1 Vector Calculus

1.1 Notation

1.1.1 Denoting a vector

Throughout the text vectors are typically noted in boldface upright characters. For example, the vector electric field is given by \( \mathbf{E} \). Scalars are noted by normal italic letters; the scalar potential is given, for example, by \( V \). It is really impossible to designate boldface letters on the blackboard so you will often see me write \( \vec{E} \) instead of \( E \).

1.1.2 Functions

We will also be dealing primarily with quantities that are a function of spatial coordinates. To be more compact, instead of writing \( \mathbf{E}(x, y, z) \) or \( \vec{E}(x, y, z) \), we will simply write \( E \) or \( \vec{E} \). It is important to remember that these quantities are functions and have a spatial (and, later in the course, time) dependence.

1.1.3 Unit vectors

Note also that unit vectors appear with a hat. For example the quantity \( \hat{x} \) means a vector with a length of one unit pointing in the direction of the \( x \)-axis. Some of you may be used to the notation \( \hat{i}, \hat{j}, \hat{k} \) to denote the unit vectors of the \( x, y, \) and \( z \) axes. Our text uses \( \hat{x}, \hat{y}, \hat{z} \).

1.2 Differential Vector Calculus

1.2.1 The \( \nabla \) operator

The \( \nabla \) (“del”) operator is given by

\[
\nabla = \left( \frac{\partial}{\partial x} \hat{x}, \frac{\partial}{\partial y} \hat{y}, \frac{\partial}{\partial z} \hat{z} \right).
\]

While not really a vector it behaves like a vector when operating on a function. Specifically, \( \nabla f \) produces a vector (the gradient), \( \nabla \cdot A \) produces a scalar (the divergence), and \( \nabla \times A \) produces a vector (the curl).

1.2.2 The gradient

When operating with \( \nabla \) on a scalar-valued function a vector-valued function is produced. For example, in two-dimensions, if we have

\[
f = x^2 + y^2,
\]

then

\[
\nabla f = 2x \hat{x} + 2y \hat{y}.
\]

Here one can visualize \( f \) as a surface above the \( x-y \) plane. The gradient then is a vector field that indicates the direction and magnitude of the largest change in slope of this surface.
1.2.3 The divergence

The divergence is represented as the scalar product of $\nabla$ with a vector field, e.g., $\nabla \cdot \mathbf{A}$. The divergence is a scalar-valued function. Writing out the components we have

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(4)

The divergence represents the sum of the rate of change of all components of a vector field. Again, we will work in two-dimensions since these functions can be more easily drawn. For example if

$$\mathbf{A} = x \hat{x} + y \hat{y},$$

then

$$\nabla \cdot \mathbf{A} = 2$$

(6)

One can visualize the divergence as a measure of how much the fields lines emanate from (positive divergence) or point to (negative divergence) a single point. Remember that the divergence is a function – in the case of

$$\mathbf{A} = x^2 \hat{x} + y^2 \hat{y}$$

(7)

$$\nabla \cdot \mathbf{A} = 2x + 2y.$$  

(8)

1.2.4 The curl

The curl is represented as a vector product of $\nabla$ and a vector field, e.g., $\nabla \times \mathbf{A}$. The vector product can easily be remembered as computing the determinant of a matrix:

$$\nabla \cdot \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix},$$

(9)

or, in component form:

$$\nabla \cdot \mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}. $$

(10)

The curl can be visualized as a measure of the rotation of a vector field at a particular point. This rotation is measured by a vector whose direction is given by the right hand rule and magnitude is given by the amount of rotation. Note that for a vector field that has no $z$ component the curl is purely in the $z$ direction, i.e. orthogonal to the plane of rotation. A simple vector field with nonzero curl is

$$\mathbf{A} = -y \hat{x} + x \hat{y}. $$

(11)

Here,

$$\nabla \times \mathbf{A} = 2 \hat{z} $$

(12)

Note that if the signs on each of the components of $\mathbf{A}$ are flipped the field “rotates” the opposite direction and the sign of the curl changes. Again it is important to stress that $\nabla \times \mathbf{A}$ is a vector field, i.e. a vector-valued function that can vary over space. In this case it is a constant field with length 2 in the $\hat{z}$ direction. Note that for the field $\mathbf{A} \nabla \cdot \mathbf{A} = 0$ – the vector field does not point to or emanate from any particular place.

1.2.5 Figures

Below are a few figures illustrating the functions mentioned in the above text.
Figure 1: The function $f = x^2 + y^2$ and its gradient $\nabla f = 2x \hat{x} + 2y \hat{y}$.

Figure 2: The function $f = y^2 - (x - 2)^2$ and its gradient $\nabla f = -2(x - 2) \hat{x} + 2y \hat{y}$.

Figure 3: The two vector fields $x \hat{x} + y \hat{y}$ (left) and $-x \hat{x} - y \hat{y}$ (right). The left has positive divergence while the right has a negative divergence.
Figure 4: The vector field \( \mathbf{A} = x^2 \hat{x} + y^2 \hat{y} \) and its divergence \( \nabla \cdot \mathbf{A} = 2x + 2y \). The divergence is plotted here as a function of \( x \) and \( y \). Note that the divergence has large positive values in quadrant 1 of the graph and large negative values in quadrant 3 of the graph. The divergence is the same and relatively smaller in quadrants 2 and 4.

Figure 5: The two vector fields \(-y \hat{x} + x \hat{y}\) (left) and \(y \hat{x} - x \hat{y}\) (right). The left has a curl that points in the \( \hat{z} \) direction while the right has a curl that points in the \(-\hat{z}\) direction.