3. A LITTLE ABOUT GROUP THEORY

3.1 Preliminaries

It is an apparent fact that nature exhibits many symmetries, both exact and approximate. A symmetry is an invariance property of a system under a set of transformations. For example, our faces have approximate reflection symmetry, because we look approximately the same in a photograph as in a mirror. As another example, a sphere has rotational symmetry because it looks the same no matter how it is rotated.

Symmetry transformations of physical systems have properties analogous to those of a mathematical *group*. These properties are: If we successively perform two symmetry transformations we obtain a unique symmetry transformation; the transformations are associative; and inverse and identity transformations exist.

We have already mentioned in chapter 1 a theorem, called *Noether's theorem*, which relates symmetry principles to conservation laws. Noether's theorem says that if a physical system can be described by a classical Lagrangian which is invariant under a continuous group of transformations, then the system has a conserved current. If a classical field is quantized, the resulting quantum field theory usually has the same symmetry. However, the quantized theory may have an *anomaly*, which breaks the classical symmetry. We briefly discuss anomalies in Section 4.8.

As two examples of symmetry, we note that the fundamental interactions of nature are apparently invariant under the group of translations and the group of rotations in three dimensions. Noether's theorem relates symmetry under translations to the law of conservation of momentum, and the symmetry under rotations to the law of conservation of angular momentum. The translation and rotation groups are examples of *Lie* groups, which we define in Section 3.3.

The Lagrangian of the standard model is invariant under the group of gauge transformations $SU(3) \times SU(2) \times U(1)$. (We define gauge transformations in chapter 4.) The standard model is also invariant under the proper Poincaré group, which includes translations in space and time, rotations, and proper Lorentz transformations. (Improper Lorentz transformations include space and time reflections.) The Poincaré group is a Lie group. The groups $SU(3)$, $SU(2)$, and $U(1)$ are *special unitary* groups, which are also Lie groups. A unitary group is a group of unitary matrices, and a special unitary group is a group of unitary matrices with determinants equal to unity. In order to understand the standard model, we have to have some familiarity with the Lie groups and their Lie algebras, especially unitary groups. (We discuss unitary groups further in Section 3.5.)

In addition to the space–time symmetries of the proper Poincaré group, some theories have additional symmetries under finite transformation groups. Of these, we single out space reflection or parity P, time inversion T, and charge conjugation C. We do not give much discussion of these groups, but they play an important role in the standard model. As we have already mentioned, none of the groups P, C, and T is an exact symmetry of the standard model, but the combined symmetry CPT, taken in any order, is exact.

In this chapter we briefly discuss groups in general, then Lie groups and their algebras, and the unitary groups and their algebras. We concentrate on group representations (which we define in the Wection 3.2), especially irreducible unitary representations. We can have unitary representations of many different groups, not only of unitary groups. The treatment in this chapter may seem to be a little condensed for those who only know a little about group theory. More details can be found in many places, for example, in a book on unitary symmetry (Lichtenberg, 1978). We do not discuss the Poincaré group in any detail.

A group G is a set of elements which satisfy four postulates:

1) A law of combination, often called a product, is defined so that if a and b belong to G, the product ab is a unique element of G.
2) Multiplication is *associative*, i.e., $a(bc) = (ab)c$.
3) An *identity* e exists such that $ea = ae = a$.
4) An *inverse* a^{-1} exists to any element a such that $a^{-1}a = aa^{-1} = e$.

The number of elements of a group may be finite, in which case the group is called a finite group, or infinite. If all the elements of a group commute with one another, the group is said to be *abelian*. Otherwise the group is *nonabelian*. A subgroup of a group is a subset of elements which is itself a group under the same multiplication law. Every group has at least two subgroups: itself and the group consisting only of the identity. These are called *improper* subgroups; any others are called *proper* subgroups. A group H is *homomorphic* to a group G if there is a mapping of the elements of G onto the elements of H. The groups are *isomorphic* if the mapping is one-to-one.

An element a belonging to G is said to be *conjugate* to an element b in G if there exists an element u in G such that $a = ubu^{-1}$. Let H be a subgroup of G, and let h be in H and g be in G. Form the product elements $h' = ghg^{-1}$ for all h. Then the h' form a group H' which is isomorphic to H. If, for all g in G, the elements of H and H' are identical, then H is called an *invariant* or self-conjugate subgroup of G.

The group G is said to be the *direct product* of two groups H and H' if every h in H commutes with every h' in H' and if every g in G can be written uniquely as a product of an element in H and an element in H'. The direct product is written in the form $G = H \times H'$.

3.2 Group representations

A *representation* of a group is a homomorphism between the group and a group of linear operators which operate on a vector space. We can think of the vectors in this space as being the states (wave functions) of a quantum mechanical system. A finite-dimensional *matrix representation* of a group is a homomorphism between the group and a group of matrices. We often simply use the word "representation" to mean a matrix representation. If a representation is isomorphic to the group, it is said to be *faithful*. We shall consider only representations by square matrices. If G is a group with elements g, then we often denote the corresponding element of the representation by $D(g)$.

The matrices of a representation are a special case of linear operators which act on a vector space. If the matrices are n-by-n, the vectors (wave functions) are column matrices

with n entries, and their hermitian conjugates are row matrices. The vectors are members of an n-dimensional vector space, and therefore the matrices are said to be n-dimensional.

