-\not{\ell}+m)\gamma_5] = \text{Tr}[(-\not{\ell}-\not{k}+m)(+\not{\ell}+m)]$$

$$= 4[(\ell+k)\ell+m^2]$$

$$\gamma_5\not{p}\gamma_5 = -\not{p}$$

$$\gamma_5^2 = 1$$

$$\equiv 4N\ .$$

Combining the denominators we have:

$$\frac{1}{(\ell+k)^2+m^2}\,\frac{1}{\ell^2+m^2} = \int_0^1 dx\,\frac{1}{(q^2+D)^2}$$

$$D = x(1-x)k^2+m^2$$

changing the integration variable we get:

$$q = \ell+xk$$

$$i\Pi_{\Psi\,\text{loop}}(k^2) = 4g^2\int_0^1 dx\int\frac{d^4q}{(2\pi)^4}\,\frac{N}{(q^2+D)^2}$$

where $N = (q+(1-x)k)(q-xk)+m^2$

$$i\Pi_{\Psi\,\text{loop}}(k^2) = 4g^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{N}{(q^2+D)^2}$$

where $N = (q+(1-x)k)(q-xk)+m^2$

the integral diverges in 4 spacetime dimensions and so we analytically continue it to $d = 4-\varepsilon$; we also make the replacement $g \to g\tilde{\mu}^{\varepsilon/2}$ to keep the coupling dimensionless:

$$N = q^2 - x(1-x)k^2 + m^2 + (1-2x)kq$$

0

we use the usual formulas to get:

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2+D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)} D^{-(b-a-d/2)}$$

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum_{k=1}^n k^{-1} + O(x)\right]$$

$$A^{\varepsilon/2} = 1 + \tfrac{\varepsilon}{2}\ln A + O(\varepsilon^2)$$

$$\tilde{\mu}^\varepsilon \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2+D)^2} = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} - \ln(D/\mu^2)\right],$$

$$\tilde{\mu}^\varepsilon \int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(q^2+D)^2} = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} + \frac{1}{2} - \ln(D/\mu^2)\right](-2D),$$

$$\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$$

Putting things together:

$$i\Pi_{\Psi\,\mathrm{loop}}(k^2) = 4g^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\,\frac{N}{(q^2+D)^2}$$

$$N = q^2 - x(1-x)k^2 + m^2 + (1-2x)kq$$

0

$$\tilde{\mu}^\varepsilon \int \frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2} = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} - \ln(D/\mu^2)\right] ,$$

$$\tilde{\mu}^\varepsilon \int \frac{d^dq}{(2\pi)^d}\,\frac{q^2}{(q^2+D)^2} = \frac{i}{16\pi^2}\left[\frac{2}{\varepsilon} + \tfrac{1}{2} - \ln(D/\mu^2)\right](-2D) ,$$

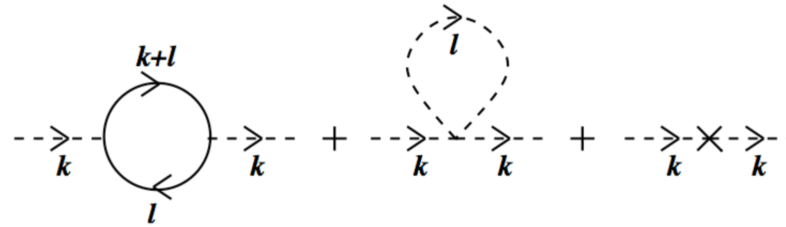
$$D = x(1-x)k^2 + m^2$$

we find:

$$\Pi_{\Psi\,\mathrm{loop}}(k^2) = -\frac{g^2}{4\pi^2}\Bigg[\frac{1}{\varepsilon}(k^2+2m^2) + \tfrac{1}{6}k^2 + m^2 - \int_0^1 dx\,\Big(3x(1-x)k^2 + m^2\Big)\ln(D/\mu^2)\Bigg]$$

the divergent piece has the form that permits cancellation by the counterterms!

