Error Analysis for the Euler Method

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We want to do a very careful estimate of the error for the simple differeantial equation

$$dy/dx = f(x). \tag{1}$$

This example is simple because the right hand side depends on x, not $y(x)$. So solving this differential equation is just doing an integral.

The solution can be written as

$$y(x_2) - y(x_1) = \int_{x_1}^{x_2} y(x)dx \tag{2}$$

In our case, we will compare this integral with the first step of the Euler method, so let $x_1 = x$ and $x_2 = x + \Delta x$, where Δx is our step size.

The Euler approximation to our differential equation is

$$y_{Euler}(x + \Delta x) = y(x) + f(x)\Delta x \tag{3}$$

The exact solution is

$$y_{Exact}(x + \Delta x) = y(x) + \int_{x}^{x+\Delta x} f(x')dx' \tag{4}$$

We will use a Taylor series expansion for $f(x')$:

$$f(x') = f(x) + (x' - x)f'(x) + (1/2!)(x' - x)^2 f''(x) + (1/3!)(x' - x)^3 f'''(x) + \dots \tag{5}$$

Integrating term by term, we have for the integral on the RHS of Eq. (4)

$$f(x)\Delta x + (1/2!)f'(x)(\Delta x)^2 + (1/3!)f''(x)(\Delta x)^3 + \dots \tag{6}$$

Thus, the first term in Eq. (6) is the last term in Eq. (3) and we have

$$y_{Exact}(x + \Delta x) - y_{Euler}(x + \Delta x) = (1/2!)f'(x)(\Delta x)^2 + (1/3!)f''(x)(\Delta x)^3 + \dots \tag{7}$$

In our first example of the Euler program, $f(x) = 2x$. Thus, $f'(x) = 2$ and $f''(x) = 0$, so the second term on the RHS of Eq. (7) is 0. Why is the error order Δx, not $(\Delta x)^2$? That is because the above expression is for each step. In our example, we integrate x from 1 to 2, so the number of steps $n = (1/\Delta x)$. Thus, the total error is Δx for this case.

In our second example, $f(x) = 3x^2$. $f'(x) = 6x$ and $f''(x) = 6$, so the second error term does not vanish. If you very carefully evalute the two error terms you will find that the first error term is

$$3(\Delta x)^2 * n(1 + (n-1) * \Delta x/2) \tag{8}$$

where $n = 1/\Delta x$. Using the fact that $n * \Delta x = 1$, this first term can be written as

$$3\Delta x(3/2 - \Delta x/2) \quad \text{or} \quad 9/2 * \Delta x - 3/2 * (\Delta x)^2. \tag{9}$$

Actually, only this term requires care. The f'' term is easily seen to be $(\Delta x)^2$. Adding the two terms, our final result is

$$9/2 * \Delta x - 1/2 * (\Delta x)^2. \tag{10}$$

Before we did the careful estimate, we used a program polyfit to fit the result as a function of step size with a second order polynomial. That result agrees with the error calculation above. It is also possible to plot the error divided by dt and find that the coefficient of the second term is negative, not positive as we would get from just looking at the second term in the Taylor expansion.