A similarity transformation is a transformation by means of a matrix S which leaves unaltered the algebra of the transformed system. A similarity transformation acts differently on a representation D and on a vector V, namely

$$D' = SDS^{-1}, \quad V' = SV. \tag{3.1}$$

If a representation can be brought into the following form by a similarity transformation:

$$D(g) = \begin{pmatrix} D_1(g) & X(g) \\ 0 & D_2(g) \end{pmatrix} \tag{3.2}$$

for all g, then the representation is called *reducible*. If not, it is *irreducible*. If $X(g) = 0$, the representation is fully reducible. We shall restrict our considerations to cases in which reducible representations are fully reducible, and we shall omit the word "fully." A theorem (Schur's lemma) states: A matrix which commutes with all matrices of an irreducible representation is a multiple of the unit matrix.

We next discuss the importance of irreducible unitary representations of groups within the Hamiltonian formalism, as using this formalism is somewhat easier than using the Lagrangian formalism. Let us consider an n-dimensional irreducible unitary representation of a group G. The unitary matrices act on a set of n linearly-independent vectors, which can be chosen to be orthonormal. The members of this orthonormal set (or basis) constitute a *multiplet*.

Let a unitary representation of a symmetry group (that is, a group of transformations which leaves the physical system invariant) be denoted by U_a, where a stands for all the parameters which specify individual group elements. If we have any transition matrix (ϕ, ψ), where ϕ and ψ are state vectors (or wave functions) describing physical states, then the transformed states $\phi' = U_a\phi$ and $\psi' = U_a\psi$ satisfy the condition

$$(\phi', \psi') = (U_a\phi, U_a\psi) = (U_a^{-1}U_a\phi, \psi) = (\phi, \psi). \tag{3.3}$$

Thus, unitary transformations are important in quantum mechanics because they leave transition matrixes invariant.

If the Hamiltonian H of a physical system is invariant under a symmetry group G, then all members of a multiplet belonging to an irreducible unitary representation have the same energy, as we now show. Now consider the eigenvalue equation

$$H\psi_n = E_n\psi_n. \tag{3.4}$$

If we operate on this equation with U_a, we get

$$U_aH\psi_n = U_aHU_a^{-1}U_a\psi_n = E_nU_a\psi_n. \tag{3.5}$$

Now if H' and ψ'_n are defined as

$$H' = U_aHU_a^{-1}, \quad \psi'_n = U_a\psi_n, \tag{3.6}$$

our equation becomes

$$H'\psi'_n = E_n\psi'_n. \tag{3.7}$$

But because U_a is a symmetry group, by definition it leaves the Hamiltonian H invariant, so that $H' = H$ This implies that U_a commutes with the Hamiltonian:

$$HU_a = U_aH, \text{ or} [H, U_a] = 0, \tag{3.8}$$

where $[H, U_a] = HU_a - U_aH$ is called the *commutator* of H and U_a. Then Eq. (3.7) becomes simply

$$H\psi'_n = E_n\psi'_n, \tag{3.9}$$

so that the transformed wave functions ψ'_n are also eigenfunctions of the Hamiltonian with the same energy eigenvalue. But the transformed wave functions are in general linear combinations of all members of the original multiplet. Therefore, in order for Eq. (3.9) to be true, all members of the multiplet must have the same energy eigenvalue. We mention that if the representation is reducible, the new wave functions are not in general linear combinations of *all* the wave functions belonging to the representation, so that all the wave functions do not need to have the same energy.

It should be clear from the above arguments that if any operator A commutes with the U_a, then all members of a multiplet have the same eigenvalue of the operator A. Thus, for example, let us consider the rotation group $R(3)$. Not only is the Hamiltonian invariant under rotations, so that all members of a multiplet have the same energy, but U_a also commutes with the operator J^2, so that all members of a multiplet have the same eigenvalue of J^2, namely, $J(J + 1)$.

3.3 Lie groups

We have noted that a group may have a finite or infinite number of elements. A *Lie* group has a continuously infinite number of elements characterized by a finite number of parameters which can vary continuously. Furthermore, if an element of a Lie group is characterized by a set of r parameters collectively denoted by a ($a = a_1, a_2...a_r$), and another element is characterized by a set of parameters b, then the product element is characterized by a set of parameters c which are *analytic functions* of a and b.

As an example of a Lie group, consider the rotations in two dimensions. These are characterized by a parameter θ. The transformation is

$$\begin{aligned} x'_1 &= x_1 \cos\theta - x_2 \sin\theta, \\ x'_2 &= x_1 \sin\theta + x_2 \cos\theta. \end{aligned} \tag{3.10}$$

The transformation can be written in matrix form as

$$x' = R(\theta)x, \tag{3.11}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.12}$$

The rotation matrix $R(\theta)$ is the group element characterized by the single parameter θ. Rotations in two dimensions constitute an abelian group, but in more dimensions the rotation group is nonabelian. Note that the groups $R(n)$ are faithful representations of themselves in n dimensions.

The group multiplication law for rotations in 2 dimensions can be stated as follows: If

$$R(\theta) = R(\theta_2)R(\theta_1), \tag{3.13}$$

then

$$\theta = \theta_2 + \theta_1. \tag{3.14}$$

The rotation groups are *compact*. This means that the parameters vary over a *finite, closed* region. For example, the parameter θ of the two-dimensional rotation group varies over the interval $0 \leq \theta \leq 2\pi$.