Collecting contributions of all diagrams:



$$\Pi_{\Psi\,\text{loop}}(k^2) = -\frac{g^2}{4\pi^2}\left[\frac{1}{\varepsilon}(k^2+2m^2)+\tfrac{1}{6}k^2+m^2-\int_0^1 dx\,\Big(3x(1-x)k^2+m^2\Big)\ln(D/\mu^2)\right]$$

$$\Pi_{\varphi\,\text{loop}}(k^2) = \frac{\lambda}{(4\pi)^2}\left[\frac{1}{\varepsilon}+\tfrac{1}{2}-\tfrac{1}{2}\ln(M^2/\mu^2)\right]M^2$$

$$\Pi_{\text{ct}}(k^2) = -(Z_\varphi-1)k^2-(Z_M-1)M^2$$

we get:

$$Z_\varphi = 1-\frac{g^2}{4\pi^2}\left(\frac{1}{\varepsilon}+\text{finite}\right),$$

$$Z_M = 1+\left(\frac{\lambda}{16\pi^2}-\frac{g^2}{2\pi^2}\frac{m^2}{M^2}\right)\left(\frac{1}{\varepsilon}+\text{finite}\right),$$

we can impose $\Pi(-M^2)=0$ by writing:

$$\Pi(k^2) = \frac{g^2}{4\pi^2}\left[\int_0^1 dx\,\Big(3x(1-x)k^2+m^2\Big)\ln(D/D_0)+\kappa_\varphi(k^2+M^2)\right]$$

$$D_0 = -x(1-x)M^2+m^2$$

we can impose $\Pi(-M^2) = 0$ by writing:

$$\Pi(k^2) = \frac{g^2}{4\pi^2}\left[\int_0^1 dx\,\left(3x(1-x)k^2 + m^2\right)\ln(D/D_0) + \kappa_\varphi(k^2+M^2)\right]$$

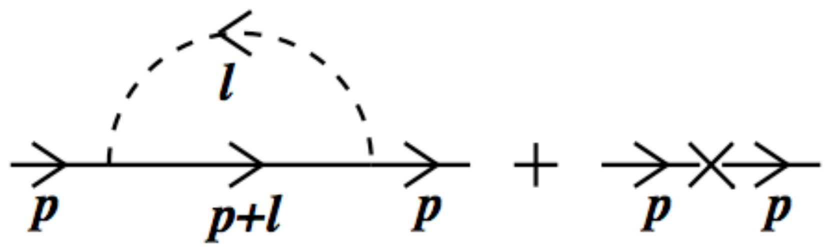
$$D_0 = -x(1-x)M^2 + m^2$$

fixed by imposing: $\Pi'(-M^2) = 0$

$$\kappa_\varphi = \int_0^1 dx\; x(1-x)[3x(1-x)M^2 - m^2]/D_0$$

no $O(\lambda)$ correction in the OS scheme!

Let's evaluate diagrams contributing to the fermion propagator:



$$i\Sigma_{1\,\text{loop}}(\not{p}) = (-g)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma_5 \tilde{S}(\not{p}+\not{\ell})\gamma_5\right]\tilde{\Delta}(\ell^2)$$

$$\tilde{S}(\not{p}) = \frac{-\not{p}+m}{p^2+m^2-i\epsilon}$$

$$\tilde{\Delta}(\ell^2) = \frac{1}{\ell^2+M^2-i\epsilon}$$

$$\Sigma_{\text{ct}}(\not{p}) = -(Z_\Psi - 1)\not{p} - (Z_m - 1)m$$

$$\mathcal{L}_{\text{ct}} = i(Z_\Psi - 1)\overline{\Psi}\not{\partial}\Psi - (Z_m - 1)m\overline{\Psi}\Psi - \tfrac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_M - 1)M^2\varphi^2$$

$$i\Sigma_{1\,\text{loop}}(\not p) = (-g)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\left[\gamma_5 \tilde S(\not p + \not \ell)\gamma_5\right]\tilde\Delta(\ell^2)$$

$$\tilde S(\not p) = \frac{-\not p + m}{p^2 + m^2 - i\epsilon}$$

$$\tilde\Delta(\ell^2) = \frac{1}{\ell^2 + M^2 - i\epsilon}$$

combining the denominators we have:

$$i\Sigma_{1\,\text{loop}}(\not p) = -g^2\int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\frac{N}{(q^2+D)^2}$$

$$q = \ell + xp$$

$$N = \not q\,(\to 0) + (1-x)\not p + m\,,$$

$$D = x(1-x)p^2 + xm^2 + (1-x)M^2$$

the integral diverges in 4 spacetime dimensions and so we analytically continue it to $d = 4-\varepsilon$; we also make the replacement $g \to g\tilde\mu^{\varepsilon/2}$ to keep the coupling dimensionless...

