**Introductory Laboratory 0: Buffon's Needle**

**Introduction:** In geometry, the circumference of a circle divided by its diameter is a fundamental constant, denoted by the symbol \( \pi \). Curiously, this number has no simple form. Expressed as a decimal, the digits of \( \pi \),

\[
\pi = 3.141592653589793238462643383279502884197169399375105820974944.\ldots
\]

continue forever without settling into a regular pattern. This mysterious contrast between the simple geometric explanation of the value of \( \pi \), and the complexity of the numerical value has inspired books, movies, and even cults. Because no simple mathematical form exists for \( \pi \), we must rely on methods which are able to to calculate \( \pi \) iteratively where the approximation to its value becomes better and better with each iteration. The first such estimate was made by Archimedes roughly 200 years BC by calculating the circumference of polygons which fit just inside and just outside of a circle. Currently, the world's best estimate of \( \pi \) contains 1.24 trillion decimal places. In this introductory lab we will explore an experimental method to calculate \( \pi \) introduced by the French naturalist Georges Buffon in 1733.

**Method:** Buffon's method for estimating the value of \( \pi \) relies on the following observation. Suppose you mark a surface with parallel lines separated by a distance \( d \). Now imagine throwing a straight object (a needle, or a toothpick) of length \( l \) randomly onto that surface. It can be shown that in the case where \( l < d \), the probability that the needle lands such that it crosses a line is \( p = \frac{2l}{\pi d} \).

We can compute the probability \( p \) experimentally by repeatedly throwing a needle on to the grid. If we throw the needle \( N \) times and record \( n \) hits and \( m \) misses \((N = n + m)\), we can approximate \( p \) as \( p = \frac{n}{N} \). We can then approximate \( \pi \) as \( \pi = \frac{2l}{dp} \) or \( \pi = \frac{2lN}{dn} \). The more throws of the needle we make the better our approximation to \( \pi \) becomes.
Questions:

[1] Can you derive the expression for the probability \( p \)? [Hint: draw a graph with the distance the needle lands from a line on one axis and the angle the needle lands with on the other axis. Which points on this graph produce a hit?]

Let’s do this calculation explicitly (please look at the answer only after you have tried doing the derivation yourself):

Consider a toothpick which lands between two lines. The center of the toothpick must be a distance \( y \leq \frac{d}{2} \) from the nearer of the two lines. Call the angle of the toothpick with respect to the horizontal axis \( \theta \). \(-\frac{\pi}{2} \leq \theta < \pi\). Projecting in the horizontal direction, the length of the toothpick from its center towards the nearer vertical line is \( x = \frac{l}{2} \cos(\theta) \). If \( \frac{l}{2} \cos(\theta) = x \geq y \), the toothpick crosses the line, otherwise it doesn’t.

Consider the simple extreme cases:

If \( \theta = 0 \), \( x = \frac{l}{2} \) so the toothpick crosses the nearest line if \( y \leq \frac{l}{2} \).

If \( \theta = \frac{\pi}{2} \), \( x = 0 \) so the toothpick only crosses the nearest line if \( y = 0 \)

In between, the toothpick crosses the line if \( \frac{l}{2} \cos(\theta) \geq y \).
Let’s plot the *phase space* for the problem (whether the toothpick crosses the line as a function of $y$ and $\theta$.

All four quadrants are the same ($0 \leq \theta \leq \pi$ vs $-\pi \leq \theta \leq 0$ and $0 \leq y \leq \frac{d}{2}$ and $-\frac{d}{2} \leq y \leq 0$) so let’s just consider the quadrant where both $\theta$ and $y$ are positive.

In this quadrant, if $\frac{l}{2} \cos(\theta) \geq y$ then the toothpick crosses the line.

We can draw this result as shown. To the left of the curved blue line, the toothpick crosses the nearest line. To the right it does not cross. Then the probability that the toothpick crosses the nearest line is just the area of the region to the left of the blue line divided by the total area of the rectangle bounded by the dashed and dotted red lines:

The area of the rectangle is:

$$A_{\text{Rectangle}} = \frac{\pi}{2} \times \frac{d}{2} = \frac{\pi d}{4}. \quad (\text{eq. 1})$$

The area to the left of the blue line is:

$$A_{\text{crossing}} = \frac{\pi}{2} \left[\frac{l}{2} \cos(\theta) \right]_{\theta = 0}^{\theta = \frac{\pi}{2}} = \frac{l}{2} \left( \sin\left(\frac{\pi}{2}\right) - \sin(0) \right) = \frac{l}{2}. \quad (\text{eq. 2})$$

