Lorentz Covariance and Generators of Lorentz Group

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On pg. 17 of Srednicki, we make use of the equation

$$U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda) \tag{1}$$

Let $\Lambda' = 1 + \delta\omega'$. First, we do not assume that Λ is close to the identity. We will expand both sides of Eq. 1 for small $\delta\omega'$

$$\begin{aligned} LHS &= U(\Lambda)^{-1}[I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}M^{\mu\nu}]U(\Lambda) & (2)\\ &= I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) & (3)\\ & & (4) \end{aligned}$$

On the other hand, the RHS involves $\Lambda^{-1}\Lambda'\Lambda$. Putting in indices,

$$\begin{aligned} (\Lambda^{-1}\Lambda'\Lambda)^{\rho}{}_{\sigma} &= (\Lambda^{-1})^{\rho}{}_{\mu}\Lambda'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} & (5)\\ &= (\Lambda^{-1})^{\rho}{}_{\mu}(\delta^{\mu}{}_{\nu} + \delta\omega'^{\mu}{}_{\nu})\Lambda^{\nu}{}_{\sigma} & (6)\\ &= (\Lambda^{-1})^{\rho}{}_{\mu}\Lambda^{\mu}{}_{\sigma} + (\Lambda^{-1})^{\rho}{}_{\mu}\delta\omega'^{\mu}{}_{\nu}\Lambda^{\nu}{}_{\sigma} & (7)\\ &= \delta^{\rho}{}_{\sigma} + \delta\omega'^{\mu}{}_{\nu}(\Lambda_{\mu}{}^{\rho}\Lambda^{\nu}{}_{\sigma}) & (8) \end{aligned}$$

where to get to the second line we just use the fact that Λ' is close the identity, and to get to the fourth line we use Srednicki, Eq. (2.5)

$$(\Lambda^{-1})^{\rho}{}_{\nu} = \Lambda_{\nu}{}^{\rho}. \tag{9}$$

(In general, an element of Λ^{-1} is the element of Λ with the order of indices reversed and whether the index is up or down preserved.) Now, we can compute the RHS of Eq. 1, as we see that $(\Lambda^{-1}\Lambda'\Lambda)^{\rho}{}_{\sigma}$ is a Lorentz tranformation close to the identity. (Note that we have to lower the index ρ to contract with $M^{\rho\sigma}$ and we have chosen lower μ on $\delta\omega'$ and raise it on Λ.

$$RHS = U(\Lambda^{-1}\Lambda'\Lambda) = I + \frac{i}{2\hbar}\delta\omega'_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma} \tag{10}$$

Note that $\delta\omega'_{\mu\nu}$ on each side of the equation is antisymmetric in μ and ν. The LHS clearly multiplies a term antisymmetric in μ and ν. What about the RHS? We will now do a manipulation in excruciating detail, so that we never have to do so again. Consider

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}. \tag{11}$$

We will now interchange μ and ν, so we have

$$\begin{aligned}
\Lambda^{\nu}{}_{\rho}\Lambda^{\mu}{}_{\sigma}M^{\rho\sigma} &= \Lambda^{\nu}{}_{\sigma}\Lambda^{\mu}{}_{\rho}M^{\sigma\rho} && (12)\\
&= -\Lambda^{\nu}{}_{\sigma}\Lambda^{\mu}{}_{\rho}M^{\rho\sigma} && (13)\\
&= -\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}. && (14)
\end{aligned}$$

In the first line, we used the fact that ρ and σ are just dummy variables that are summed over, so we can interchange their names. To get the second line, we use the antisymmetry of M. To get to the third line, we just interchanged the order of the two elements of the Lorentz transformation, which we can do because as long as we explicitly indicate the indices, they are just real numbers. Thus, we have shown that $\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}$ is antisymmetric under interchange of μ and ν. Thus, the two antisymmetric tensors multiplying $\delta\omega'_{\mu\nu}$ must be equal and we have:

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}M^{\rho\sigma}, \tag{15}$$

which is Eq. (S2.14). It is worth emphasizing that we did not yet assume that Λ is close to the identity. We should also emphasize how natural this result is. Under a Lorentz transformation, each index of $M^{\mu\nu}$ gets its own factor of Λ. This is exactly how we expect a multi-index object to transform.

In the next step, we will let Λ be close to the identity. That is $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$ and

$$U(\Lambda) = I + \frac{i}{2\hbar}\delta\omega_{\rho\sigma}M^{\rho\sigma} \equiv I + \frac{i}{2\hbar}\delta\omega M, \tag{16}$$

where we have introduced a simplified notation for the second term. We next expand Eq/ 15 keeping linear terms.

$$(I - \frac{i}{2\hbar}\delta\omega M)M^{\mu\nu}(I + \frac{i}{2\hbar}\delta\omega M) = (\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma})M^{\rho\sigma} \tag{17}$$

The leading term on each side is $M^{\mu\nu}$, so we will just concentrate on the terms linear in $\delta\omega$.

$$\begin{aligned}
\frac{i}{2\hbar}[M^{\mu\nu}, \delta\omega M] &= \delta\omega^{\mu}{}_{\rho}M^{\rho\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} && (18)\\
&= \delta\omega^{\mu}{}_{\sigma}M^{\sigma\nu} + \delta\omega^{\nu}{}_{\sigma}M^{\mu\sigma} && (19)
\end{aligned}$$

We note that the final expression for the RHS is antisymmetric under the interchange of μ and ν.

$$\begin{aligned}
\frac{i}{2\hbar}[M^{\mu\nu}, \delta\omega_{\rho\sigma}M^{\rho\sigma}] &= g^{\mu\rho}\delta\omega_{\rho\sigma}M^{\sigma\nu} + g^{\nu\rho}\delta\omega_{\rho\sigma}M^{\mu\sigma} && (20)\\
&= \delta\omega_{\rho\sigma}(g^{\mu\rho}M^{\sigma\nu} - g^{\nu\rho}M^{\sigma\mu}) && (21)\\
&= \frac{1}{2}\delta\omega_{\rho\sigma}[(g^{\mu\rho}M^{\sigma\nu} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma)] && (22)
\end{aligned}$$

In the last expression, we have explicitly made the expression that multiplies $\delta\omega_{\rho\sigma}$ antisymmetric under the interchange $(\rho \leftrightarrow \sigma)$. Multiplying both sides by $-i\hbar$, we have

$$
\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= (-i\hbar)\,(g^{\mu\rho}M^{\sigma\nu} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma) &(23)\\
&= i\hbar(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma) &(24)
\end{aligned}
$$

Note that in the first term on the RHS, the first indices μ and ρ from the two factors of M are paired on the metric tensor and the second two indices from the LHS are paired on M and they are in the same order on M is they appear on the LHS.

You will derive Eqs. (S2.17) and (S2.18) for homework.

To summarize, the commutation relations for the generators of the Poincare group are:

$$
\begin{aligned}
[J_i, J_j] &= i\hbar\varepsilon_{ijk}J_k,\\
[J_i, K_j] &= i\hbar\varepsilon_{ijk}K_k,\\
[K_i, K_j] &= -i\hbar\varepsilon_{ijk}J_k,\\
[J_i, H] &= 0,\\
[J_i, P_j] &= i\hbar\varepsilon_{ijk}P_k,\\
[K_i, H] &= i\hbar c P_i,\\
[K_i, P_j] &= i\hbar\delta_{ij}H/c,\\
[P_i, P_j] &= 0,\\
[P_i, H] &= 0.
\end{aligned}
$$