$$\int \frac{d^d\bar q}{(2\pi)^d}\frac{(\bar q^2)^a}{(\bar q^2 + D)^b} = \frac{\Gamma(b-a-\frac12 d)\Gamma(a+\frac12 d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac12 d)}D^{-(b-a-d/2)}$$

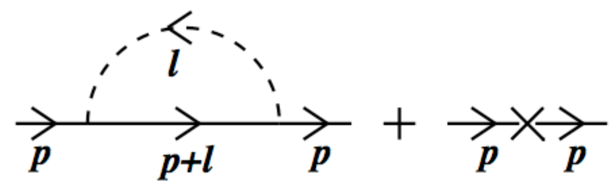
$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum\nolimits_{k=1}^{n} k^{-1} + O(x)\right]$$

$$A^{\varepsilon/2} = 1 + \tfrac{\varepsilon}{2}\ln A + O(\varepsilon^2)$$

using the usual formulas we get:

$$\Sigma_{1\,\text{loop}}(\not p) = -\frac{g^2}{16\pi^2}\left[\frac{1}{\varepsilon}(\not p + 2m) - \int_0^1 dx\,\Big((1-x)\not p + m\Big)\ln(D/\mu^2)\right]$$

Adding contributions of both diagrams:



$$\Sigma_{1\,\text{loop}}(\not p) = -\frac{g^2}{16\pi^2}\left[\frac{1}{\varepsilon}(\not p + 2m) - \int_0^1 dx\,\Big((1-x)\not p + m\Big)\ln(D/\mu^2)\right]$$

$$\Sigma_{\text{ct}}(\not p) = -(Z_\Psi - 1)\not p - (Z_m - 1)m$$

we get:

$$Z_\Psi = 1 - \frac{g^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right),$$

$$Z_m = 1 - \frac{g^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right),$$

we can impose $\Sigma(-m) = 0$ by writing:

$$\Sigma(\not p) = \frac{g^2}{16\pi^2}\left[\int_0^1 dx\,\Big((1-x)\not p + m\Big)\ln(D/D_0) + \kappa_\Psi(\not p + m)\right]$$

fixed by imposing: $\Sigma'(-m) = 0$

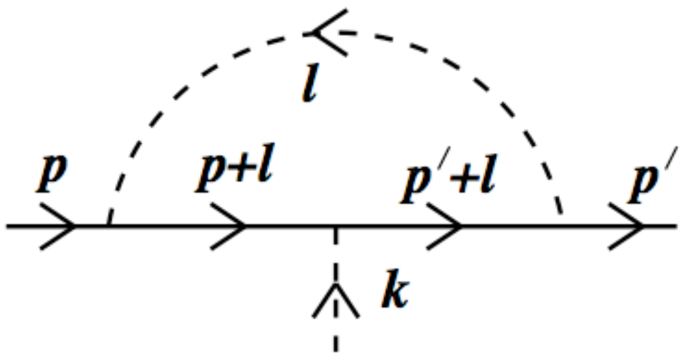
$$\kappa_\Psi = -2\int_0^1 dx\; x^2(1-x)m^2/D_0$$