So the probability of crossing is:

$$p = \frac{A_{\text{crossing}}}{A_{\text{Rectangle}}} = \frac{\frac{l}{2}}{\frac{\pi d}{4}} = \frac{4l}{2\pi d} = \frac{2l}{\pi d}. \quad (\text{eq. 3})$$

If we rearrange we can calculate $\pi$ from $p$:

$$p = \frac{2l}{\pi d} \Rightarrow \pi = \frac{2l}{pd}, \quad (\text{eq. 4})$$

Which was what we wanted to show.
[2] If the distance \( l \) and \( d \) are measured with a precision \( \sigma_l \) and \( \sigma_d \), and the probability \( P \) is known with uncertainty \( \sigma_P \), what is the uncertainty \( \sigma_\pi \) in the result to \( \pi \)?

Remember our basic relationship for error propagation:

\[
\sigma_f(x_1, x_2, x_3, \ldots) = \left( \sum_{i} \frac{\partial f}{\partial x_i} \sigma_{x_i}^2 \right)^{1/2}.
\]  

(eq. 5)

In this case, the function

\[
\pi(p, l, d) = f(p, l, d) = \frac{2l}{p d},
\]

so

\[
\frac{\partial f}{\partial p} = \frac{2l}{d} \left( -\frac{1}{p^2} \right) = -\frac{\pi}{p}, \quad \frac{\partial f}{\partial l} = \frac{2}{dp} = \frac{\pi}{l}, \quad \frac{\partial f}{\partial d} = \frac{2l}{p} \left( -\frac{1}{d^2} \right) = -\frac{\pi}{d}.
\]  

(eq. 7)

Plug in to eq. 5:

\[
\sigma_\pi(p, l, d) = \left( \frac{-\pi}{p} \sigma_p^2 + \frac{\pi^2}{l} \sigma_l^2 + \frac{-\pi^2}{d} \sigma_d^2 \right)^{1/2}.
\]

(eq. 8)

We can simplify this equation by looking at the relative error rather than the absolute error. Divide through by \( \bar{\pi} \) and eliminate extra \( - \) signs:

\[
\frac{\sigma_\pi(p, l, d)}{\bar{\pi}} = \left( \frac{1}{p} \sigma_p^2 + \frac{1}{l} \sigma_l^2 + \frac{1}{d} \sigma_d^2 \right)^{1/2}.
\]

(eq. 9)

Put \( \sigma_p^2 \) inside \( \ldots \sigma_p^2 \):

\[
\frac{\sigma_\pi(p, l, d)}{\bar{\pi}} = \left( \frac{\sigma_p^2}{p} + \frac{\sigma_l^2}{l} + \frac{\sigma_d^2}{d} \right)^{1/2}.
\]

(eq. 10)

This equation for propagation of relative error is a true as long as the individual errors are uncorrelated with each other (for the more complicated cases, see https://en.wikipedia.org/wiki/Propagation_of_uncertainty). It is also the reason that working with relative error is easier than working with absolute error.

[3] Inferred vs. Relative Error:

From your measurements, you can now calculate both the measured relative error and the theoretical or inferred relative error for \( \pi \):

Given your measured values of \( \{p_1, p_2, \ldots p_N\} \) calculate \( \bar{p} = \frac{1}{N} \sum_{i=1}^{N} p_i \), and \( \sigma_p = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (p_i - \bar{p})^2} \), similarly calculate \( \bar{l}, \sigma_l \) and \( \bar{d}, \sigma_d \) for your measured values of \( \{l_1, l_2, \ldots l_M\} \).
and \(\{d_1, d_2, ... d_Q\}\).

First look at \(\frac{\sigma_p}{p}\), \(\frac{\sigma_p}{p}\) and \(\frac{\sigma_d}{d}\): Is one of these much larger than the others? If so it is the dominant source of error (and you can essentially neglect the others). In your lab write of give these individual values, discuss their relative significance and why they have the relationship they do. **You should make and present this analysis in every experiment you do in this course!**

Now calculate the theoretical or inferred relative error using eq. 10:

\[
\frac{\sigma_{\pi}}{\pi} = \left( \left| \frac{\sigma_p}{p} \right|^2 + \left| \frac{\sigma_l}{l} \right|^2 + \left| \frac{\sigma_d}{d} \right|^2 \right)^{\frac{1}{2}}.
\]  

(eq. 10)

What is this value?

