Laboratory #20: Mechanical Resonance

Goal: Study a driven, harmonic oscillator, resonance behavior, and the effect of damping.


(A) Physics:

Given a mass $m$ on a spring. The spring provides a restoring force $F = -kx$ towards the equilibrium position at $x=0$, where $k$ is the spring constant. Friction (in our case) is proportional to the velocity $dx/dt$. This yields the following equation of motion

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \frac{k}{m} x = 0.$$  (1)

The solution of this equation,

$$x(t) = x_{\text{max}} \cos \omega_o t = x(t = 0) \cdot e^{-\gamma t/2} \cos \omega_o t,$$  (2)

describes an oscillation with a frequency of $\omega_o = 2\pi f_o$, where $\omega_o^2 = k/m$, and with an amplitude $x_{\text{max}}$ that decays exponentially with time $t$. The energy $U$ of the system is given by $U = \frac{1}{2} k x_{\text{max}}^2$, i.e., proportional to the square of the amplitude. A quantity that is often used when describing oscillations is the quality factor $Q$, defined as 2 times the stored energy $U$, divided by the decrease of $U$ per period $T = 1/f_o$, thus

$$Q = \frac{2\pi U}{(dU/dt) \cdot T} = \frac{2\pi e^{-\gamma t}}{\gamma T e^{-\gamma}} = \frac{\omega_o}{\gamma}.$$  (3)

When the oscillator is driven by an external force, $F(t) = F_0 \cos \omega t$, the math becomes more complicated. The equation of motion is like eq.1, but the zero on the right is replaced by $F(t)/m$. One then finds as the solution

$$x(t) = x_{\text{max}}(\omega) \cos(\omega t + \Phi(\omega)),$$  (4)

where \( x_{\text{max}} \) is the amplitude, and \( \Phi \) is the phase difference between the driving force and the response of the system. Both are functions of the driving frequency \( \omega \). Mathematical analysis (see, e.g., [FEY66]) yields

\[
x_{\text{max}}(\omega) = \frac{F_0}{m} \frac{\omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}},
\]  

(5)

and

\[
\tan(\Phi(\omega)) = -\frac{\gamma \omega}{\omega_0^2 - \omega^2}.
\]  

(6)

When the driving frequency \( \omega \) approximately equals the free frequency \( \omega_0 \), the amplitude reaches a maximum. The full width \( \Gamma \) at half the maximum of the resonance curve that you get when you plot energy \( U(\omega) \) (or the square of the amplitude) vs. \( \omega \), is related to the quality factor \( Q \) by

\[
Q = \frac{\omega_0}{\Gamma}.
\]  

(7)

For this relation it is assumed that \( \Gamma \ll \omega_0 \), in which case it is also true that \( \gamma \approx \Gamma \).

Try to derive eq.7, using eqs. 3 and 5. Also, convince yourself that the phase shift between the oscillation and the driving force goes through 90° when the resonance is crossed.

(B) \hspace{1cm} \text{Experiment:}

Determine the spring constant \( k \), and the mass \( m \) of the pendulum. Calculate and measure the free frequency \( \omega_0 \) and compare the two values. Use the driver motor. Check the readout for the driving frequency with a stopwatch.

Friction is applied with an eddy current brake (how does this work? Is friction in this case indeed proportional to velocity?). Adjusting the spacing of the permanent magnets varies the amount of friction. Try to measure the damping time and calculate \( \gamma \).

Map out the resonance (amplitude and phase) as a function of the driving frequency for \( f \) from 0.5 Hz to about 4 Hz. Do this with and without damping. Using eq.7, determine the respective quality factors from the resonance curve, and also calculate from eq.3 the quality factors from the measured damping factors \( \gamma \). Compare the measurements with the calculated, frequency-dependent amplitude and phase shift of eqs.5 and 6.
Adjustment procedure for the ME-9210A apparatus

- Retract the damper magnets.
- Level the apparatus, using the two adjustable feet, such that the mass runs exactly between the damper magnets.
- Run the oscillator at some high frequency and adjust the string length such that the indicator light at the sensor is on half the time. Course adjustment with the clip on the string, fine adjustment with the screw at the very top of the apparatus.
- The amplitude of the driver can be varied with the slide on the big wheel in the back.

References:
