Application of Gauss’ Law to infinite line of charge

The integral form of Gauss’ Law,

\[
\oint E \cdot da = \frac{Q_{enc}}{\epsilon_0},
\]

(1)
can be particularly useful for determining the electric field produced by various charge configurations. In these instances one usually writes an expression for the left hand side of this equation in terms of \(E\) and then solves for \(E\). In order to do this, one must choose a closed “Gaussian surface” over which to integrate the electric field. The general guidelines for picking a Gaussian surface, while not necessarily appropriate for every problem, are:

- the surface should include the point at which the electric field is to be determined, and
- the surface should be composed of parts or faces where either
  - \(E\) is parallel to \(da\) and the magnitude of \(E\) is constant, (In this case, \(\oint E \cdot da = EA\), where \(A\) is the area of the surface \(S\).) or
  - \(E\) is perpendicular to \(da\). (In this case \(\oint S E \cdot da = 0\) over the surface \(S\).)

**Example:** Assume that an infinite line of charge has linear charge density \(\lambda\). What is the magnitude of the electric field a distance \(s\) away from the line?

A cylinder with radius \(s\) and length \(l\) is a good Gaussian surface to solve this problem. It includes the set of points a distance \(s\) from the line. By symmetry we know that \(E\) must point radially outward and have a constant magnitude at a fixed distance from the line. This means that \(E\) is perpendicular to \(da\) on the ends of the cylinder and \(E\) is constant and parallel to \(da\) everywhere around the “side” (the curved surface of the cylinder).

To get the surface integral of \(E\) over the Gaussian surface we can write

\[
\oint E \cdot da = \int_{ends} E \cdot da + \int_{side} E \cdot da.
\]

(2)

Note \(\int_{ends} E \cdot da = 0\) since \(E\) is perpendicular to \(da\) everywhere on the ends of the cylinder. We also have that \(\int_{side} E \cdot da = EA\) where \(A = 2\pi sl\) (the area of the curved surface). This last relation is only true because the magnitude of \(E\) and the orientation of \(E\) with respect to \(da\) is the same over the entire curved surface. Therefore

\[
\oint E \cdot da = 0 + E(2\pi sl) = 2\pi slE
\]

(3)

We now have an expression for the left hand side of Eq. (1). To use it to obtain \(E\), we need an expression for the right hand side of Eq. (1). This is the charge enclosed by the cylinder, which is \(\lambda l\). (Remember \(\lambda\) has units of charge per length.)
Putting everything together we have:

\[ \oint E \cdot da = \frac{Q_{enc}}{\varepsilon_0}, \]  
\[ 2\pi s l E = \frac{\lambda l}{\varepsilon_0}. \]  

This implies that

\[ E = \frac{\lambda}{2\pi \varepsilon_0 s}, \]  

and, including what we know about the direction of \( \mathbf{E} \),

\[ \mathbf{E} = \frac{\lambda}{2\pi \varepsilon_0 s} \hat{s}. \]  

Remember that the key to this problem is creating a surface over which we can write the surface integral of the electric field in terms of the magnitude of the electric field. Use a cylinder for linear arrangements of charge, a box for planar arrangements of charge, and a sphere for spherical arrangements of charge. See pp. 70-74 of the text for additional examples.