Separately, you will calculate the measured error as follows:

From \(\{p_1, p_2, ... p_N\}\), \(\bar{l}\) and \(\bar{d}\), calculate \(\{\pi_1, \pi_2, ... \pi_N\}\), using eq. 6 \((\pi_i = \frac{2l}{p_i d})\). Your estimated value of

\[
\bar{\pi} = \frac{1}{N} \sum_{i=1}^{N} \pi_i.
\]  

(eq. 11)

The measured relative error is now:

\[
\frac{\sigma_{\pi}^\prime}{\bar{\pi}} = \frac{1}{\bar{\pi}} \sqrt{\frac{1}{\bar{\pi}(N-1)} \sum_{i=1}^{N} (\pi_i - \bar{\pi})^2}.
\]  

(eq. 12)

You expect that the inferred and measured relative errors should be about the same (they need not be exactly equal). Compare the two values and their difference. Is this difference significant? Why or why not? If one relative error is much bigger than the other, you should think carefully about why and explain this discrepancy in your report.

[4] Which uncertainties in your experiment would you label as systematic? Which as statistical? [Hint—look up the concept of counting error in Wikipedia or your statistics notes and textbook. Also consider measurement errors like the accuracy and precision of your ruler, or the possibility of interaction between your toothpicks (what would the effect of clumping toothpicks be on your result)?

[5] If you make \(N\) throws of a single toothpick \(\{p_1, p_2, ..., p_N\}\) resulting in \(n\) hits and \(m\) misses, what is the uncertainty in \(p\)? Consider each throw to be a measurement of the value of \(p\) which results in either a hit \((p_i = 1)\) or a miss \((p_i = 0)\) and apply the formulas for estimating the error on the mean of a distribution from repeated measurements (see statistics handouts from first class) to show that:
\[ \sigma_p^2 = \frac{p(1-p)}{N-1}. \]  

(eq. 13)

As above, I’ll present this derivation (please look at the answer only after you have tried it yourself):

\[ \bar{p} = \frac{1}{N} \sum_{i=1}^{N} p_i, \]  

(eq. 14)

and

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (p_i - \bar{p})^2. \]  

(eq. 15)

Now, expand the square on the right-hand side of equation 15:

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \sum_{i=1}^{N} p_i^2 - 2p_i \bar{p} + \bar{p}^2. \]  

(eq. 16)

Since \( p_i \epsilon \{0,1\} \), \( p_i^2 = p_i \), so:

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \sum_{i=1}^{N} p_i - 2p_i \bar{p} + \bar{p}^2. \]  

(eq. 17)

Expand \(-2p_i \bar{p} = -p_i \bar{p} - p_i \bar{p}^2: \)

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \sum_{i=1}^{N} p_i - p_i \bar{p} - p_i \bar{p} + \bar{p}^2. \]  

(eq. 18)

Factor out \(-p_i\) in the first two terms:

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \sum_{i=1}^{N} p_i(1-\bar{p}) - p_i \bar{p} + \bar{p}^2. \]  

(eq. 19)

\( \bar{p} \) is a constant, so pull \( 1-\bar{p}, \bar{p} \) and \( \bar{p}^2 \) out of the summation:

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \{(1-\bar{p}) \sum_{i=1}^{N} p_i - \bar{p} \sum_{i=1}^{N} p_i + \bar{p}^2 \sum_{i=1}^{N} 1 \}. \]  

(eq. 20)

\[ \sum_{i=1}^{N} p_i = N\bar{p} \] and \( \sum_{i=1}^{N} 1 = N \), so:

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \{(1-\bar{p})N\bar{p} - \bar{p}N\bar{p} + \bar{p}^2N \}. \]  

(eq. 21)

But \( \bar{p}N\bar{p} = \bar{p}^2N \), so the second and third terms cancel, and:

\[ \sigma_{p_i}^2 = \frac{1}{N-1} \{(1-\bar{p})N\bar{p} \}. \]  

(eq. 22)

Remember that the uncertainty of the mean squared \( \sigma_{\bar{p}}^2 \) is \( \frac{1}{N} \) times the uncertainty of the individual measurements squared \( \sigma_{p_i}^2 \), so:

\[ \sigma_{\bar{p}}^2 = \frac{\sigma_{p_i}^2}{N} = \frac{1}{N} \frac{N\bar{p}(1-\bar{p})}{N-1} = \frac{\bar{p}(1-\bar{p})}{N-1}. \]  

(eq. 23)
[6] If we assume that \( l \) and \( d \) are perfectly well known \((i.e., \sigma_l = 0, \sigma_d = 0)\), how many throws \((N)\) do you need to measure \( \pi \) with 1\% uncertainty \((i.e., for \frac{\sigma_\pi}{\pi} = 0.01)\)? Call the uncertainty \( \delta \equiv \frac{\sigma_\pi}{\pi} \).

