P309 Intermediate Physics Laboratory
Lecture 1: Experimental Uncertainties

Reading:
• Squires “Practical physics”, pages 1-30
• Handouts from http://physics.indiana.edu/~courses/p309/s09
  • “Experimental uncertainties”
  • “Basic statistics”

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**“The” Scientific Method**

- Experimental physics has as its essential goal the precise measurement of the parameters of the natural universe. These can then be compared to our models (“theories”) of the natural universe to select the models that most accurately represent reality.

- The key branch point in this process is this one: “Do measurement and theory agree?” As scientists, we need to be able to answer this question in an objective and quantitative way. As such, we need not only an accurate measurement but a reliable estimate of the uncertainty in that measurement.

- Example:

  - **Theory:**
    - \( R_E = 6.378 \times 10^6 \) m
    - \( M_E = 5.974 \times 10^{24} \) kg
    - \( G = 6.6743 \times 10^{-11} \text{ m}^3/\text{kg}/\text{s}^2 \)
    - \( g = \frac{GM_E}{R_E^2} = 9.8017 \text{ m/s}^2 \)

  - **Measure time for object to fall:**
    - \( h = 3.0 \) m
    - \( t = 0.80 \) s
    - \( g = \frac{2x}{t^2} = 9.37 \text{ m/s}^2 \)

Experimenter has responsibility to provide not only the best estimate for the parameter being measured but also an accurate, quantitative estimate of the uncertainty in the value quoted.
Experimental Uncertainties

- Note on terminology: Sometimes (ok, actually, most often...) you will hear the uncertainty in a measurement referred as “the error on the measurement”. This is a short hand. Errors (that is, mistakes) in measurements should be corrected long before they are ever reported. Uncertainty, on the other hand, is something we have to live with. When you see phrases like “the error on the measurement” or and “error bar” think “the uncertainty on the measurement” and “the uncertainty range”. This jargon (using “error” when we mean “uncertainty”) is so ingrained in the way people talk you should just learn the meaning of “error” in this context.
Experimental Uncertainties

• There are two categories of measurement uncertainty
  - Statistical uncertainties are due to random noise in the measurement process. If you repeat the measurement many times this noise will eventually average out to zero.
  - Systematic uncertainties are due to the limitations of the methods or apparatus you are using to make the measurement. These affect all measurements in the same way and hence do not average out when the measurements are repeated.

• Example: Suppose you use a stop watch which reads to 1/100th of a second to measure a time \( t \)
  - Trial to trial your reactions might be different resulting in a variation in the measured time. This would be a random error (see, there I go using the word “error” when I really mean uncertainty) and so making a large number of trials will help reduce the size of the statistical uncertainty in the final quoted time \( t \)
  - But, you’ll never know the time to much better than \( \pm 0.01 \) s which is the limitation of the watch. This is a systematic uncertainty.
  - Also, your reaction time is probably slow making your measurement of the time \( t \) high by some amount. How well can you correct for that? This contributes to a systematic uncertainty.
Experimental Uncertainties

- **Systematic uncertainties** often require the experimenter to make good judgements and cross-checks about the precision of their apparatus and methods. Each source of systematic uncertainty is unique and there is no general prescription for how they should be estimated. This is part of the “art of the experimenter”.

- **Statistical uncertainties** (“random errors”), on the other hand, can be treated precisely mathematically.

- Suppose you measure some quantity $\mu$ using a process which has some random noise in it accumulating results $x_i$ which might look like this:

  ![Distribution after some very large number of measurements](image)

  This kind of chart is called a “histogram”.

- This distribution has a central value (quantified by the mean) and a width (quantified by the standard deviation). You would report the mean as the best estimate of $\mu_{\text{true}}$. The width is related to the size of the statistical uncertainty you would report.
Statistical Uncertainty

The final statistical error you would quote would be given by

Final result to quote would be \( \mu_{\text{TRUE}} \simeq \bar{x} \pm \sigma_m \)

Notice that error goes down like \( 1/\sqrt{N} \). Repeating a measurement four times yields an error bar which is twice as small

You can work out the error on a single measurement:

\[
\sigma_m = \frac{\sigma}{\sqrt{N}}
\]
Answering the question: Do measurement and theory agree?

- With a reliable estimate of the uncertainty in a measurement we can now make a quantitative answer to the question we really want to answer: Do measurement and theory agree? More precisely stated: “What is the likelihood that my measurement is a fluctuation of the true value predicted by the theory?”

- Common to assume measurements follow a “bell curve”, (aka Gaussian or normal) distribution:

\[ P(x)dx \equiv f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

Table 32.1: Area of the tails \( \alpha \) outside ±\( \delta \) from the mean of a Gaussian distribution.

<table>
<thead>
<tr>
<th>( \alpha ) (%)</th>
<th>( \delta )</th>
<th>( \alpha ) (%)</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>31.73</td>
<td>1( \sigma )</td>
<td>20</td>
<td>1.28( \sigma )</td>
</tr>
<tr>
<td>4.55</td>
<td>2( \sigma )</td>
<td>10</td>
<td>1.64( \sigma )</td>
</tr>
<tr>
<td>0.27</td>
<td>3( \sigma )</td>
<td>5</td>
<td>1.96( \sigma )</td>
</tr>
<tr>
<td>6.3×10^{-3}</td>
<td>4( \sigma )</td>
<td>1</td>
<td>2.58( \sigma )</td>
</tr>
<tr>
<td>5.7×10^{-5}</td>
<td>5( \sigma )</td>
<td>0.1</td>
<td>3.29( \sigma )</td>
</tr>
<tr>
<td>2.0×10^{-7}</td>
<td>6( \sigma )</td>
<td>0.01</td>
<td>3.89( \sigma )</td>
</tr>
</tbody>
</table>

Figure 32.4: Illustration of a symmetric 90% confidence interval (unshaded) for a measurement of a single quantity with Gaussian errors. Integrated probabilities, defined by \( \alpha \), are as shown.
Propagation of errors

• Final skill to master regarding error bars. If I have a function of several variables $f(x_1, x_2, x_3, ...)$ and I make measurements of $x_1, x_2, x_3, ...$ with uncertainties $\sigma_1, \sigma_2, \sigma_3, ...$ what is the overall uncertainty in $f$?

$$\sigma_f^2 = \left| \frac{\partial f}{\partial x_1} \right|^2 \sigma_{x_1}^2 + \left| \frac{\partial f}{\partial x_2} \right|^2 \sigma_{x_2}^2 + \left| \frac{\partial f}{\partial x_3} \right|^2 \sigma_{x_3}^2 + ...$$

• Read section on “Common sense in errors” in Squires. For complicated functions its sometimes easier to evaluate these terms numerically by shifting variables one-by-one by their errors and reevaluating $f$

• Some rules that are worth committing to memory:

$$f = a + b \quad \rightarrow \quad \sigma_f^2 = \sigma_a^2 + \sigma_b^2 \quad \text{sums and differences: errors add in quadrature}$$

$$f = ab \quad \rightarrow \quad \frac{\sigma_f^2}{f^2} = \frac{\sigma_a^2}{a^2} + \frac{\sigma_b^2}{b^2} \quad \text{products and quotients: percent errors add in quadrature}$$