Complex Numbers and AC Circuits

We pretend that there is an (imaginary) number, $i$, which, multiplied by itself, equals $-1$:

$$i^2 = \sqrt{-1}$$

A so-called complex number, $z = x + iy$, has both, a real part ($\text{Re}(z) = x$) and an imaginary part ($\text{Im}(z) = y$). The complex conjugate $z^*$ of $z$ one obtains by flipping the sign of all terms with an $i$ in them, i.e., $z^* = x - iy$.

Leonhard Euler (1707 – 1783) discovered the relation, which relates complex numbers to the (periodic) trigonometric functions (and which is the reason why we are doing what we are doing!):

$$e^{i\alpha} = \cos \alpha + i \cdot \sin \alpha$$

Consider the voltage $U(t)$ that appears somewhere in some AC circuit. This is a periodic function, so it makes sense to try to write this as

$$U(t) = U_0 \cdot e^{i\omega t}$$

This is a complex number, but instead of specifying its real and imaginary parts, we specify its amplitude $U_0$ and its phase $\omega t$. Here, $\omega$ is the angular frequency, which is related to the frequency $f$ by $\omega = 2\pi f$.

When we have any complex number $z$, in any form, we can easily calculate its amplitude $|z|$ and its phase $\varphi$:

$$|z| = \sqrt{z \cdot z^*}$$

$$\tan \varphi = i \frac{z - z^*}{z + z^*}$$

Insert $z = U(t)$ from eq.3 into eqs. 4 and 5 to and see what happens!

When dealing with DC currents, we encountered Ohm’s law, which states that the resistance $R$ is the ratio between voltage and current, or $R = \frac{U}{I}$. With AC currents both, $U$ and $I$ are complex numbers, so the resistance is now also complex. A complex resistance we call impedance and denote it by the symbol $Z$.

The building blocks of AC circuits are resistors ($R$, [$\Omega$]), inductors (coils, $L$, [H=Henry]) and capacitors ($C$, [F=Farad]). Their respective impedances are

$$Z_R = R \quad Z_L = i\omega L \quad Z_C = \frac{1}{i\omega C}$$

Now, we are ready to apply all this to an example.
In “Electrical Measurements II” we studied a voltage divider that contained a coil and a resistor (see figure). The output voltage $U_2$ can be calculated exactly as in the DC case, but now using impedances instead of resistor values:

$$ U_2(t) = U_1(t) \cdot \frac{Z_R}{Z_R + Z_L} \quad (7) $$

The alternating input voltage we simply set to

$$ U_1(t) = U_{10} e^{i\omega t} \quad (8) $$

Inserting the appropriate expressions for $Z_R$ and $Z_L$ (eq. 6), and using eq. 4, we find the amplitude of the output voltage as a function of frequency

$$ U_{20}(\omega) = \sqrt{U_2 \cdot U_2^*} = \sqrt{U_1 \cdot U_1^*} \cdot \frac{R}{\sqrt{R + i\omega L}} \cdot \frac{R - i\omega L}{\sqrt{R - i\omega L}} = \frac{U_{10}}{\sqrt{1 + \omega^2 \frac{L^2}{R^2}}} \quad (9) $$

If $\omega = 0$, the input and output amplitudes are the same, but for large $\omega$, the output goes to zero. For this reason, this circuit is called a “Low-pass filter”.

The phase shift $\phi$ between input and output can be calculated with eq. 5:

$$ \tan \phi = \frac{U_2 - U_2^*}{U_2 + U_2^*} = \frac{\omega L}{R} \quad (10) $$

But when we compare these theoretical expressions with our measurement, we see that there is a “little” problem: theory and data look totally different!

The reason for this is that our coil not only has a certain inductance $L$, but also a small capacitance $C$ in parallel. Thus, the correct $Z_{coil}$ is given by

$$ Z_{coil} = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{i\omega L - \frac{1}{i\omega C}}{i\omega L + \frac{1}{i\omega C}} = \frac{i\omega L}{1 - \omega^2 LC} \quad (11) $$

The remarkable thing is that there is a value for $\omega$ for which $Z_{coil}$ becomes infinite and the output goes to zero.

Replacing $Z_L$ in eq. 7 by $Z_{coil}$ you can get the correct expressions for $U_{20}$ and $\tan \phi$: try it!