As above, I’ll present this derivation \((please look at the answer only after you have tried it yourself):\)

From equation 10,
\[
\delta^2 = \frac{\sigma_\pi^2}{\pi^2} = \left| \frac{\sigma_P}{\bar{p}} \right|^2 + \left| \frac{\sigma_l}{l} \right|^2 + \left| \frac{\sigma_d}{d} \right|^2.
\]  
(eq. 24)

Since we assume \( \sigma_l = 0, \sigma_d = 0 \):
\[
\delta^2 = \left| \frac{\sigma_P}{\bar{p}} \right|^2.
\]  
(eq. 25)

From equation 23, \( \sigma_P^2 = \frac{\bar{p}(1-\bar{p})}{N-1} \), so:
\[
\delta^2 = \frac{\bar{p}(1-\bar{p})}{\bar{p}^2(N-1)} = \frac{1-\bar{p}}{\bar{p}(N-1)}.
\]  
(eq. 26)

Solve for \( \delta \):
\[
N - 1 = \frac{1-\bar{p}}{\delta^2 \bar{p}},
\]  
(eq. 27)

So:
\[
N = \frac{1}{\delta^2} \frac{1-\bar{p}}{\bar{p}} + 1.
\]  
(eq. 28)

[7] If we assume that \( l \) and \( d \) are both measured to be \(4.0 \pm 0.1\)cm, how many throws \((N)\) does it take to make the uncertainty resulting from your measurement of \( P \) equal to the uncertainty from the measurements of \( l \) and \( d \)? \([Hint, see point 2 above and eq. 28]\). At this point we refer to the measurement as \textit{systematics-limited} since the statistical error is no longer the largest contribution to the relative error. Repeating the measurement further makes little sense until the systematic errors can be reduced via improved methods and instruments.

[8] In the experiment, you can choose to make the distance \( d \) which separates the lines either nearly equal to \( l \) or much larger than \( l \). What are the advantages and disadvantages of each? Which choice of \( \frac{l}{d} \) gives you the smallest relative error for a given number of repetitions \((N)\)? \([Hint, what value of p gives the smallest possible value for \frac{1-\bar{p}}{\bar{p}} in equation 28? Why is this the optimum?What ratio of \frac{l}{d} gives this value of p (use equation 3)?]\)
**Procedure:**
Count out 20 toothpicks and measure their length. Compute the average length $l$ of the toothpicks in your sample and estimate the uncertainty in this length. Next draw a set of parallel lines on a piece of paper at a distance $d$ apart. Be sure to choose a distance $d$ such that $d > l$. A good choice is to pick a value for $d$ that will give you a large hit probability, do pick $d$ as close to $l$ as you can while ensuring that $l < d$. Estimate your uncertainty in $d$. Devise a way to throw your toothpicks such that they land with random positions and orientations on the paper. Toss any toothpicks that do not land on the paper again. After all 20 sticks have been thrown, record the number of hits. Let's call this set of 20 toothpicks a trial. Repeat this process as many times as you can. Finish at least $N = 50$ trials.

**Analysis:**

[1] Using the estimate $p = \frac{n}{N}$ compute your estimate of $\pi$ after each trial (include all preceding trials in the calculation). Plot the value of $\pi$ as a function of the trial number. Do you see the trend you expect?

[2] Make a histogram which shows the number of times a trial yielded, 1, 2, 3, ..., 20 hits on a line. Calculate the mean and standard deviation of this distribution. For a “counting experiment” like this one, the standard deviation of the distribution is roughly equal to the square-root of the mean. How does this expectation compare with your measurements?

Calculate the uncertainty in the mean of this distribution. Based on this calculation what do you estimate for $\bar{p}$ and for the error in $\bar{p}$ (remember that $\sigma_{\bar{p}} = \frac{\sigma_p}{\sqrt{N}}$)? What value of $\pi$ does this value of $\bar{p}$ yield? What is the uncertainty in $\pi$, $\sigma_{\pi}$ from this measurement?

[3] Use the results derived in Question 4 to estimate $p$ and its uncertainty. How does this compare to the result in part [2] above? What result do you get for $\pi$ using this method?

[4] Exchange your data with as many lab groups as you can. Using all of the data available to you, what is your estimate for $\pi$? What is the uncertainty in your experimental calculation? Which contributions to the uncertainties matter most? The limited statistics? The measurement of $l$? $d$?