On the other hand, the translation groups are not compact because the parameters are unbounded. For example, a translation in 1 dimension,

$$x' = x + a,$$

is characterized by a parameter a which can vary from $-\infty$ to ∞. Likewise, the group of Lorentz transformations is not compact because the group is characterized by a parameter v (the velocity) which varies in the interval $0 \leq v < c$, which is open at one end. Rotations and Lorentz transformations are both subgroups of the Lorentz group.

The concepts of simple and semisimple Lie groups are important but somewhat complicated. An oversimplified definition, which is adequate for our purposes, is that a Lie group is *simple* if it is nonabelian and has no proper invariant Lie subgroups. It is *semisimple* if it is nonabelian and has no abelian invariant Lie subgroups. Clearly, a simple group is also semisimple. If a group is the direct product of two or more groups H, H',..., then the subgroups H, H',... are invariant. The direct product of simple and/or semisimple Lie groups is semisimple.

Recall that the local gauge group of the standard model is $SU(3) \times SU(2) \times U(1)$. This group is not semisimple because it has an abelian invariant subgroup $U(1)$. However, the group $SU(3) \times SU(2)$ is semisimple. The groups $SU(3)$ and $SU(2)$ are simple.

3.4 Lie algebras

Let us consider a Lie group of transformations. We obtain the *Lie algebra* of the group by considering group elements which differ only infinitessimally from the identity. From these elements we can construct operators called *generators* which allow us to obtain a unitary representation of the group. More precisely, we obtain all the elements of the group which can be generated by continuous transformations from the identity. There is one generator for each parameter of the group. Methods for obtaining the generators of a Lie group have been discussed in many places (see, e.g., Lichtenberg, 1978).

Let the generators of a Lie group be X_i, $i = 1, 2...r$, where the group is characterized by r real parameters a_i. If the generators are Hermitian, a unitary representation of an arbitrary group element U_a is given by

$$U_a = e^{-i\sum a_i X_i}.$$

It can be shown that the X_i form a Lie algebra, which means that they satisfy the algebraic equations

$$[X_i, X_j] = i \sum_{k=1}^{r} c_{ij}^k X_k. \tag{3.15}$$

Here the *commutator* $[A, B] = AB - BA$ and the c_{ij}^k are constants called the *structure constants* of the group. (Some people call other constants $b_{ij}^k = ic_{ij}^k$, the group structure constants.) There is no significance to the fact that we write c_{ij}^k with both lower and upper indices in Eq. (3.15). We do this because in the future we shall use the *summation convention* of omitting the summation sign and summing over a repeated upper and lower index (in any order). The structure constants of a Lie algebra can differ with different choices of generators.

As we see from Eq. (3.15), a Lie algebra has the property that the commutator of any two members of the algebra (generators of the Lie group) is a linear combination of the members of the Lie algebra. We also see that the algebra is in general neither commutative nor associative. A *representation* of a Lie algebra is a set of matrices which obey the commutation relations of the algebra.

If a Lie group is abelian, all the commutators of its Lie algebra vanish, i.e. all its structure constants are zero. The maximum number of commuting generators of a Lie group is called the *rank* of the group. Since any generator commutes with itself, every Lie group is at least rank one. The k commuting generators of a rank k Lie group can be simultaneously diagonalized in a matrix representation.

If a Lie group of rank k is semisimple and compact, then one can construct from the members of its Lie algebra k nonlinear *invariant* operators, called *Casimir* operators, which commute with every member of the algebra.

As an example, let us consider the Lie algebra and Casimir operators of the familiar rotation group in three dimensions $R(3)$. This group is characterized by 3 parameters (for example, the Euler angles). Therefore, it has three generators, which can be taken to be the familiar angular momentum operators J_x, J_y, and J_z. They satisfy the Lie algebra

$$[J_x, J_y] = iJ_z, \quad \hbar = 1, \tag{3.16}$$

and cyclic permutations. This group is rank one because none of the J_i commutes with any other. It is also semisimple (actually, simple), so that it has one Casimir operator J^2 given by

$$J^2 = J_x^2 + J_y^2 + J_z^2. \tag{3.17}$$

A representation of the Casimir operator in n dimensions commutes with all the members of an irreducible representation in n dimensions. Therefore, by Schur's lemma, a representation of the Casimir operator is a multiple of the unit matrix.

It is convenient to denote the generators of $R(3)$ by J_1, J_2, and J_3, and write

$$[J_i, J_j] = i\epsilon_{ij}^k J_k, \quad i, j, k = 1, 2, 3, \tag{3.18}$$

where $\epsilon_{ij}^k = \epsilon_{ijk}$ is completely antisymmetric in its indices and $\epsilon_{123} = 1$. Then we see that the structure constants of $R(3)$ are given by ϵ_{ijk} or $i\epsilon_{ijk}$, depending on whether the

structure constants are defined with or without the i. It is easy to show from the definition of rotations that the number of parameters of $R(n)$ is $(n^2-n)/2$.