Next, let's evaluate the diagram contributing to the Yukawa vertex:

vertex factor:

$$i(iZ_g g)\gamma_5 = -Z_g g\gamma_5$$

$$Z_g = 1 + O(g^2)$$



$$k = p' - p$$

$$i\mathbf{V}_Y(p',p) = -Z_g g\gamma_5 + i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p',p) + O(g^5)$$

$$i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p',p) = (-g)^3\left(\tfrac{1}{i}\right)^3 \int \frac{d^d\ell}{(2\pi)^d}\left[\gamma_5\tilde{S}(\not{p}'+\not{\ell})\gamma_5\tilde{S}(\not{p}+\not{\ell})\gamma_5\right]\tilde{\Delta}(\ell^2)$$

$$\tilde{S}(\not{p}) = \frac{-\not{p}+m}{p^2+m^2-i\epsilon}$$

$$\tilde{\Delta}(\ell^2) = \frac{1}{\ell^2+M^2-i\epsilon}$$

$$i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p',p) = (-g)^3\left(\tfrac{1}{i}\right)^3 \int \frac{d^d\ell}{(2\pi)^d}\left[\gamma_5\tilde{S}(\not{p}'+\not{\ell})\gamma_5\tilde{S}(\not{p}+\not{\ell})\gamma_5\right]\tilde{\Delta}(\ell^2)$$

$$\tilde{S}(\not{p}) = \frac{-\not{p}+m}{p^2+m^2-i\epsilon}$$

$$\tilde{\Delta}(\ell^2) = \frac{1}{\ell^2+M^2-i\epsilon}$$

numerator:

$$N = (\not{p}'+\not{\ell}+m)(-\not{p}-\not{\ell}+m)\gamma_5$$

combining the denominators we have:

$$i\mathbf{V}_{Y,\,1\,\mathrm{loop}}(p',p) = -ig^3\int dF_3 \int \frac{d^4q}{(2\pi)^4}\,\frac{N}{(q^2+D)^3}$$

$$q = \ell + x_1 p + x_2 p' \,,$$

$$N = [\not{q} - x_1\not{p} + (1-x_2)\not{p}' + m][-\not{q} - (1-x_1)\not{p} + x_2\not{p}' + m]\gamma_5 \,,$$

$$D = x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2\,p\cdot p' + (x_1+x_2)m^2 + x_3M^2 \,.$$

using $\not{q}\not{q} = -q^2$

divergent — finite — 0

$$N = q^2\gamma_5 + \widetilde{N} + (\text{linear in } q)$$

$$\widetilde{N} = [-x_1\not{p} + (1-x_2)\not{p}' + m][-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma_5$$

We proceed in a usual way, and get:

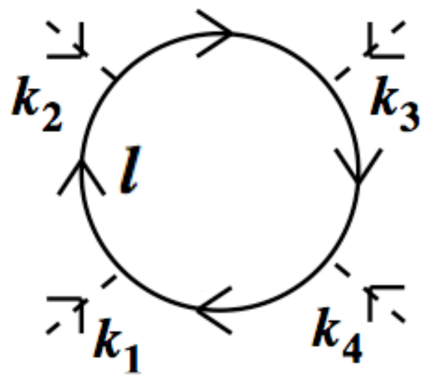
$$i\mathbf{V}_{Y,1\,\mathrm{loop}}(p',p) = \frac{g^3}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - \tfrac{1}{4} - \tfrac{1}{2}\int dF_3\,\ln(D/\mu^2)\right)\gamma_5 + \tfrac{1}{4}\int dF_3\,\frac{\widetilde{N}}{D}\right]$$

$$i\mathbf{V}_Y(p',p) = -Z_g g\gamma_5 + i\mathbf{V}_{Y,1\,\mathrm{loop}}(p',p) + O(g^5)$$

$$Z_g = 1 + \frac{g^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \mathrm{finite}\right)$$

finite piece fixed by imposing some condition, e.g.: $\mathbf{V}_Y(0,0) = ig\gamma_5$

Finally, let's evaluate diagrams contributing to the 4-point vertex:

