Exam 1 will be in class, Wednesday, February 10. It will cover the section on AC circuits and Chapter 8 of the Griffiths text.

0. If you didn’t do it last week: Demonstrate that Eqs. 8.21 and 8.18 in Griffiths are, in fact, equal. You need to do this for just one component, say, the $x$ component. In other words, start by writing down

$$f_x = (\nabla \cdot \mathbf{T})_x - \varepsilon_0 \mu_0 \frac{\partial S_x}{\partial t}$$

and show explicitly, line by line, that this is equivalent to

$$f_x = \varepsilon_0 \left[ (\nabla \cdot \mathbf{E}) E_x + (\mathbf{E} \cdot \nabla) E_x \right] + \frac{1}{\mu_0} \left[ (\nabla \cdot \mathbf{B}) B_x + (\mathbf{B} \cdot \nabla) B_x \right] - \frac{1}{2} \frac{\partial}{\partial x} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})_x.$$

In doing so, use Eq. 8.20 to demonstrate that the $x$ component of the divergence of $\mathbf{T}$ is as written in the text. In other words, show also that the expression just above Eq. 8.21 for $(\nabla \cdot \mathbf{T})_x$ is correct.

1. Fun with tensors: Remember in class we discussed the Kronecker delta $\delta_{ij}$, which can be treated as a second rank tensor with $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$, and also the Levi-Civita tensor $\epsilon_{ijk}$, where $\epsilon_{ijk} = 1$ if $ijk$ is an even permutation of 123, $\epsilon_{ijk} = -1$ if $ijk$ is an odd permutation of 123, and $\epsilon_{ijk} = 0$ otherwise.

(a) Demonstrate that

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

(b) Use this, along with the fact that a cross product can be expressed as $\epsilon_{ijk} A_j B_k$ to prove the vector product identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}).$$

If you were to do this by computing all the components of the cross product, this would take a lot of writing. It can be done in 2-3 easy lines using tensor arithmetic and the identity above.

2. (a) Consider two equal point charges $q$, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell’s stress tensor over this plane, determine the force of one charge on the other.

(b) Repeat for charges that are opposite in sign.

3. Consider an infinite parallel-plate capacitor, with the lower plate (at $z = -d/2$) carrying the charge density $-\sigma$, and the upper plate (at $z = +d/2$) carrying the charge density $+\sigma$.

(a) Determine all nine elements of the stress tensor $\mathbf{T}$ in the region between the plates. You can write your answer as a $3 \times 3$ matrix with elements $T_{ij}$.

(b) Using the equation

$$\mathbf{F} = \oint \mathbf{T} \cdot d\mathbf{a}$$

compute the force per unit area on the top plate. (Note that the above equation is correct since $\mathbf{S} = 0$ in this case.) Does this agree with what you expect?

(c) What is the momentum per unit area crossing the $x,y$ plane?

(d) At the plates, this momentum you computed in (c) is absorbed, and the plates recoil. Find the recoil force per unit area on the top plate and compare with your answer to (b). (This force is the same force you calculated in (b), it was just determined in a different way: from the momentum in the field, rather than directly from the field itself.)
4. A charged parallel plate capacitor with plate area $A$, separation $d$, and uniform electric field $E = E \hat{z}$ is placed in a uniform magnetic field $B = B \hat{x}$ (parallel to the plates, which are parallel to the $x, y$ plane).

(a) Find the electromagnetic momentum between the plates.

(b) Now a resistive wire is connected between the plates, along the $z$ axis so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system during the discharge?

(c) Instead of turning off the electric field as we did in (b), supposed that we slowly reduce the magnetic field. This will induce an electric field by Faraday’s law which, in turn, exerts a force on the plates. Show that the total impulse is equal to the momentum originally stored on the plates.

5. Study Ex. 8.4 in Griffiths. Suppose that instead of turning off the magnetic field, we turn off the electric field by connecting a radial spoke between the cylinders. (This is not that different from how we turned off the electric field in the capacitor for the problem above.) From the magnetic force on the current in the spoke, determine the total angular momentum delivered to the cylinders as they discharge. (They now rotate together.) Compare this to the initial angular momentum stored in the fields:

$$L_{\text{em}} = -\frac{1}{2}\mu_0 n I Q (R^2 - a^2) \hat{z}. \quad (4)$$

Like parts (b) and (c) of the problem above this illustrates two completely different ways of “harvesting” the angular (in the previous problem, linear) momentum from the field.

6. Suppose that you had an electric charge $q_e$ and a magnetic monopole $q_m$. The field of the electric charge is given by

$$E = \frac{1}{4\pi \epsilon_0} \frac{q_e}{r^2} \hat{r}. \quad (5)$$

Suppose the field of the magnetic monopole is

$$B = \frac{\mu_0 q_m}{4\pi} \frac{\hat{r}}{r^2}. \quad (6)$$

Assume the two charges are separated by a distance $d$ – you can place the electric charge at the origin and the magnetic charge a distance $d$ away from the origin on the $z$-axis. Show that the total angular momentum stored in the fields is (remarkably, independent of $d$ and) equal to

$$L_{\text{em}} = \frac{\mu_0}{4\pi} q_e q_m. \quad (7)$$

Doing so requires a little work, but the result is worth it. Start by using the law of cosines to express the $B$ field in terms of the radial coordinate $r$ and the $\hat{r}$ and $\hat{z}$ unit vectors. Why is the result so cool? Quantum mechanics requires quantization of $L$ in units of $\hbar$. This would imply that that product $q_e q_m$ must be quantized. If there existed a magnetic monopole with magnetic charge $q_m$, quantum mechanics could be used to explain the quantization of electric charge we observe in nature.

7. A point charge $q$ is a distance $a > R$ from the axis of an infinite solenoid (radius $R$, $n$ turns per unit length, current $I$). Show that the linear momentum and angular momentum in the fields are

$$p_{\text{em}} = \frac{\mu_0 q n I R^2}{2a} \hat{y}; \quad L_{\text{em}} = 0. \quad (8)$$

Put $q$ on the $x$ axis, with a solenoid along $z$; treat the solenoid as a nonconductor, so you don’t need to worry about induced charges on its surface.