3.5 Unitary groups and algebras

The unitary group in n dimensions $U(n)$ is the group of $n \times n$ matrices U_a satisfying

$$U_a^\dagger = U_a^{-1}, \tag{3.19}$$

where a stands for the parameters of the group, the dagger denotes the Hermitian conjugate matrix, and the superscript $^{-1}$ denotes the inverse. By definition, for any matrix A, we have $(A^\dagger)_{ij} = A^*_{ji}$, with the asterisk denoting the complex conjugate.

A complex matrix in n dimensions is specified by $2n^2$ real numbers. If the matrix is unitary, there are n^2 relations among these numbers, so that $U(n)$ is characterized by n^2 parameters. The group $U(1)$ is one-dimensional and is characterized by only one parameter. Each element of $U(1)$ is a phase $e^{i\theta}$.

The special unitary groups $SU(n)$ have matrices with determinants equal to unity. This provides another relation so that $SU(n)$ is characterized by n^2-1 parameters. The rank of $SU(n)$ is $n-1$. The $SU(n)$ groups are semisimple and compact, so that $SU(n)$ has $n-1$ Casimir operators.

Like $R(3)$, $SU(2)$ has 3 parameters and is of rank 1. In fact, the generators of $SU(2)$ satisfy the same Lie algebra as the generators of $R(3)$. This implies that the two groups are locally isomorphic (i.e., the mapping of a neighborhod of one onto a neighborhood of the other is one-to-one) and globally homomorphic. In fact, the homomorphism is two-to-one from $SU(2)$ onto $R(3)$.

The groups $U(n)$ and $SU(n)$ are matrix groups, and so are faithful representations of themselves. In discussing representations of the unitary groups, we usually confine ourselves to $SU(n)$. The reason is that the algebra of $U(n)$ is the same as the algebra of $SU(n) \times U(1)$, and all the representations of $U(1)$ are one dimensional. The group $SU(n)$ has $n-1$ so-called fundamental representations. Of these, two are n-dimensional if $n > 2$. There is only one fundamental (two-dimensional) representation if $n = 2$.

The group $SU(n)$ also has a representation of n^2-1 dimensions, the same number as the number of generators of the group. This representation is called the *adjoint* representation.

We can construct n-dimensional representations of the algebra of $SU(n)$. For $n = 2$, we can chose these matrices to be the familiar Pauli spin matrices σ_1, σ_2, and σ_3, given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.20}$$

The Pauli matrices satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ij}^k \sigma_k. \tag{3.21}$$

Note the factor 2 difference between the structure constants when expressed in terms of the σ's rather than in terms of the J's, given in Eq. (3.10). This follows because $\sigma_i = 2J_i$,

and is an example of the fact that the structure constants depend on the representation of the Lie algebra. Still another representation of the Lie algebra of $SU(2)$ or $R(3)$ is by the two-dimensional matrices σ_+, σ_-, and σ_3, where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.22}$$

The matrix σ_+ is called a raising operator and σ_- is called a lowering operator because of their action on the eigenvectors of the operator σ_3 (see Section 3.6). The matrices σ_+ and σ_- can be written in terms of σ_1 and σ_2 as follows:

$$\sigma_+ = (\sigma_1 + i\sigma_2)/2; \quad \sigma_- = (\sigma_1 - i\sigma_2)/2.$$

The Casimir operator of $SU(2)$ is the same as that of $R(3)$. We can write the two-dimensional Casimir operator in terms of the Pauli matrices:

$$J^2 = \frac{1}{4}\sigma^2,$$

where

$$\sigma^2 = \sum_{i=1}^{3} \sigma_i^2 = 2(\sigma_+\sigma_- + \sigma_-\sigma_+) + \sigma_3^2. \tag{3.23}$$

We now turn to $SU(3)$. The generalization of the Pauli matrices are the so-called Gell-Mann matrices λ_i $(i = 1, 2...8)$, which are given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{3.24}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The λ_i satisfy the commutation relations

$$[\lambda_i, \lambda_j] = 2if_{ij}^k \lambda_k, \tag{3.25}$$

with summation implied. The f_{ijk} are themselves often called the structure constants of the group. They are given in Table 3.1.

There are two Casimir operators of $SU(3)$, one quadratic and the other cubic in the generators. We shall have occasion to use only the quadratic Casimir operator F^2, which is given by

$$F^2 = \frac{1}{4}\lambda^2 = \frac{1}{4}\sum_{i=1}^{8} \lambda_i^2. \tag{3.26}$$

Table 3.1. Nonvanishing structure constants of $SU(3)$. The f_{ijk} are antisymmetric under permutation of any two indices.

ijk	f_{ijk}	ijk	f_{ijk}
123	1	345	1/2
147	1/2	367	$-1/2$
156	$-1/2$	458	$\sqrt{3}/2$
246	1/2	678	$\sqrt{3}/2$
257	1/2		