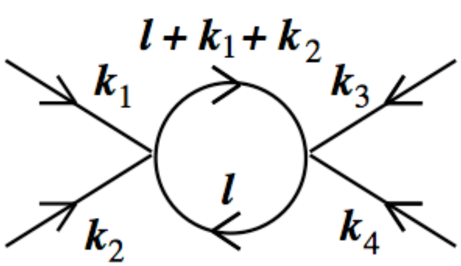


+ 5 others with permuted external lines

$$iV_{4,\,\Psi\,\text{loop}} = (-1)(-g)^4\left(\frac{1}{i}\right)^4 \int \frac{d^4\ell}{(2\pi)^4}\,\text{Tr}\Big[\tilde{S}(\not\ell)\gamma_5\tilde{S}(\not\ell-\not k_1)\gamma_5 \times \tilde{S}(\not\ell+\not k_2+\not k_3)\gamma_5\tilde{S}(\not\ell+\not k_2)\gamma_5\Big] + 5 \text{ permutations of } (k_2,k_3,k_4)\ .$$

extra -1 for fermion loop

and



$+\ k_2 \leftrightarrow k_3$

$+\ k_2 \leftrightarrow k_4$

$$V_{4,\,\varphi\,\text{loop}} = \frac{3\lambda}{16\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right)$$

was your homework

calculation is straightforward; let's calculate the divergent part only:

divergent pieces are sufficient to find beta functions of the theory

we can set external momenta to zero (divergent piece doesn't depend on these):

$$\mathrm{Tr}\,(\not{\ell}\gamma_5)^4 = 4(\ell^2)^2$$

numerator

$$i\mathbf{V}_{4,\,\Psi\,\mathrm{loop}} = (-1)(-g)^4\left(\tfrac{1}{i}\right)^4 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[\tilde{S}(\not{\ell})\gamma_5\tilde{S}(\not{\ell}-\not{k}_1)\gamma_5 \times \tilde{S}(\not{\ell}+\not{k}_2+\not{k}_3)\gamma_5\tilde{S}(\not{\ell}+\not{k}_2)\gamma_5\Big] + 5 \text{ permutations of } (k_2, k_3, k_4)\,.$$

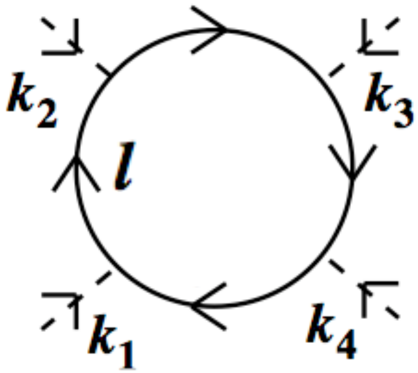
denominator

$$(\ell^2+m^2)^4$$

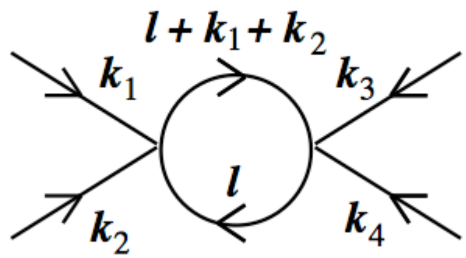
using the usual formula for the loop integral we find:

$$\mathbf{V}_{4,\,\Psi\,\mathrm{loop}} = -\frac{3g^4}{\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right)$$

Putting things together:



+ ... +



+ ...

$$\mathbf{V}_4 = -Z_\lambda \lambda + \mathbf{V}_{4,\,\Psi\,\text{loop}} + \mathbf{V}_{4,\,\varphi\,\text{loop}} + \ldots$$

$$\mathbf{V}_{4,\,\Psi\,\text{loop}} = -\frac{3g^4}{\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) \qquad \mathbf{V}_{4,\,\varphi\,\text{loop}} = \frac{3\lambda}{16\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right)$$

we find:

$$Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\left(\frac{1}{\varepsilon} + \text{finite}\right)$$

that concludes the calculation of 1-loop corrections to the Yukawa theory

Beta functions in Yukawa theory

based on S-52

The lagrangian in terms of renormalized fields and parameters:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \,,$$

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}M^2\varphi^2 \,,$$

$$\mathcal{L}_1 = iZ_g g\varphi\overline{\Psi}\gamma_5\Psi - \tfrac{1}{24}Z_\lambda\lambda\varphi^4 + \mathcal{L}_{\rm ct} \,,$$

$$\begin{aligned}\mathcal{L}_{\rm ct} = {} & i(Z_\Psi - 1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m - 1)m\overline{\Psi}\Psi \\ & - \tfrac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}(Z_M - 1)M^2\varphi^2\end{aligned}$$

can be also written in terms of bare fields and parameters (independent of μ)!

