1. **Quality factor of a series RLC oscillator:** One often uses a “quality factor” or “$Q$” to characterize a resonator, whether mechanical or electrical. The higher the $Q$ the longer the resonator oscillates, e.g. a bell with a high $Q$ rings for a long time after being struck. One definition for $Q$ (a dimensionless quantity) is

$$Q = \frac{\omega \times \text{Total Energy of Oscillator}}{\text{Average Rate of Energy Loss}}.$$ 

(a) Show that for a damped series RLC oscillator (like we studied in class) $Q = \frac{1}{R} \sqrt{\frac{L}{C}}$.

(b) There exists a critical value of $Q$. Oscillators with a $Q$ above this value will exhibit damped oscillations, while oscillators with a $Q$ below this value will be “over-damped” and slowly return to equilibrium without oscillation. What is the critical value of $Q$ at which this happens?

2. **How does $Q$ affect the properties of a circuit?** Consider a driven series RLC circuit. In addition to $Q$, the other dimensionless number that is useful for characterizing the circuit is $\omega' \equiv \omega/\omega_r$, where $\omega$ is the angular frequency of the driving AC power source and $\omega_r = 1/\sqrt{LC}$ the resonant frequency of the circuit.

(a) Show that the impedance of the RLC oscillator can be written in terms of $Q$ and $\omega'$ as

$$Z = \left[1 + iQ \left(\omega' - \frac{1}{\omega'}\right)\right] R$$

(b) Recall from class that the rate of energy dissipation in the circuit has a factor of $\cos \delta = \frac{R}{|Z|}$ where $\delta$ is the phase between the current and the voltage in the circuit. Plot $R/|Z|$ for values of $Q = 1, 3, 10, 30$ as a function of $\log \omega'$ from where the values of $\omega'$ range from 0.1 to 10 (log $\omega'$ goes from -1 to 1).

(c) Now plot the phase angle between the current and the voltage over the same range in $\omega'$ for the four values of $Q$ listed above.

It should be clear from these two parts that the higher the $Q$ of the circuit, the narrower the frequency band at which there is efficient energy transfer from the driving source into the resonator; however, since higher $Q$ means less average power loss, the amplitude of the oscillations when the resonator is driven at the resonant frequency also increases with $Q$.

3. **Filters revisited:** Last week we examined how to improve the cutoff of a high-pass filter – after the addition of just a few extra components the calculation quickly became impossible. It turns out that the infinite “ladder” is not too hard to solve. Consider the following infinitely long circuit composed of an infinite number of sections.

![Diagram of an infinite RLC ladder circuit]
(a) Show that the effective impedance of this circuit $Z_{eq}$ is given by

$$Z_{eq} = \frac{Z_1}{2} + \sqrt{\left(\frac{Z_1^2}{4}\right) + Z_1 Z_2}.$$  

Do this by assuming that if the chain is really infinite, then the equivalent impedance doesn’t change by adding one additional section.

(b) If $n$ is the number of sections in the circuit, show that for large $n$ the following is true

$$\frac{V_{n+1}}{V_n} = \alpha^n; \quad \alpha \equiv \left(\frac{Z_{eq} - Z_1}{Z_{eq}}\right).$$

Do this by first computing $V_{n+1}/V_n$ using the fact that the equivalent impedance for $V_{n}$ is $Z_{eq}$.

(c) Now assume the ladder takes on the configuration of a high pass filter where $Z_1 = 1/i\omega C$ and $Z_2 = i\omega L$. Derive an expression for $\alpha$ and show that there there is a critical value of $\omega$ above which $|\alpha| = 1$ and below which $|\alpha| < 1$. What is the value of $\omega$? In the limit that $n \rightarrow \infty$, $|V_n/V_i| = 1$ above this frequency and $|V_n/V_i| = 0$ below this frequency.

4. Consider the circuit sketched below

(a) Calculate the impedance of the circuit. Write your answer in the form $Z = R + iX$. What is the value of $Z$ as $\omega \rightarrow \infty$ and $\omega \rightarrow 0$?

(b) Now assume that $R = 10 \, \Omega$, $L = 5 \, mH$, and $C = 5 \, \mu F$. What is the magnitude and phase of $Z$ at a frequency of 1 kHz? Does the current in the circuit lead or lag behind the voltage at 1 kHz?

(c) What is the power dissipated in the circuit if the current is 100 mA at 1 kHz?

5. Griffiths Problem 8.1

6. Griffiths Problem 8.9

7. Demonstrate that Eqs. 8.21 and 8.18 in Griffiths are, in fact, equal. You need to do this for just one component, say, the $x$ component. In other words, start by writing down the expression:

$$f_x = (\nabla \cdot \vec{T})_x - \epsilon_0 \mu_0 \frac{\partial S_x}{\partial t}$$

and show explicitly, line by line, that this is equivalent to

$$f_x = \epsilon_0 [(\nabla \cdot \vec{E})_x + (\vec{E} \cdot \nabla)E_x] + \frac{1}{\mu_0} [(\nabla \cdot \vec{B})B_x + (\vec{B} \cdot \nabla)B_x] - \frac{1}{2} \frac{\partial}{\partial x} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \epsilon_0 \frac{\partial}{\partial t} (\vec{E} \times \vec{B})_x.$$  

In doing so, use Eq. 8.20 to demonstrate that the $x$ component of the divergence of $\vec{T}$ is as written in the text. In other words, show also that the expression just above Eq. 8.21 for $(\nabla \cdot \vec{T})_x$ is correct.