We now introduce a different representation for the generators, which is suitable to generalization to any $SU(n)$. We introduce the notation H_a, $(a = 1, 2...n-1)$ for the mutually commuting generators (which can be simultaneously diagonalized) and the notation E_{ab} for the $n^2 - n$ nondiagonal generators. These are $n \times n$ matrices with matrix elements given by

$$(H_a)_{jk} = \delta_{jk}[\sum_{l=1}^{a} \delta_{jl} - a\delta_{j,a+1}], \tag{3.27}$$

$$(E_{ab})_{jk} = \delta_{aj}\delta_{bk}, \quad a \neq b, \tag{3.28}$$

where δ_{ab} are elements of the unit matrix. If $a > b$, E_{ab} is a lowering operator; if $a < b$, it is a raising operator. Also, $E_{ab} = E_{ba}^{\dagger}$. In $SU(2)$,

$$H_a = \sigma_3, \quad E_{12} = \sigma_+, \quad E_{21} = \sigma_-. \tag{3.29}$$

We see from Eqs. (3.27) and (3.28) that the $SU(3)$ generators are

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.30}$$

$$E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It should be clear from the examples we have given for $SU(2)$ and $SU(3)$ that it is straightforward to write down the matrix generators of any $SU(n)$ in both the λ_i and (H_a, E_{ab}) representations (up to normalization constants).

We can use the lowering operators $E_{a+1,a}$ and their hermitian conjugates to obtain the Clebsch-Gordan coefficients of any $SU(n)$. We show in the next section how this is done.

The trace of a matrix is the sum of its diagonal elements. The n^2-1 generators of $SU(n)$ are traceless matrices in n dimensions. Any real $n \times n$ traceless matrix can be written as a linear combination of them, and any real $n \times n$ matrix can be written as a linear combination of them and the unit matrix in n dimensions.

3.6 Multiplets of unitary groups

The n-dimensional generators of $SU(n)$ operate on n-dimensional column vectors. Clearly, there are n linearly independent vectors, which we may denote by u_a, $(a = 1, 2...n)$. A convenient representation for these vectors is that the j-th row of u_a is equal to δ_{aj}. In $SU(2)$ the u_a are

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.31}$$

In $SU(3)$ they are

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.32}$$

We can order the vectors from *highest* to *lowest*, such that u_a is *higher* than u_b if $a < b$. The vectors u_a of $SU(n)$ are said to belong to the first *fundamental* representation. Altogether, $SU(n)$ has $n-1$ inequivalent so-called fundamental representations, two of which have n dimensions, except for $SU(2)$, which has only one fundamental representation. All the multiplets of $SU(n)$ can be built up from the vectors of the first fundamental representation using only the raising and lowering matrices of the n-dimensional representation of the Lie algebra. Sometimes, however, it is convenient to use more than one fundamental representation in building the multiplets.

The eigenvalues of the H_a operating on a state vector is called the *weight* $\mathbf{m}$ of the vector. For example, in $SU(3)$, we see from Eqs. (3.30) and (3.32) that $(H_1, H_2)u_2 = (-1, 1)u_2$. therefore, the weight $\mathbf{m}$ of u_2 is $\mathbf{m} = (m_1, m_2) = (-1, 1)$.

We see from the definitions of E_{ab} and the u_a, that

$$E_{ab}u_c = u_a\delta_{bc}. \tag{3.33}$$

With our ordering of the vectors such that u_a is higher than u_b if $a < b$, we can see that the operators E_{ab} are indeed raising or lowering operators for $a < b$ or $a > b$ respectively. We shall restrict ourselves to the the lowering operators $E_{a+1,a}$ and their hermitian conjugates, which are raising operators.

Let us use the u_a in $SU(2)$ and $SU(3)$ to build up some other multiplets of these groups. We start with $SU(2)$, which is a case which should be familiar. To make things more concrete, u_1 and u_2 can be the spin-up and spin-down wave functions (state vectors) of a particle of spin 1/2. As another possibility, they can be the isospin wave functions of a u and d quark respectively. If we wish to consider N-particle wave functions, we start

with the wave function $u_1(1)u_1(2)...u_1(N)$, where the numbers in parentheses stand for particle 1, particle 2, etc. A simpler notation is to omit the numbers in parentheses, and by convention write the wave function of particle 1 first, etc. We shall adopt this simpler notation in the following. We also introduce the notation that the lowering operator E_{ab} operating on an N-particle state is given by

$$E_{ab} = \sum_{i=1}^{N} E_{ab}(i). \tag{3.34}$$

It is best to begin with only two particles. Let $\chi_1 = u_1 u_1$ and operate on χ_1 with the lowering operator

$$E_{21} = E_{21}(1) + E_{21}(2) = \sigma_-(1) + \sigma_-(2). \tag{3.35}$$

We get

$$E_{21}\chi_1 = u_1 u_2 + u_2 u_1 = \sqrt{2}\chi_2, \tag{3.36}$$

where we have defined χ_2 to be a normalized state. Repeating the operation, we get

$$E_{21}\chi_2 = \sqrt{2}u_2 u_2 = \sqrt{2}\chi_3. \tag{3.37}$$

If we operate on χ_3 we get 0. Thus, starting from two doublets of $SU(2)$, we have obtained a triplet state χ_i, corresponding to spin or isospin 1. We next construct a state ϕ_1 orthogonal to χ_2. We see that ϕ_1 must be given by

$$\phi_1 = (u_1 u_2 - u_2 u_1)/\sqrt{2}. \tag{3.38}$$

If we operate on ϕ_1 we get 0, as we expect, so that this state is a singlet, corresponding to spin or isospin 0. We can obtain the eigenvalues of the diagonal operators J_3 and J^2 by directly operating on the χ_i and ϕ_1.