The dictionary for couplings:

$$g_0 = Z_\varphi^{-1/2} Z_\Psi^{-1} Z_g \tilde{\mu}^{\varepsilon/2} g \,,$$

$$\lambda_0 = Z_\varphi^{-2} Z_\lambda \tilde{\mu}^{\varepsilon} \lambda \,.$$

$$g_0 = Z_\varphi^{-1/2} Z_\Psi^{-1} Z_g \tilde{\mu}^{\varepsilon/2} g \ ,$$

$$\lambda_0 = Z_\varphi^{-2} Z_\lambda \tilde{\mu}^{\varepsilon} \lambda \ .$$

$$\ln g_0 = \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n} + \ln g + \tfrac{1}{2}\varepsilon \ln \tilde{\mu} \ ,$$

$$\ln \lambda_0 = \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n} + \ln \lambda + \varepsilon \ln \tilde{\mu} \ .$$

$$\ln\left(Z_\varphi^{-1/2} Z_\Psi^{-1} Z_g\right) = \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n}$$

$$\ln\left(Z_\varphi^{-2} Z_\lambda\right) = \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n}$$

$$Z_\Psi = 1 - \frac{g^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right),$$

$$Z_g = 1 + \frac{g^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right),$$

$$Z_\varphi = 1 - \frac{g^2}{4\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right),$$

$$Z_\lambda = 1 + \left(\frac{3\lambda}{16\pi^2} - \frac{3g^4}{\pi^2\lambda}\right)\left(\frac{1}{\varepsilon} + \text{finite}\right),$$

$$G_1(g,\lambda) = \frac{5g^2}{16\pi^2} + \dots \ ,$$

$$L_1(g,\lambda) = \frac{3\lambda}{16\pi^2} + \frac{g^2}{2\pi^2} - \frac{3g^4}{\pi^2\lambda} + \dots \ ,$$

$$\ln g_0 = \sum_{n=1}^{\infty} \frac{G_n(g,\lambda)}{\varepsilon^n} + \ln g + \tfrac{1}{2}\varepsilon \ln \tilde{\mu} \,,$$

$$\ln \lambda_0 = \sum_{n=1}^{\infty} \frac{L_n(g,\lambda)}{\varepsilon^n} + \ln \lambda + \varepsilon \ln \tilde{\mu} \,.$$

bare parameters do not depend on μ

$$0 = \sum_{n=1}^{\infty} \left(g \frac{\partial G_n}{\partial g} \frac{dg}{d\ln\mu} + g \frac{\partial G_n}{\partial \lambda} \frac{d\lambda}{d\ln\mu} \right) \frac{1}{\varepsilon^n} + \frac{dg}{d\ln\mu} + \tfrac{1}{2}\varepsilon g \,,$$

$$0 = \sum_{n=1}^{\infty} \left(\lambda \frac{\partial L_n}{\partial g} \frac{dg}{d\ln\mu} + \lambda \frac{\partial L_n}{\partial \lambda} \frac{d\lambda}{d\ln\mu} \right) \frac{1}{\varepsilon^n} + \frac{d\lambda}{d\ln\mu} + \varepsilon \lambda \,.$$

must be finite as $\varepsilon \to 0$

$$\frac{dg}{d\ln\mu} = -\tfrac{1}{2}\varepsilon g + \beta_g(g,\lambda) \,,$$

$$\frac{d\lambda}{d\ln\mu} = -\varepsilon \lambda + \beta_\lambda(g,\lambda) \,.$$

$$\beta_g(g,\lambda) = g\left(\tfrac{1}{2} g \frac{\partial}{\partial g} + \lambda \frac{\partial}{\partial \lambda} \right) G_1$$

$$\beta_\lambda(g,\lambda) = \lambda\left(\tfrac{1}{2} g \frac{\partial}{\partial g} + \lambda \frac{\partial}{\partial \lambda} \right) L_1$$

$$\beta_g(g,\lambda) = g\Big(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda}\Big)G_1$$

$$\beta_\lambda(g,\lambda) = \lambda\Big(\tfrac{1}{2}g\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial\lambda}\Big)L_1$$

