A capacitor with a non-uniform dielectric

Consider a parallel-plate capacitor filled with non-uniform dielectric. The dielectric is still linear, in other words, \( P = \epsilon_0 \chi_e E \), but this time \( \chi_e \) varies as a function of position in the dielectric. Let’s say that \( \chi_e \) varies such that

\[
\epsilon_r = \epsilon_{r0} + ax.
\]

Here \( \epsilon_{r0} \) is a dimensionless constant, \( x \) is the distance from one plate, and \( a \) is constant with units 1/length. Let’s assume the area of the plates is \( A \) and their separation is \( d \). What is the capacitance?

Remember the strategy for computing capacitance: put a charge \( Q \) on the capacitor and then compute the potential difference between the plates in terms of this charge. The capacitance can then be identified by comparing this expression to \( V = Q/C \). In order to compute \( V \) we first need to find the electric field.

Let’s charge up the plates so that we have a free surface charge density of \( +\sigma_f \) on the left plate and \( -\sigma_f \) on the right plate. We can then compute the electric displacement \( \mathbf{D} \) based on the free charge, and obtain

\[
\mathbf{D} = \sigma_f \hat{x}.
\]

This then gives

\[
\mathbf{E} = \frac{1}{\epsilon} \mathbf{D} = \frac{\sigma_f}{\epsilon_0 (\epsilon_{r0} + ax)} \hat{x}.
\]

Now, to obtain the potential, we need to compute the line integral of this field from the negative plate to the positive plate. We have

\[
V = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{\sigma_f}{\epsilon_0} \int_d^0 \frac{1}{\epsilon_{r0} + ax} dx = \frac{\sigma_f}{\epsilon_0 a} \left[ \ln(\epsilon_{r0} + ad) - \ln(\epsilon_{r0}) \right]
\]

Let’s rewrite the difference of logs as a log of a ratio and substitute in \( \sigma_f = Q/A \). This allows us to identify the capacitance

\[
V = \frac{Q \ln(1 + ad/\epsilon_{r0})}{A \epsilon_0 a} \implies C = \frac{A \epsilon_0 a}{\ln(1 + ad/\epsilon_{r0})}. \]

It is interesting to examine this expression in the limit that the dielectric is uniform. In this case we have \( a \to 0 \) and \( \epsilon_r = \epsilon_{r0} \). This poses a little problem since, in the expression above, substituting in \( a = 0 \) we have \( 0 / \ln(1) = 0/0 \). It is helpful to remember that for small values of \( y \) we can write

\[
\ln(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} + ...
\]

This means that for \( a \) very small we have

\[
C \approx \frac{A \epsilon_0 a}{\ln(1 + ad/\epsilon_{r0})} \approx \frac{A \epsilon_0 a}{(ad/\epsilon_{r0}) - (ad/\epsilon_{r0})^2/2} = \frac{A \epsilon_0}{d/\epsilon_{r0} - a(d/\epsilon_{r0})^2/2}.
\]

We can see that in the limit \( a \to 0 \) this expression reduces to \( \epsilon_0 \epsilon_0 A/d \), which is exactly what we expect for a parallel plate capacitor with dielectric constant \( \epsilon_{r0} \). Exercise: Compute the bound charge densities \( \sigma_b \) on both sides of the capacitor. Which side should have the higher bound charge density? In this case, \( \rho_b \neq 0 \) – compute it also. Does its sign agree with your intuition? Integrate \( \rho_b \) over the volume and show that it is equal in magnitude but opposite in sign to the sum of the bound surface charge.