The coefficients multiplying the product wave functions $u_i u_j$ in the expressions for χ_i and ϕ_i are known as Clebsch-Gordan coefficients. In the case we have considered, these Clebsch-Gordan coefficients are unique, but in the case of the product of three or more wave functions, the Clebsch-Gordan coefficients can depend on somewhat arbitrary definitions of wave functions. We can see this as follows: If we start with the product $u_1 u_1 u_1$, we can use the lowering operator E_{21} to construct all the symmetric wave functions belonging to the same multiplet as $u_1 u_1 u_1$. The problem arises when we want to construct other multiplets. For example, consider the (unnormalized) wave function ξ given by

$$\xi = u_1 u_1 u_2 + u_1 u_2 u_1 + u_2 u_1 u_1. \tag{3.39}$$

There are *two independent* wave functions orthogonal to ξ, and the Clebsch-Gordan coefficients depend on what linear combination of these wave functions we choose. The choice in some instances is given by convention, but real questions of physics can influence what choice is convenient.

Let us now generalize to $SU(3)$. The χ_i and ϕ_1 from $SU(2)$ are also wave functions of $SU(3)$, but they are not complete multiplets. We can operate on χ_2 with E_{32} to get

$$\chi_4 = (u_1u_3 + u_3u_1)/\sqrt{2}. \tag{3.40}$$

Operating on χ_4 with E_{21}, we get

$$\chi_5 = (u_2u_3 + u_3u_2)/\sqrt{2}, \tag{3.41}$$

and operating on χ_5 with E_{32}, we get after normalizing

$$\chi_6 = u_3u_3. \tag{3.42}$$

Thus, the χ_i are a sextet of $SU(3)$. Likewise, from ϕ_1, we can obtain

$$\phi_2 = u_1u_3 - u_3u_1, \quad \phi_3 = u_2u_3 - u_3u_2, \tag{3.43}$$

so that the ϕ_i are a triplet of $SU(3)$.

We now define a new triplet of vectors $\bar{u}_a$ given by

$$\bar{u}_1 = \phi_3, \quad \bar{u}_2 = -\phi_2, \quad \bar{u}_3 = \phi_1. \tag{3.44}$$

By explicit construction we find

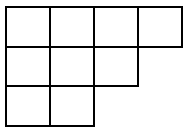
$$E_{ab}\bar{u}_c = -\bar{u}_b\delta_{ac}. \tag{3.45}$$

This differs from the action of E_{ab} on u_c. The $\bar{u}_a$ are the vectors belonging to the second fundamental representation of $SU(3)$.

3.7 Young tableaux

A *Young tableau* or *Young diagram* describes the symmetry of a collection of an integer ν identical particles, each of which can be in one of several available states. We shall confine our considerations to the description of Young diagrams in the case that the symmetry group is $U(n)$ or $SU(n)$ and the particles belong to the first fundamental representation of the group. Then the number of possible states of a particle is also n. An example is a collection of five electrons, each of which can be in one of two spin states (spin up or spin down). In this case $\nu = 5$, $n = 2$.

A Young tableau is a collection of boxes, one for each particle, arranged in rows and columns to represent the symmetry of the state: symmetric in rows, antisymmetric in columns. A *proper* tableau is one in which each row is no longer than the one above it and each column is no longer than the one to the left of it. When we refer to a Young tableau, we shall mean a proper tableau unless otherwise specified. An example of a proper tableau is the following:



(3.46)

We begin by using Young tableaux in connection with $SU(2)$. We denote the basis vectors of the fundamental doublet of $SU(2)$ by u_1 and u_2. These vectors may, for example, denote the two states of a particle with spin 1/2. Another notation for these one-particle states is by means of a Young tableau with a single box. We make the identification

$$u_1 = \boxed{1}\,, \qquad u_2 = \boxed{2} \tag{3.47}$$

The single box without a number stands for both members of the doublet. The same considerations hold for $U(2)$.

Now suppose we have a two-particle state. If it is a symmetric ψ_s, we denote it by a row, and if is an antisymmetric state ψ_a, by a column:

$$\psi_s = \begin{array}{|c|c|}\hline & \\ \hline \end{array}\,, \qquad \psi_a = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \tag{3.48}$$

These tableaux represent multiplets which are different than the two-dimensional fundamental multiplet. Consider first the symmetric state. If both particles are in the state u_1 or both are in the state u_2, the corresponding tableaux are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}\,, \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}\,.$$

There is one symmetric state with one particle having the state vector u_1 and the other having the vector u_2, namely $(u_1u_2 + u_2u_1)/\sqrt{2}$. (We adopt the convention of writing the state vector of the first particle at the left, the vector of the second particle next, etc. This convention saves us the trouble of writing $[u_1(1)u_2(2) + u_2(1)u_1(2)]/\sqrt{2}$.) This symmetric state is represented by the tableau

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\,.$$

The arrangement

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline \end{array}$$

is obviously the same as the previous arrangement, and must not be counted. It is called a *nonstandard* arrangement. Thus, the symmetric state is a triplet. There is only one antisymmetric two-particle state $(u_1u_2 - u_2u_1)/\sqrt{2}$, corresponding to the arrangement

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\,.$$

The other arrangement

$$\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$$

is nonstandard and must not be counted.