$$G_1(g,\lambda) = \frac{5g^2}{16\pi^2} + \ldots\ ,$$

$$L_1(g,\lambda) = \frac{3\lambda}{16\pi^2} + \frac{g^2}{2\pi^2} - \frac{3g^4}{\pi^2\lambda} + \ldots\ ,$$

$$\beta_g(g,\lambda) = \frac{5g^3}{16\pi^2} + \ldots\ ,$$

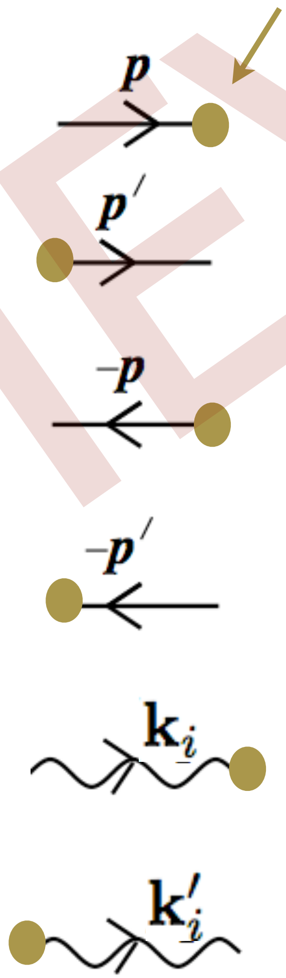
$$\beta_\lambda(g,\lambda) = \frac{1}{16\pi^2}\Big(3\lambda^2 + 8\lambda g^2 - 48g^4\Big) + \ldots\ .$$

we obtained the beta functions of the Yukawa theory.

Review of Feynman rules for QED

- external lines:

vertex and the rest of the diagram

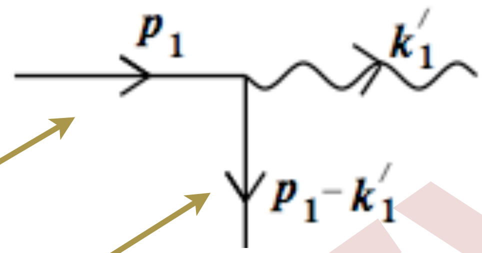


incoming electron	$u_{s_i}(\mathbf{p}_i)$
outgoing electron	$\overline{u}_{s'_i}(\mathbf{p}'_i)$
incoming positron	$\overline{v}_{s_i}(\mathbf{p}_i)$
outgoing positron	$v_{s'_i}(\mathbf{p}'_i)$
incoming photon	$\varepsilon^{\mu *}_{\lambda_i}(\mathbf{k}_i)$
outgoing photon	$\varepsilon^{\mu}_{\lambda'_i}(\mathbf{k}'_i)$

REVIEW

- vertex

the arrow for the photon can point both ways



$$ie\gamma^{\mu}$$

one arrow in and one out

REVIEW

- draw all topologically inequivalent diagrams
- for internal lines assign momenta so that momentum is conserved in each vertex (the four-momentum is flowing along the arrows)
- propagators

for each internal photon $\quad -ig^{\mu\nu}/(k^2 - i\epsilon)$

for each internal fermion $\quad -i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$

- spinor indices are contracted by starting at the end of the fermion line that has the arrow pointing away from the vertex, write $\overline{u}_{s_i'}(\mathbf{p}_i')$ or $\overline{v}_{s_i}(\mathbf{p}_i)$; follow the fermion line, write factors associated with vertices and propagators and end up with spinors $u_{s_i}(\mathbf{p}_i)$ or $v_{s_i'}(\mathbf{p}_i')$.

  follow arrows backwards!

  The vector index on each vertex is contracted with the vector index on either the photon propagator or the photon polarization vector.

- assign proper relative signs to different diagrams

  draw all fermion lines horizontally with arrows from left to right; with left end points labeled in the same way for all diagrams; if the ordering of the labels on the right endpoints is an even (odd) permutation of an arbitrarily chosen ordering then the sign of that diagram is positive (negative).

- sum over all the diagrams and get $i\mathcal{T}$

additional rules for counterterms and loops

REVIEW