The above considerations for $U(2)$ or $SU(2)$ can be generalized to any $U(n)$ or $SU(n)$ and lead us to the following definition:

A *standard arrangement* of a tableau of $U(n)$ or $SU(n)$ is a proper tableau containing a postive integer i in each box ($1 \leq i \leq n$) such that the integers increase in going from top to bottom in a column and do not decrease in going left to right in a row. Hereafter, unless we explicitly state otherwise, an arrangement will mean a standard arrangement.

An important theorem which we do not prove is that the number N of standard arrangements of a Young tableau with positive integers no greater than n is equal to the dimension of an irreducible representation of $U(n)$ or $SU(n)$.

We see that a Young tableau for $U(n)$ or $SU(n)$ consists of ν boxes in no more than n rows. The tableaux are limited to n rows because one cannot antisymmetrize a configuration of more than n particles when each particle has only n available states.

As an example, consider a collection of five electrons, each of which can be in one of two spin states (spin "up" u_1 or spin "down" u_2). In this case $\nu = 5$, $n = 2$, and the symmetry group of the spins is $SU(2)$. If we include the lepton number in our description (an additive quantum number), then the symmetry group is $U(2)$ or $SU(2) \times U(1)$. (We do not distinguish between these last two groups, as we use only the Lie algebra, which is the same for both.) The dimensionality is the same whether the lepton number is included in the description. Electrons must obey Fermi statistics, that is, their state vectors must be antisymmetric under the interchange of *all* the coordinates of any two electrons. However, the symmetry under the interchange of only the spins is given by any standard Young tableau.

Consider a three-electron state belonging to the Young tableau

This state has the same multiplicity as the one-electron state belonging to the tableau

but the lepton number is 3 in the first case and 1 in the second. If we are interested only in the $SU(2)$ of the spins and not in the lepton number, the two diagrams are equivalent in that they correspond to the same quantum numbers.

To avoid the complication of the quantum number associated with the $U(1)$ subgroup of $U(n)$, we often restrict ourselves to $SU(n)$. Then all columns with n boxes may be removed from a given tableau, as there is only one way to antisymmetrize a state of n particles, each of which has n degrees of freedom. The number of states N_n is equal to the result obtained by counting the number of ways one can put positive integers $\leq n$ in the remaining boxes, consistent with the rules that numbers must not decrease going from left to right in any row and must increase going from top to bottom in any column.

We see that in the case of $SU(n)$, a tableau with ν boxes can also denote states of the corresponding symmetry containing a different number of particles. We can therefore divorce the concept of the symmetry of a state corresponding to a Young tableau from the concept of particles belonging to the first fundamental representation.

A tableau with no more than $n-1$ rows can be specified by a set of $n-1$ integers p_i, which is the number of boxes in row i minus the number of boxes in row $i+1$. The multiplicity N_n of any diagram is a function of the p_i. It is a complicated combinatorial problem to find the number of standard arrangements of a tableau with a given $\mathbf{p}$. The formula for $N_n(p_1, p_2 ... p_{n-1}) = N_n(\mathbf{p})$ is known for any n but we write it down just for $n = 2$ and $n = 3$. For $SU(2)$ we have

$$N_2(p) = p + 1. \tag{3.49}$$

The number of states of a given angular momentum j is $2j + 1$. Then, using Eq. (3.40), we can make the identification

$$p = 2j. \tag{3.50}$$

For $SU(3)$ the formula is

$$N_3(\mathbf{p}) = (p_1 + 1)(p_2 + 1)(p_1 + p_2 + 2)/2. \tag{3.51}$$

These formulas give the number of states in a multiplet belonging to an irreducible representation of the group with the symmetry specified by the Young tableau $\mathbf{p}$.

We see that for $SU(2)$ the number N_2 can be any positive integer. However, for $SU(n)$ with $n > 2$, the numbers N_n include only a proper subset of the positive integers. For example, in $SU(3)$, the numbers N_3 have the values 1, 3, 6, 8, 10, 15, etc, as determined either by counting the standard arrangements of Young tableaux or from Eq. (3.51), substituting non-negative integers for p_1 and p_2.

All the formulas $N_n(\mathbf{p})$ are symmetric under the interchange

$$p_i \leftrightarrow p_{n-i},$$

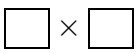
that is,

$$N_n(p_1, p_2 ... p_{n-1}) = N_n(p_{n-1} ... p_1). \tag{3.52}$$

Two Young tableaux which transform into each other under this transformation are called *conjugate tableaux*, and the irreducible representations which act on them are called *conjugate representations*. The first fundamental representation of $SU(3)$ is characterized by $(p_1, p_2) = (1, 0)$; the second fundamental representation is the conjugate representation, and is characterized by $(p_1, p_2) = (0, 1)$. However, it is common to characterize these (and other representations) by a single number which gives their multiplicity: in the case of the first and second representations, we use 3 and $\bar{3}$ respectively. Conjugate representations have the same dimensionality, but conjugate tableaux do not necessarily have the same number of boxes. If a tableau is unchanged under the transformation of interchanging p_i and p_{n-i}, it is self-conjugate, and likewise for the corresponding representation. Since all representations of $SU(2)$ consist of a single row, all representations of $SU(2)$ are self-conjugate.

We now show how to build all irreducible representations of $SU(n)$ starting from the first fundamental one. First we note that a product of two or more representations is in

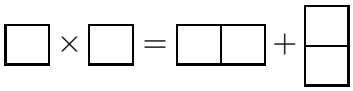
general reducible. We can see this by considering the basis vectors on which they act. It is simplest to begin with $SU(2)$. The product states are written using Young tableaux as



These stand for the four product states

$$u_1u_1, \quad u_1u_2, \quad u_2u_1, \quad u_2u_2.$$

But we know that to obtain the basis vectors of irreducible representations we must take the linear combinations which correspond to the symmetric and antisymmetric Young tableaux. This result is true in any $SU(n)$. We write



(3.53)

In $SU(2)$, the multiplicities are

$$SU(2): \quad 2 \times 2 = 3 + 1. \tag{3.54}$$

In $SU(3)$, we have

$$SU(3): \quad 3 \times 3 = 6 + \bar{3}, \tag{3.55}$$

and in any $SU(n)$ we have

$$SU(n): \quad n \times n = n(n+1)/2 + n(n-1)/2. \tag{3.56}$$

In the above examples we have found the irreducible representations contained in the product of two irreducible representations. This decomposition is called the *Clebsch–Gordan series*. If the decomposition contains no representation more than once, the product is called *simply reducible*. If $n > 2$, the decomposition of $n \times \bar{n}$ is different from the decomposition of $n \times n$. We have

$$SU(n): \quad n \times \bar{n} = (n^2 - 1) + 1, \tag{3.57}$$

which is different from the decomposition given in Eq. (3.57). In particular, in $SU(3)$ we have

$$3 \times \bar{3} = 8 + 1. \tag{3.58}$$

We now tell how to find the Clebsch–Gordan series for the product of any two representations of $SU(n)$. We do not give the proof, as it is quite complicated (even the recipe is complicated).

Recipe. We draw the Young tableaux of the two representations, marking each box of the second with the number of the row to which it belongs. We then attach the boxes of

the second tableau in all possible ways to the first tableau, subject to the following rules for the combined tableaux:

1) Each tableau should be proper.
2) No tableau should have a column with more than n boxes, and we can remove all columns with n boxes.
3) The numbers must not decrease from left to right in a row.
4) The numbers must increase from top to bottom in a column.
5) We can make a path by moving along each row from the right, starting at the top. At each point of the path, the number of boxes encountered with the number i must be less or equal to the number of boxes with $i-1$.

As an example, if we follow the rules, we find the irreducible representations contained in 8×8 of $SU(3)$ to be

$$8 \times 8 = 27 + 10 + \bar{10} + 8 + 8 + 1. \tag{3.59}$$

We see that the Clebsch–Gordan series contains two equivalent representations, namely, the two 8's. This means that the product of 8×8 is not simply reducible.

The product of any two representations of $SU(2)$ is simply reducible, but this result does not hold for any $n > 2$. Even for $SU(2)$, the product of three representations is not necessarily simply reducible. For example, in $SU(2)$, we have

$$2 \times 2 \times 2 = 4 + 2 + 2. \tag{3.60}$$

If two or more equivalent representations appear in the reduction of a product, the use of group theory alone is not sufficient to enable us to label the states. We must know something of the physics of the problem in order to obtain the most useful labeling. The Clebsch–Gordan coefficients are also not determined uniquely without additional input.

For example, the reduction given in Eq. (3.60) can come about from obtaining the irreducible multiplets from three electron spins. One of the two doublets arises by combining the spins of the first two electrons to give a triplet and then combining the third spin to give a doublet. The other doublet arises from combining the first two spins to form a singlet and then combining the third spin to give a doublet. Group theory says nothing about how the first two spins in fact combine, as that depends on the forces involved. As another example, in $SU(3)$ we have

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1. \tag{3.61}$$

The way the two 8's are chosen in either the decomposition (3.59) or (3.61) depends on the physics.

3.8 Evaluating quadratic Casimir operators

The group $SU(n)$ has $n-1$ Casimir operators, one of which is quadratic in the generators. The quadratic Casimir of $SU(n)$ may be evaluated by making use of the raising and lowering operators. For $SU(2)$ this operator is J^2 and is given by

$$J^2 = \frac{1}{4}(2\sigma_+\sigma_- + 2\sigma_-\sigma_+ + \sigma_z^2). \tag{3.62}$$

We take advantage of the fact that J^2 is an invariant operator to operate on the particular state for which σ_+ vanishes. We eliminate the term $\sigma_+\sigma_-$ by making use of the commutation relation

$$[\sigma_+, \sigma_-] = \sigma_z. \tag{3.63}$$

This enables us to write J^2 (when operating on our particular state) only in terms of the diagonal operator σ_z. Since σ_z is an additive quantum number, it is straightforward to evaluate J^2 for any representation. Details are given in books on quantum mechanics. The answer is $J^2 = j(j+1)$, or, since $p = 2j$,

$$J^2 = \frac{1}{2}p(\frac{1}{2}p + 1). \tag{3.64}$$

The same method works for any $SU(n)$ except that the algebra is more complicated. In particular, for $SU(3)$ we get

$$F^2 = p_1 + p_2 + \frac{1}{3}(p_1^2 + p_2^2 + p_1 p_2). \tag{3.65}.$$